Relation between $\underline{E}$ and $\underline{H}$ in a uniform plane wave $\nabla \cdot \underline{E} = \frac{1}{\varepsilon} \nabla \cdot \underline{D} = 0$

$\Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$ no variation of E_z in the z direction (direction of propagation)

$\Rightarrow \frac{\partial E_z}{\partial z} = 0 \Rightarrow E_z$ be either 0 or constant in space.

$\Rightarrow \frac{\partial^2 E_z}{\partial t^2} = 0 \Rightarrow$ E_z can at most be a linearly increasing function of time. (Inna, p643)

$\Rightarrow E_z = 0$ $\therefore$ a uniform plane wave propagating in the z direction has no z component of

Assuming only E_x exists

$$\nabla \times \underline{E} = -\mu_0 \frac{\partial \underline{H}}{\partial t}$$

$$\nabla \times \underline{E} = \begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \frac{\partial E_x}{\partial z} \underline{a_y} = -\mu_0 \frac{\partial H_y}{\partial t} \underline{a_y}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z}$$

$$H_y = -\frac{1}{\mu_0} \int \frac{\partial E_x}{\partial z} dt + C$$

if $E_x = E_x(z - v_p t)$, $\zeta = z - v_p t$

then
$$\begin{cases} \frac{\partial E_x(z - v_p t)}{\partial z} = \frac{\partial E_x(\zeta)}{\partial \zeta} \frac{\partial \zeta}{\partial z} = \frac{\partial E_x(\zeta)}{\partial \zeta} \\ \frac{d\zeta}{dt} = -v_p, \quad dt = -\frac{d\zeta}{v_p} \end{cases}$$

$$H_y = -\frac{1}{\mu_0} \int \frac{\partial E_x}{\partial z} dt + C$$

$$= -\frac{1}{\mu_0} \int \frac{\partial E_x(\zeta)}{\partial \zeta} \left(-\frac{d\zeta}{v_p} \right) + C$$

$$= \frac{1}{\mu_0 v_p} E_x(\zeta) + C$$

C indicates that a field could have component that is independent of t which would not be a part of wave motion (z- $v_p t$).

$H_y = \frac{1}{\mu_0 v_p} E_x$ in free space, $\boldsymbol{v_p = 1/\sqrt{\mu_0 \varepsilon_0}}$

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0 = 120\pi = 377 \qquad [\Omega]$$ **intrinsic impedance of the free space**

intrinsic impedance of an medium $\eta \equiv \sqrt{\mu / \varepsilon} \qquad [\Omega]$

Similarly,

$$\frac{E_y}{H_x} = -\sqrt{\frac{\mu_0}{\varepsilon_0}} = -\eta_0$$

$$\underline{E} \cdot \underline{H} = E_x H_x + E_y H_y = \eta_0 H_y H_x - \eta_0 H_x H_y = 0$$

$\therefore \quad \underline{E} \perp \underline{H}$

$$\underline{E} \times \underline{H} = \begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ E_x & E_y & 0 \\ H_x & H_y & 0 \end{vmatrix} = \left(E_x H_y - E_y H_x\right)\underline{a_z}$$

Electromagetinc energy flow and the Poynting vector

$$\underline{F} = q\left(\underline{E} + \underline{\mathrm{v}} \times \underline{B}\right)$$

$$dW = q\left(\underline{E} + \underline{\mathrm{v}} \times \underline{B}\right) \cdot \underline{dl}; \qquad \underline{dl} = \underline{\mathrm{v}}dt, \quad dW = Pdt$$

$$P = q\left(\underline{E} + \underline{\mathrm{v}} \times \underline{B}\right) \cdot \underline{d\mathrm{v}}$$

$$= q\underline{\mathrm{v}} \cdot \underline{E}$$

$$dP = \rho_v dv \underline{\mathrm{v}} \cdot \underline{E} \qquad q \to \rho_v dv$$

$$= \left(\rho_v \underline{\mathrm{v}}\right) dv \cdot \underline{E}$$

$$= \underline{J}\, dv \cdot \underline{E} \qquad \underline{J} = \rho_v \underline{\mathrm{v}}$$

power density $p = \frac{dP}{dv} = \underline{J} \cdot \underline{E}$

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}, \qquad \underline{J} = \nabla \times \underline{H} - \frac{\partial \underline{D}}{\partial t}$$

$$\underline{J} \cdot \underline{E} = \underline{E} \cdot \left(\nabla \times \underline{H}\right) - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

$$\nabla \cdot \left(\underline{E} \times \underline{H}\right) = \underline{H} \cdot \left(\nabla \times \underline{E}\right) - \underline{E} \cdot \left(\nabla \times \underline{H}\right)$$

$$\underline{J} \cdot \underline{E} = \underline{H} \cdot \left(\nabla \times \underline{E}\right) - \nabla \cdot \left(\underline{E} \times \underline{H}\right) - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

$$\underline{J} \cdot \underline{E} = -\underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} - \nabla \cdot \left(\underline{E} \times \underline{H}\right), \qquad \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

Poynting vector $\underline{S} = \underline{E} \times \underline{H} \qquad \left[W / m^2\right]$

for the time-varying fields in a linear, homogeneous, and isotropic medium, $\underline{B} = \mu \underline{H}, \underline{D} = \varepsilon \underline{E}$

$$\underline{H} \cdot \frac{\partial \underline{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\underline{B} \cdot \underline{H}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu|\underline{H}|^2\right)$$

$$\underline{E} \cdot \frac{\partial \underline{D}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\underline{D} \cdot \underline{E}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon|\underline{E}|^2\right)$$

$$\underline{J} \cdot \underline{E} = -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu|\underline{H}|^2\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon|\underline{E}|^2\right) - \nabla \cdot \left(\underline{E} \times \underline{H}\right)$$

differential form of Poynting's theorem

$$\int_v \underline{J} \cdot \underline{E} dv = -\frac{\partial}{\partial t}\int_v \left(\frac{1}{2}\mu|\underline{H}|^2 + \frac{1}{2}\varepsilon|\underline{E}|^2\right) dv - \int_v \nabla \cdot \left(\underline{E} \times \underline{H}\right) dv$$

Poynting theorem $-\int_v \left(\underline{E} \times \underline{H}\right) \cdot \underline{ds} = \int_v \underline{J} \cdot \underline{E} dv + \frac{\partial}{\partial t}\int_v \left(\frac{1}{2}\mu|\underline{H}|^2 + \frac{1}{2}\varepsilon|\underline{E}|^2\right) dv$ conservation of energy

Electromagnetic power flow into a closed surface at any instant equals the sum of the time rates of increase of the stored electric and magnetic energies plus the ohmic power dissipated (or electric power generated, if the surface enclosed a source) within the enclosed volume.

$\int_v \left(\underline{E} \times \underline{H}\right) \cdot \underline{ds}$ the flow of energy inward or outward through the surface enclosing the volume v

$\int_v \underline{E} \cdot \underline{J} dv$ Joule's law : the instantaneous power dissipated in the volume v(ohmic power loss)

$\frac{\partial}{\partial t}\int_v \left(\frac{1}{2}\mu|\underline{H}|^2 + \frac{1}{2}\varepsilon|\underline{E}|^2\right) dv$ the rate of decrease of the electromagnetic energy stored in volume v

electric energy density $w_e = \frac{1}{2}\varepsilon|\underline{E}|^2$ $\left[J/m^3\right]$

magnetic energy density $w_m = \frac{1}{2}\mu|\underline{H}|^2$ $\left[J/m^3\right]$

$$-\int_v \underline{S} \cdot \underline{ds} = \int_v \underline{J} \cdot \underline{E} dv + \frac{\partial}{\partial t}\int_v \left(w_e + w_m\right) dv$$

Time-harmonic uniform plane waves in a lossless medium

$$\underline{E}(x,y,z,t) = \underline{E}(\underline{r},t) = Re\left\{\hat{\underline{E}}(\underline{r})e^{j\omega t}\right\} = \frac{1}{2}\left(\hat{\underline{E}}(\underline{r})e^{j\omega t} + \hat{\underline{E}}^*(\underline{r})e^{-j\omega t}\right)$$

$$\underline{H}(x,y,z,t) = \underline{H}(\underline{r},t) = Re\left\{\hat{\underline{H}}(\underline{r})e^{jwt}\right\} = \frac{1}{2}\left(\hat{\underline{H}}(\underline{r})e^{j\omega t} + \hat{\underline{H}}^*(\underline{r})e^{-j\omega t}\right)$$

$$\underline{J}(x,y,z,t) = \underline{J}(\underline{r},t) = Re\left\{\hat{\underline{J}}(\underline{r})e^{jwt}\right\}$$

$$\rho(x,y,z,t) = \rho(\underline{r},t) = Re\left\{\hat{\rho}(\underline{r})e^{jwt}\right\}$$

$$\frac{\partial}{\partial t} \rightarrow j\omega$$

$$\nabla \times \underline{\hat{E}} = -j\omega\mu\underline{\hat{H}} \qquad \nabla \times \underline{\hat{H}} = j\omega\varepsilon\underline{\hat{E}}$$

$$\nabla \cdot \underline{\hat{D}} = \rho \qquad \nabla \cdot \underline{\hat{B}} = 0$$

$$\nabla \times \nabla \times \underline{\hat{E}} = -j\omega\mu\nabla \times \underline{\hat{H}}$$

$$\nabla\left(\nabla \cdot \underline{\hat{E}}\right) - \nabla^2 \underline{\hat{E}} = -j\omega\mu\left(-j\omega\varepsilon\underline{\hat{E}}\right)$$

$$\nabla^2 \underline{\hat{E}} + \omega^2 \mu\varepsilon\underline{\hat{E}} = 0, \qquad \nabla^2 \underline{\hat{E}} + k^2 \underline{\hat{E}} = 0$$

$k = \omega\sqrt{\mu\varepsilon}$ phase constant; wave number; propagation constant

$$\nabla \times \nabla \times \underline{\hat{H}} = j\omega\varepsilon\nabla \times \underline{\hat{E}}$$

$$\nabla\left(\nabla \cdot \underline{\hat{H}}\right) - \nabla^2 \underline{\hat{H}} = j\omega\varepsilon\left(-j\omega\mu\underline{\hat{H}}\right)$$

$$\nabla^2 \underline{H} + \omega^2 \mu\varepsilon\underline{\hat{H}} = 0, \qquad \nabla^2 \underline{H} + k^2 \underline{\hat{H}} = 0$$

$$\underline{\hat{E}} = \hat{E}_x \underline{a_x} + \hat{E}_y \underline{a_y} + \hat{E}_z \underline{a_z}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\hat{E}_x + k^2 \hat{E}_x = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\hat{E}_y + k^2 \hat{E}_y = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\hat{E}_z + k^2 \hat{E}_z = 0$$

Considering only $\hat{\boldsymbol{E}}_x$

Let $\hat{E}_x(x, y, z) = \hat{f}(x)\hat{g}(y)\hat{h}(z)$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\hat{E}_x + k^2 \hat{E}_x = 0$$

$$\hat{f}''\hat{g}\hat{h} + \hat{f}\hat{g}''\hat{h} + \hat{f}\hat{g}\hat{h}'' + k^2 \hat{f}\hat{g}\hat{h} = 0$$

$$\frac{\hat{f}''}{\hat{f}} + \frac{\hat{g}''}{\hat{g}} + \frac{\hat{h}''}{\hat{h}} + k^2 = 0$$

$$\begin{cases} \dfrac{\hat{f}''}{\hat{f}} = -k_x^2 & \hat{f}'' + k_x^2 \hat{f} = 0 \\ \dfrac{\hat{g}''}{\hat{g}} = -k_y^2 & \hat{g}'' + k_y^2 \hat{g} = 0 \\ \dfrac{\hat{h}''}{\hat{h}} = -k_z^2 & \hat{h}'' + k_z^2 \hat{h} = 0 \end{cases}$$

$$\Rightarrow \quad \begin{aligned} \hat{f} &= Ae^{-jk_x x} + Be^{jk_x x} \\ \hat{g} &= Ce^{-jk_y y} + De^{jk_y y} \\ \hat{h} &= Ee^{-jk_z z} + Fe^{jk_z z} \end{aligned}$$

Assuming only $e^{-jk_x x}$ term exists

$$\hat{E}_x(x,y,z) = \hat{f}(x)\hat{g}(y)\hat{h}(z) = ACEe^{-j(k_x x + k_y y + k_z z)}$$

$$\Rightarrow \hat{E}_x(x,y,z) = \hat{E}_{0x} e^{-j(k_x x + k_y y + k_z z)}$$

$$\Rightarrow \begin{cases} \hat{E}_x(x,y,z) = \hat{E}_{0x} e^{-j(k_x x + k_y y + k_z z)} \\ \hat{E}_y(x,y,z) = \hat{E}_{0y} e^{-j(k_x x + k_y y + k_z z)} \\ \hat{E}_z(x,y,z) = \hat{E}_{0z} e^{-j(k_x x + k_y y + k_z z)} \end{cases}$$

$\hat{E}_{0x}, \hat{E}_{0y}, \hat{E}_{0z}$ are constant

$$\underline{k} = k_x \underline{a}_x + k_y \underline{a}_y + k_z \underline{a}_z = k\underline{a}_n, \qquad k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$\underline{r} = x\underline{a}_x + y\underline{a}_y + z\underline{a}_z$$

$$\underline{k} \cdot \underline{r} = k_x x + k_y y + k_z z$$

$$\begin{aligned} \underline{\hat{E}}(x,y,z) &= \hat{E}_x \underline{a}_x + \hat{E}_y \underline{a}_y + \hat{E}_z \underline{a}_z \\ &= \left[\hat{E}_{0x} \underline{a}_x + \hat{E}_{0y} \underline{a}_y + \hat{E}_{0z} \underline{a}_z\right] e^{-j(k_x x + k_y y + k_z z)} \\ &= \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}} \end{aligned}$$

$\underline{\hat{E}}_0$ is a constant vector

$$\begin{aligned} \underline{E}(\underline{r},t) &= \mathrm{Re}\left\{\underline{\hat{E}}(x,y,z)e^{j\omega t}\right\} \\ &= \mathrm{Re}\left\{\underline{\hat{E}}_0 e^{j(\omega t - \underline{k}\cdot\underline{r})}\right\} \end{aligned}$$

Generally speaking, for $\nabla^2 \underline{\hat{E}} + k^2 \underline{\hat{E}} = 0$

$$\underline{\hat{E}}(\underline{r}) = \underline{\hat{E}}(x,y,z) = \underline{\hat{E}}_0^+ e^{-j\underline{k}\cdot\underline{r}} + \underline{\hat{E}}_0^- e^{j\underline{k}\cdot\underline{r}}$$

$\underline{\hat{E}}(x,y,z) = \hat{E}_x(z)\underline{a}_x$ of a uniform plane wave is uniform over the *xy* plane

$\Rightarrow$ independent of the *x* and *y* coordinates

$$\Rightarrow \quad \frac{\partial \hat{E}_x}{\partial x} = \frac{\partial \hat{E}_x}{\partial y} = 0$$

$$\frac{\partial^2 \hat{E}_x}{\partial z^2} + k^2 \hat{E}_x = 0$$

$$\hat{E}_x(z) = \underbrace{\hat{E}_{x0}^+ e^{-jkz}}_{\hat{E}_x^+(z)} + \underbrace{\hat{E}_{x0}^- e^{jkz}}_{\hat{E}_x^-(z)}$$

$$E_x(z,t) = Re\left\{\left(\hat{E}_{x0}^+ e^{-jkz} + \hat{E}_{x0}^- e^{jkz}\right)e^{j\omega t}\right\}$$

Relation between $\underline{\hat{E}}$ and $\underline{\hat{H}}$ in a uniform plane wave

$$\nabla \cdot \underline{\hat{E}} = 0 = \nabla \cdot \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}}$$
$$= e^{-j\underline{k}\cdot\underline{r}} \nabla \cdot \underline{\hat{E}}_0 + \underline{\hat{E}}_0 \cdot \nabla e^{-j\underline{k}\cdot\underline{r}}$$
$$= \underline{\hat{E}}_0 \cdot \nabla e^{-j\underline{k}\cdot\underline{r}}$$
$$= \underline{\hat{E}}_0 \cdot [-j\underline{k} e^{-j\underline{k}\cdot\underline{r}}]$$
$$= -j\left(\underline{k} \cdot \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}}\right)$$
$$= -j\left(\underline{k} \cdot \underline{\hat{E}}\right)$$

$$\underline{k} \cdot \underline{\hat{E}} = 0 \quad \Rightarrow \quad \underline{k} \perp \underline{\hat{E}} \quad \Rightarrow \quad \underline{k} \perp \underline{E}$$

$$\nabla e^{-j\underline{k}\cdot\underline{r}} = \left(\frac{\partial}{\partial x}\underline{a}_x + \frac{\partial}{\partial y}\underline{a}_y + \frac{\partial}{\partial z}\underline{a}_z\right) e^{-j\left(k_x x + k_y y + k_z z\right)}$$
$$= -j\left(k_x \underline{a}_x + k_y \underline{a}_y + k_z \underline{a}_z\right) e^{-j\left(k_x x + k_y y + k_z zt\right)}$$
$$= -j\underline{k} e^{-j\left(k_x x + k_y y + k_z zt\right)}$$
$$= -j\underline{k} e^{-j\left(\underline{k}\cdot\underline{r}t\right)} \qquad \nabla \rightarrow \pm j\underline{k}$$

$$\nabla \times \varphi \underline{A} = \varphi \nabla \times \underline{A} - \underline{A} \times \nabla \varphi$$

$$\nabla \cdot \varphi \underline{A} = \varphi \nabla \cdot \underline{A} + \underline{A} \cdot \nabla \varphi$$

uniform plane wave $\nabla \cdot \underline{\hat{E}}_0 = 0$

$$-j\omega\mu_0 \underline{\hat{H}} = \nabla \times \underline{\hat{E}} = \nabla \times \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}}$$
$$-j\omega\mu_0 \underline{\hat{H}} = -\underline{\hat{E}}_0 \times \nabla e^{-j\underline{k}\cdot\underline{r}}$$

$$j\omega\varepsilon_0 \underline{\hat{E}} = \nabla \times \underline{\hat{H}} = \nabla \times \underline{\hat{H}}_0 e^{-j\underline{k}\cdot\underline{r}}$$
$$j\omega\varepsilon_0 \underline{\hat{E}} = -\underline{\hat{H}}_0 \times \nabla e^{-j\underline{k}\cdot\underline{r}}$$

$$\underline{\hat{H}} = -\frac{1}{-j\omega\mu_0} \underline{\hat{E}}_0 \times -j\underline{k} e^{-j\underline{k}\cdot\underline{r}}$$
$$= \frac{\underline{k} \times \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}}}{\omega\mu_0} = \frac{\underline{k} \times \underline{\hat{E}}}{\omega\mu_0}$$
$$= \frac{k}{\omega\mu_0} \underline{a}_n \times \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}}$$
$$= \frac{1}{\eta_0} \underline{a}_n \times \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}} = \frac{1}{\eta_0} \underline{a}_n \times \underline{\hat{E}}$$
$$= \underline{\hat{H}}_0 e^{-j\underline{k}\cdot\underline{r}}$$
$$\frac{k}{\omega\mu_0} = \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\mu_0} = \sqrt{\frac{\varepsilon_0}{\mu_0}} = \frac{1}{\eta_0}$$

$$\underline{\hat{E}} = -\frac{1}{j\omega\varepsilon_0} \underline{\hat{H}}_0 \times -j\underline{k} e^{-j\underline{k}\cdot\underline{r}}$$
$$= \frac{\underline{\hat{H}}_0 e^{-j\underline{k}\cdot\underline{r}} \times \underline{k}}{\omega\varepsilon_0} = \frac{\underline{\hat{H}} \times \underline{k}}{\omega\varepsilon_0}$$
$$= \underline{\hat{H}}_0 e^{-j\underline{k}\cdot\underline{r}} \times \frac{k}{\omega\varepsilon_0} \underline{a}_n$$
$$= \underline{\hat{H}}_0 e^{-j\underline{k}\cdot\underline{r}} \times \eta_0 \underline{a}_n = \underline{\hat{H}} \times \eta_0 \underline{a}_n$$
$$= \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}}$$
$$\frac{k}{\omega\varepsilon_0} = \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\varepsilon_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0$$

$$\underline{\hat{H}} = \frac{1}{\eta_0} \underline{a}_n \times \underline{\hat{E}}$$

$$\underline{\hat{E}} = \underline{\hat{H}} \times \eta_0 \underline{a}_n$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad [\Omega]$$

$$[\mu_0] = [ML/Q^2], \qquad [\varepsilon_0] = [Q^2T^2/ML^3]$$

$$\Rightarrow \left[\sqrt{\frac{\mu_0}{\varepsilon_0}}\right] = \left[\frac{ML/Q^2}{Q^2T^2/ML^3}\right]^{1/2} = \left[\frac{M^2L^4}{Q^4T^2}\right]^{1/2} = \left[\frac{ML^2}{Q^2T}\right]$$

$$[\Omega] = \left[\frac{ML^2}{Q^2T}\right]$$

Poynting Vector

Complex Poynting vector

$$\underline{S} = \underline{E} \times \underline{H} = \frac{1}{2}\left[\underline{\hat{E}}(\underline{r})e^{j\omega t} + \underline{\hat{E}}^*(\underline{r})e^{-j\omega t}\right] \times \frac{1}{2}\left[\underline{\hat{H}}(\underline{r})e^{j\omega t} + \underline{\hat{H}}^*(\underline{r})e^{-j\omega t}\right]$$

$$= \frac{1}{4}\left[\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}^*(\underline{r}) + \underline{\hat{E}}^*(\underline{r}) \times \underline{\hat{H}}(\underline{r})\right] + \frac{1}{4}\left[\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}(\underline{r})e^{j2\omega t} + \underline{\hat{E}}^*(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})e^{-j2\omega t}\right]$$

since $\left[\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})\right]^* = \underline{\hat{E}}^*(\underline{r}) \times \underline{\hat{H}}(\underline{r}), \qquad \left[\underline{\hat{E}}^*(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})e^{-j2\omega t}\right]^* = \underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}(\underline{r})e^{j2\omega t}$

$$\underline{S} = \frac{1}{2}Re\left\{\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})\right\} + \frac{1}{2}Re\left\{\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}(\underline{r})e^{j2\omega t}\right\}$$

time-average Poynting vector $\left\langle \underline{S_{av}} \right\rangle = \frac{1}{T}\int_0^T \underline{S}dt = \frac{1}{2}Re\left\{\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})\right\}$

complex Poynting vector $\underline{\hat{S}} = \underline{\hat{E}} \times \underline{\hat{H}}^*$

$$\underline{\hat{S}} = \underline{\hat{E}} \times \underline{\hat{H}}^* = \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}} \times \left(\frac{1}{\eta_0}\underline{a}_n \times \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}}\right)^*$$

$$= \underline{\hat{E}}_0 \times \frac{1}{\eta_0}\left(\underline{a}_n \times \underline{\hat{E}}_0\right)^* = \frac{1}{\eta_0}\left(\underline{\hat{E}}_0 \cdot \underline{\hat{E}}_0^*\right)\underline{a}_n - \frac{1}{\eta_0}\left(\underbrace{\underline{\hat{E}}_0 \cdot \underline{a}_n}_{=0}\right)\underline{\hat{E}}_0^*$$

$$= \frac{1}{\eta_0}\left|\underline{\hat{E}}_0\right|^2 \underline{a}_n$$