2.4 The 2nd Variation

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$$\delta^2 J = \int_{t_0}^{t_f} \frac{1}{2!}\left[\left(\frac{\partial^2 V}{\partial x^2}\right)[\delta x(t)]^2 + 2\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)\delta x(t)\delta \dot{x}(t) + \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)[\delta \dot{x}(t)]^2\right]dt$$

$$= \frac{1}{2}\int_{t_0}^{t_f}\left\{\left[\left(\frac{\partial^2 V}{\partial x^2}\right) - \frac{d}{dt}\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)\right][\delta x(t)]^2 + \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)[\delta \dot{x}(t)]^2\right\}dt$$

The result is obtained by using integration by parts when $\delta x(t_0) = \delta x(t_f) = 0$.

Since $\delta x(t)$ and $\delta \dot{x}(t)$ are arbitrary in the above equation, we can use $\delta^2 J$ to determine whether $J(x(t))$ is a relative optimum at $x = x^*$.
relative min.$\Rightarrow$ $\delta^2 J > 0$

$$\begin{cases} \dfrac{\partial^2 V}{\partial \dot{x}^2} > 0 \\ \left(\dfrac{\partial^2 V}{\partial x^2}\right) - \dfrac{d}{dt}\left(\dfrac{\partial^2 V}{\partial x \partial \dot{x}}\right) > 0 \end{cases}$$

relative max.$\Rightarrow$ $\delta^2 J < 0$

$$\begin{cases} \dfrac{\partial^2 V}{\partial \dot{x}^2} < 0 \\ \left(\dfrac{\partial^2 V}{\partial x^2}\right) - \dfrac{d}{dt}\left(\dfrac{\partial^2 V}{\partial x \partial \dot{x}}\right) < 0 \end{cases}$$

$$\delta^2 J = \int_{t_0}^{t_f} \frac{1}{2!}\left[\left(\frac{\partial^2 V}{\partial x^2}\right)[\delta x(t)]^2 + 2\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)\delta x(t)\delta \dot{x}(t) + \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)[\delta \dot{x}(t)]^2\right]dt$$

$$= \frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix}\delta x(t) & \delta \dot{x}(t)\end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 V}{\partial x^2} & \dfrac{\partial^2 V}{\partial x \partial \dot{x}} \\ \dfrac{\partial^2 V}{\partial x \partial \dot{x}} & \dfrac{\partial^2 V}{\partial \dot{x}^2} \end{bmatrix} \begin{bmatrix}\delta x(t) \\ \delta \dot{x}(t)\end{bmatrix} dt$$

Euler-Lagrange equation

$$\frac{\partial V}{\partial x} - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) = 0$$

For convenience, we denote $V_x = \dfrac{\partial V}{\partial x}$, and $V_{\dot{x}} = \dfrac{\partial V}{\partial \dot{x}}$.

Similarly, $\dfrac{\partial V}{\partial t}$ is denoted as V_t. Also, for the 2nd partial derivatives, we can use the two variables in the subscript. $V_{xx} = \frac{\partial^2 V}{\partial x^2}$, $V_{x\dot{x}} = \frac{\partial^2 V}{\partial x \partial \dot{x}}$, $V_{\dot{x}\dot{x}} = \frac{\partial^2 V}{(\partial \dot{x})^2}$, $V_{t\dot{x}} = \frac{\partial^2 V}{\partial t \partial \dot{x}}$.

> Supplement
>
> $$\frac{d}{dt} f(x,y,z,t)$$
> $$= \frac{\partial}{\partial t} f(x,y,z,t) + \frac{\partial}{\partial x} f(x,y,z,t)\frac{dx}{dt} + \frac{\partial}{\partial y} f(x,y,z,t)\frac{dy}{dt} + \frac{\partial}{\partial z} f(x,y,z,t)\frac{dz}{dt}$$
> $$= \frac{\partial}{\partial t} f(x,y,z,t) + \frac{\partial}{\partial x} f(x,y,z,t)\dot{x} + \frac{\partial}{\partial y} f(x,y,z,t)\dot{y} + \frac{\partial}{\partial z} f(x,y,z,t)\dot{z}$$

$$\frac{\partial V}{\partial x} - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) = 0$$

$$\because \ \frac{d}{dt} V_{\dot{x}}(x,\dot{x},t) = \frac{\partial}{\partial t} V_{\dot{x}}(x,\dot{x},t) + \frac{\partial}{\partial x} V_{\dot{x}}(x,\dot{x},t)\frac{dx}{dt} + \frac{\partial}{\partial \dot{x}} V_{\dot{x}}(x,\dot{x},t)\frac{d\dot{x}}{dt} = V_{t\dot{x}} + V_{x\dot{x}}\dot{x} + V_{\dot{x}\dot{x}}\ddot{x}$$

$$\therefore \ V_x - V_{t\dot{x}} - V_{x\dot{x}}\dot{x} - V_{\dot{x}\dot{x}}\ddot{x} = 0$$

If $V_x = 0$, then $\frac{d}{dt} V_{\dot{x}} = 0$. It means $V_{\dot{x}}$ is not a function of t.

$$V_{\dot{x}} = V_{\dot{x}}(x,\dot{x})$$

Different cases for *Euler-Lagrange equation*

Case 1: $V = V(\dot{x}(t), t) \Rightarrow V_x = 0$

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) = 0$$
$$\Rightarrow V = \int g(\dot{x}(t))\,d\dot{x} + f(t)$$
$$\Rightarrow \frac{\partial}{\partial t} V_{\dot{x}} + \frac{\partial}{\partial \dot{x}} V_{\dot{x}} \frac{d\dot{x}}{dt} = 0$$

$$\because \frac{\partial}{\partial t} V_{\dot{x}} = 0$$

$$\therefore \frac{\partial V_{\dot{x}}}{\partial \dot{x}} = 0$$

when $V_{\dot{x}} = C_1$

$$\Rightarrow V = C_1\dot{x} + f(t)$$

Case 2: $V = V(\dot{x}(t)) \Rightarrow V_x = 0, V_t = 0$

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) = 0 \Rightarrow V_{\dot{x}} = C_1 \Rightarrow V = C_1\dot{x} + C_2$$

Case 3: $V = V(x(t), \dot{x}(t)) \Rightarrow V_t = 0$

$$\Rightarrow V_x - V_{x\dot{x}}\dot{x} - V_{\dot{x}\dot{x}}\ddot{x} = 0$$

$$\dot{x}\left[V_x - V_{x\dot{x}}\dot{x} - V_{\dot{x}\dot{x}}\ddot{x}\right] = 0 \Leftrightarrow \frac{d}{dt}\left(V - V_{\dot{x}}\dot{x}\right) = 0 \Rightarrow V - V_{\dot{x}}\dot{x} = C$$

Derivation of $\dot{x}\left[V_x - V_{x\dot{x}}\dot{x} - V_{\dot{x}\dot{x}}\ddot{x}\right] = \frac{d}{dt}\left(V - V_{\dot{x}}\dot{x}\right)$ If $V = V(x(t),\dot{x}(t))$, then $V_x = V_x(x(t),\dot{x}(t))$, $V_{\dot{x}} = V_{\dot{x}}(x(t),\dot{x}(t))$, and $\frac{d}{dt}(V - V_{\dot{x}}\dot{x}) = \frac{d}{dt}(V) - \frac{d}{dt}(V_{\dot{x}}\dot{x}) = (\dot{x}V_x + \ddot{x}V_{\dot{x}}) - \left[\left(\frac{d}{dt}V_{\dot{x}}\right)\dot{x} + V_{\dot{x}}\ddot{x}\right]$ $= \dot{x}V_x + \ddot{x}V_{\dot{x}} - \left[(V_{x\dot{x}}\dot{x} + V_{\dot{x}\dot{x}}\ddot{x})\dot{x} + V_{\dot{x}}\ddot{x}\right] = \dot{x}\left[V_x - V_{x\dot{x}}\dot{x} - V_{\dot{x}\dot{x}}\ddot{x}\right]$

Case 4: $V = V(x(t),t)$ omitted

P27. Necessary condition for a relative optimum

$$dV = 0, \quad \frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial \dot{x}} = 0$$

Requirement for minimum: $d^2V < 0$

P39. (last week) $\delta^2 J = \frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix}\delta x & \delta\dot{x}\end{bmatrix} \Pi \begin{bmatrix}\delta x \\ \delta\dot{x}\end{bmatrix} dt$

P40. Requirement for minimum $\left(\frac{\partial^2 V}{\partial \dot{x}^2}\right) > 0$

Example 2.9

$J = \int_{t_0}^{t_f} \sqrt{1+\dot{x}^2(t)}\,dt$ show that the straight line is the minimum path between 2 points.

Pf: $V = \sqrt{1+\dot{x}^2(t)}$, $\quad \frac{\partial V}{\partial \dot{x}} = \frac{\dot{x}(t)}{\sqrt{1+\dot{x}^2(t)}}$

$$\because \frac{\partial^2 V}{\partial \dot{x}^2} = \frac{1}{\sqrt{1+\dot{x}^2(t)}} - \frac{1}{2}\frac{2\dot{x}(t)\cdot\dot{x}(t)}{\left(1+\dot{x}^2(t)\right)^{3/2}} = \frac{1}{\left(1+\dot{x}^2(t)\right)^{3/2}} > 0$$

∴The extremum is a minimum.

2.5 Extrema of Functions with Conditions (Constraints)

Example 2.10

Obj.(objective): maximize the volume *V* of cylindrical vessel (container, tank)

s.t. (subject to: condition): material (sheet metal of thickness *t*) for making the vessel is limited, volume of the material = *A* · *t*, where *A* is a constant.

Sol: Capacity of a cylindrical vessel

$$V = \frac{\pi}{4} D^2 h$$

Volume of the material used for making the vessel:

$$2\left(\frac{\pi}{4} D^2\right)t + (\pi D h)t = At$$

The objective is to maximize.

$$V = V(D,h) = \frac{\pi}{4} D^2 h$$

s.t. $A = \frac{\pi}{2} D^2 + \pi D h = \text{constant}$

(i) Direct method (2 independent variables D and h)

Since $A = \frac{\pi}{2} D^2 + \pi D h$, we have

$$h = \left(A - \frac{\pi}{2} D^2\right) \Big/ \pi D \qquad (1)$$

Substituting (1) into V, the capacity of the vessel now is a function of D only.

$$V(D) = \frac{\pi}{4} D^2 \left(A - \frac{\pi}{2} D^2\right) \Big/ \pi D = \frac{1}{4} A D - \frac{\pi}{8} D^3$$

$$\frac{dV^*}{dD} = \frac{1}{4} A - \frac{3}{8} \pi D^{*2} = 0 \qquad D^* = \sqrt{\frac{2A}{3\pi}}$$

Is $V(D^*)$ a relative min. or max.?

$$\because \left(\frac{d^2 V}{dD^2}\right)^* = \frac{-3}{4} \pi D^* < 0 \quad (D > 0)$$

$\therefore V(D^*)$ is a max.

(ii) *Euler-Lagrange Method:*

Max. $V = \frac{\pi}{4} D^2 h$

s.t. $g = \frac{\pi}{2} D^2 + \pi D h - A = 0$

Define the Lagrangian $\mathcal{L}$ as $\mathcal{L} = V + \lambda g$ λ: Lagrange multiplier

$$\mathcal{L} = \frac{\pi}{4} D^2 h + \lambda\left(\frac{\pi}{2} D^2 + \pi D h - A\right)$$

Now there is an additional variable λ in the Lagrangian function $\mathcal{L}$.

$$\mathcal{L} = \mathcal{L}(D,h,\lambda) = \frac{\pi}{4} D^2 h + \lambda\left(\frac{\pi}{2} D^2 + \pi D h - A\right)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial D^*} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial h^*} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda^*} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\pi}{2} D^* h^* + \lambda^* \left(\pi D^* + \pi h^*\right) = 0 & (1) \\ \dfrac{\pi}{4} D^{*^2} + \lambda^* \left(\pi D^*\right) = 0 & (2) \\ \dfrac{\pi}{2} D^{*^2} + \pi D^* h^* - A = 0 \qquad (g = 0) & (3) \end{cases}$$

where "∗" denotes optimum.

$$(2) \Rightarrow \lambda^* = \frac{-D^*}{4} \qquad (4)$$

Substituting (4) into (1), we have

$$\frac{\pi}{2} D^* h^* - \frac{D^*}{4}\left(\pi D^* + \pi h^*\right) = 0$$

$$\frac{1}{4} D^* h^* - \frac{1}{4} D^{*^2} = 0$$

$$h^* = D^* \qquad (5)$$

Substituting (5) into (3), we have

$$A = \frac{3}{2}\pi D^{*^2} \qquad D^* = \sqrt{\frac{2A}{3\pi}}$$

$$V^* = \frac{\pi}{4} D^{*^2} h^* = \frac{\pi}{4} \cdot \frac{2A}{3\pi}\sqrt{\frac{2A}{3\pi}} \qquad V^* = \frac{1}{3\sqrt{6\pi}} A^{\frac{3}{2}}$$

General case of m constraints:

Obj.: optimize $f\left(x_1, x_2, \ldots, x_n\right)$

s.t.: (constraints)

$$\begin{aligned} g_1\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ g_2\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ &\vdots \\ g_m\left(x_1, x_2, \ldots, x_n\right) &= 0 \end{aligned}$$

Define the Lagrangian $\mathcal{L} = f + \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_m g_m = f + \boldsymbol{\lambda}^T \mathbf{g}$

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} g_1\left(x_1, x_2, \ldots, x_n\right) \\ g_2\left(x_1, x_2, \ldots, x_n\right) \\ \vdots \\ g_m\left(x_1, x_2, \ldots, x_n\right) \end{bmatrix}$$

$\lambda_1, \lambda_2, \ldots, \lambda_m$: Lagrange multipliers.

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \lambda_1} = 0 & \Rightarrow \quad g_1 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda_2} = 0 & \Rightarrow \quad g_2 = 0 \\ \quad \vdots & \qquad \vdots \\ \dfrac{\partial \mathcal{L}}{\partial \lambda_m} = 0 & \Rightarrow \quad g_m = 0 \end{cases} \qquad \text{constraints} \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}} = \mathbf{0} \Leftrightarrow \mathbf{g} = \mathbf{0}$$

By solving the set of $(n+m)$ eq., we can find the opt..

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} & \cdots & \dfrac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial g_m}{\partial x_1} & \dfrac{\partial g_m}{\partial x_2} & \cdots & \dfrac{\partial g_m}{\partial x_n} \end{bmatrix}$$