- Plane waves in lossy media
  (conducting media)

$$\nabla\times\hat{\underline{H}} = j\omega\varepsilon\hat{\underline{E}} + \hat{\underline{J}} = j\omega\varepsilon\hat{\underline{E}} + \sigma\hat{\underline{E}} = (j\omega\varepsilon+\sigma)\hat{\underline{E}} = j\omega\underbrace{\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}_{\hat{\varepsilon}}\hat{\underline{E}}$$

$$= j\omega\hat{\varepsilon}\hat{\underline{E}} \qquad \hat{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$$

$$\nabla\times\nabla\times\hat{\underline{E}} = -j\omega\mu\nabla\times\hat{\underline{H}}$$

$$\nabla\left(\nabla\cdot\hat{\underline{E}}\right) - \nabla^2\hat{\underline{E}} = -j\omega\mu\left(j\omega\hat{\varepsilon}\hat{\underline{E}}\right), \qquad \nabla\cdot\hat{\underline{E}} = 0 \quad \text{for source-free medium}$$

$$\nabla^2\hat{\underline{E}} + \omega^2\mu\hat{\varepsilon}\hat{\underline{E}} = 0$$

$$\nabla^2\hat{\underline{E}} + \hat{k}^2\hat{\underline{E}} = 0$$

$$\hat{k} = \omega\sqrt{\mu\hat{\varepsilon}} = \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} = \omega\sqrt{\mu\varepsilon}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{1/2} \qquad \left[m^{-1}\right]$$

similarly

$$\nabla\times\nabla\times\hat{\underline{H}} = j\omega\hat{\varepsilon}\nabla\times\hat{\underline{E}}$$

$$\nabla\left(\nabla\cdot\hat{\underline{H}}\right) - \nabla^2\hat{\underline{H}} = j\omega\hat{\varepsilon}\left(-j\omega\mu\hat{\underline{H}}\right)$$

$$\nabla^2\hat{\underline{H}} + \omega^2\mu\hat{\varepsilon}\hat{\underline{H}} = 0$$

$$\nabla^2\hat{\underline{H}} + \hat{k}^2\hat{\underline{H}} = 0$$

uniform plane wave propagation in the z direction, with only $\hat{E}_x$ existing

$$\hat{\underline{E}}(\underline{r}) = \hat{\underline{E}}(x,y,z) = \hat{E}_x(z)\underline{a}_x$$

$$\frac{\partial^2\hat{E}_x}{\partial z^2} + \hat{k}^2\hat{E}_x = 0 \qquad \Leftrightarrow \qquad \frac{\partial^2\hat{E}_x}{\partial z^2} - \hat{\gamma}^2\hat{E}_x = 0$$

$$\hat{E}_x(z) = \hat{E}_{x0}^{+}e^{-j\hat{k}z} + \hat{E}_{x0}^{-}e^{+j\hat{k}z}$$

$$= \hat{E}_{x0}^{+}e^{-\gamma z} + \hat{E}_{x0}^{-}e^{+\gamma z}$$

$$\gamma = j\hat{k} = j\omega\sqrt{\mu\hat{\varepsilon}} = j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$= j\omega\sqrt{\mu\varepsilon}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{1/2} = \alpha + j\beta$$

loss $\Rightarrow$ γ is complex $\Rightarrow \gamma = \alpha + j\beta \Rightarrow k = \beta - j\alpha$

$$\Rightarrow \quad \hat{E}_x(z) = \underbrace{\hat{E}_{x0}^{+}e^{-\alpha z}e^{-j\beta z}}_{\hat{E}_x^{+}(z)} + \underbrace{\hat{E}_{x0}^{-}e^{\alpha z}e^{j\beta z}}_{\hat{E}_x^{-}(z)}$$

$$E_x(z,t) = \underbrace{E_{x0}^{+} e^{-\alpha z} \cos(\omega t - \beta z)}_{E_x^{+}(z,t)} + \underbrace{E_{x0}^{-} e^{\alpha z} \cos(\omega t + \beta z)}_{E_x^{-}(z,t)}$$

$$= E_{x0}^{+} e^{-\alpha z} \cos\left[\omega\left(t - \frac{z}{v_p}\right)\right] + E_{x0}^{-} e^{\alpha z} \cos\left[\omega\left(t + \frac{z}{v_p}\right)\right]$$

phase velocity $v_p = \dfrac{\omega}{\beta}$

$$\left|\frac{\hat{E}_x^{+}(z)}{\hat{E}_x^{+}(z+d)}\right| = \frac{\hat{E}_{x0}^{+} e^{-\alpha z}}{\hat{E}_{x0}^{+} e^{-\alpha(z+d)}} = e^{\alpha d}$$

$$\alpha d = \ln\left[\left|\frac{\hat{E}_x^{+}(z)}{\hat{E}_x^{+}(z+d)}\right|\right] \qquad [nepers]$$

$$\alpha(\mathrm{dB}) \equiv 20 \log_{10}\left[\left|\frac{\hat{E}_x^{+}(z)}{\hat{E}_x^{+}(z+d)}\right|\right]$$

$$\begin{cases} \text{attenuation constant } \alpha = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{1/2} & [np/m] \\ \text{phase constant } \beta = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{1/2} & [rad/m] \end{cases}$$

prove:

$$\hat{k} = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)} = \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta = \cos\theta(1 - j\tan\theta) = \cos\theta(1 - jx)$$

$$x = \tan\theta \quad \Rightarrow \quad \begin{cases} \sin\theta = \dfrac{x}{\sqrt{x^2+1}} \\ \cos\theta = \dfrac{1}{\sqrt{x^2+1}} \end{cases}$$

$$\begin{cases} \cos(\theta/2) = \sqrt{\dfrac{1+\cos\theta}{2}} \\ \sin(\theta/2) = \sqrt{\dfrac{1-\cos\theta}{2}} \end{cases}$$

$$1 - jx = \frac{e^{-j\theta}}{\cos\theta} = 1 - j\frac{\sigma}{\omega\varepsilon}$$

$$(1-jx)^{1/2} = \frac{e^{-j\theta/2}}{(\cos\theta)^{1/2}} = \frac{\cos(\theta/2) - j\sin(\theta/2)}{(\cos\theta)^{1/2}} = \frac{\sqrt{\dfrac{1+\cos\theta}{2}} - j\sqrt{\dfrac{1-\cos\theta}{2}}}{(\cos\theta)^{1/2}}$$

$$= \frac{\left(\dfrac{1+\dfrac{1}{\sqrt{x^2+1}}}{2}\right)^{1/2} - j\left(\dfrac{1-\dfrac{1}{\sqrt{x^2+1}}}{2}\right)^{1/2}}{\left(\dfrac{1}{\sqrt{x^2+1}}\right)^{1/2}} = \frac{1}{\sqrt{2}}\left[\left(\sqrt{x^2+1}+1\right)^{1/2} - j\left(\sqrt{x^2+1}-1\right)^{1/2}\right]$$

$$= \frac{1}{\sqrt{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} - j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2}\right\}$$

$$\hat{k} = \omega\sqrt{\mu\varepsilon\left(1-j\frac{\sigma}{\omega\varepsilon}\right)} = \omega\sqrt{\mu\varepsilon}\cdot\frac{1}{\sqrt{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} - j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2}\right\}$$

$$= \omega\sqrt{\frac{\mu\varepsilon}{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} - j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2}\right\}$$

$$\gamma = j\hat{k} = j\omega\sqrt{\mu\hat{\varepsilon}} = j\omega\sqrt{\mu\varepsilon\left(1-j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$= j\omega\sqrt{\frac{\mu\varepsilon}{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} - j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2}\right\}$$

$$= \omega\sqrt{\frac{\mu\varepsilon}{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2} + j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2}\right\} = \alpha + j\beta$$

$$\begin{cases} \alpha = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2} \\ \beta = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} \end{cases}$$

complex intrinsic impedance of a conducting medium

$$\hat{\eta}_c = |\hat{\eta}_c|e^{j\phi_\eta} = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon\left(1-j\frac{\sigma}{\omega\varepsilon}\right)}} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}} e^{j(1/2)tan^{-1}[\sigma/(\omega\varepsilon)]}$$

$\sigma << \omega\varepsilon$ $\Rightarrow$ lossless media

$\beta \approx \omega\sqrt{\mu\varepsilon}$, $\eta_c \approx \sqrt{\mu/\varepsilon}$

$\sigma >> \omega\varepsilon$ $\Rightarrow$ perfect conductor

$\eta_c \to 0$ as $\sigma \to \infty$

loss tangent $tan\,\delta = \dfrac{\sigma}{\omega\varepsilon}$

good conductor $tan\,\delta >> 1$

poor conductor (good insulator) $tan\,\delta << 1$

$$\hat{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right) = \varepsilon' - j\varepsilon''$$

for a good dielectric $\varepsilon'' << \varepsilon'$

loss tangent $tan\,\delta = \dfrac{\varepsilon''}{\varepsilon'} << 1$

$$\hat{k} = \omega\sqrt{\mu\hat{\varepsilon}} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')} = \omega\sqrt{\mu\varepsilon'\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)} = \omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{1/2}$$

$$\cong \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{2}\left(-j\frac{\varepsilon''}{\varepsilon'}\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(-j\frac{\varepsilon''}{\varepsilon'}\right)^2 + \cdots\right]$$

$$\cong \omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{\varepsilon''}{2\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2 + \cdots\right]$$

$$\cong \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] - j\omega\sqrt{\mu\varepsilon'}\left(\frac{\varepsilon''}{2\varepsilon'}\right) \qquad \because \frac{\varepsilon''}{\varepsilon'} << 1$$

$$\gamma = jk = \alpha + j\beta$$

attenuation constant $\alpha \approx \omega\sqrt{\mu\varepsilon'}\left(\dfrac{\varepsilon''}{2\varepsilon'}\right)$ $\quad [np/m]$

phase constant $\beta \approx \omega\sqrt{\mu\varepsilon'}\left[1 + \dfrac{1}{8}\left(\dfrac{\varepsilon''}{\varepsilon'}\right)^2\right]$ $\quad [rad/m]$

phase velocity $v_p = \dfrac{\omega}{\beta} \cong \dfrac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \dfrac{1}{8}\right]$ $\quad [m/s]$

intrinsic impedance $\hat{\eta}_c = \sqrt{\dfrac{\mu}{\hat{\varepsilon}}} = \sqrt{\dfrac{\mu}{\varepsilon'\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)}} = \sqrt{\dfrac{\mu}{\varepsilon'}}\left(1 - j\dfrac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$

$$\cong \sqrt{\frac{\mu}{\varepsilon'}}\left[1 + j\frac{\varepsilon''}{2\varepsilon'} - \frac{3}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2 + \cdots\right]$$

$$\approx \sqrt{\frac{\mu}{\varepsilon'}}\left[1 - \frac{3}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] + j\sqrt{\frac{\mu}{\varepsilon'}}\left(\frac{\varepsilon''}{2\varepsilon'}\right) \qquad [\Omega]$$

for a good conductor $\sigma >> \omega\varepsilon$

loss tangent $\tan\delta = \dfrac{\sigma}{\omega\varepsilon} >> 1$

$$\hat{k} = \omega\sqrt{\mu\hat{\varepsilon}} = \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} \cong \omega\sqrt{\mu\varepsilon\left(-j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$\cong \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}e^{j\pi/4}$$

$$= \sqrt{\omega\mu\sigma}\left(\frac{1-j}{\sqrt{2}}\right)$$

$$\gamma = j\hat{k} = (1+j)\sqrt{\pi f\mu\sigma} = \alpha + j\beta$$

attenuation constant $\alpha \cong \sqrt{\pi f\mu\sigma}$

phase constant $\beta \approx \sqrt{\pi f\mu\sigma}$

intrinsic impedance

$$\hat{\eta}_c = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}} \cong \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= \sqrt{\frac{\omega\mu}{\sigma}}e^{j\pi/4} = \frac{(1+j)}{\sqrt{2}}\sqrt{\frac{2\pi f\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}}$$

$$\hat{\eta}_c = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)(\sigma\delta)^{-1}$$

skin depth $\delta = \dfrac{1}{\alpha} = \dfrac{1}{\sqrt{\pi f\mu\sigma}}$

※ Permittivity is a measure of the degree to which the material is permeated by the electric field in which it is immersed. (Hecht, p41)

※ A measure of the ability of a material to resist the formation of an electric field within it. Also called *dielectric constant*, *relative permittivity*. http://www.answers.com/topic/dielectric-constant

※ In electromagnetism, **permittivity** ε is a measure of how much a medium changes to absorb energy when subject to an electric field. It is defined as the ratio **D** / **E** where **D** is the electric displacement by the medium and **E** is the electric field strength. http://encyclopedia.laborlawtalk.com/Permittivity

※ Relative permittivity is a measure of the polarizability of a material relative to free space. (Paul, p135) Rleative permittivity is a measure of the polarization of the constituent atoms and molecules. (Ramo, p66)

※ Electric susceptibility is a measure of the ability of the material to become polarized and differs from one dielectric to another. (Rao, p219)

※

radiation pressure and electromagnetic moment

$$\nabla \times \underline{\hat{E}} = -j\omega\mu\underline{\hat{H}} \qquad \nabla \times \underline{\hat{H}} = \underline{\hat{J}} + j\omega\varepsilon\underline{\hat{E}}$$

$$\left(\nabla \times \underline{\hat{E}}\right)\cdot \underline{\hat{H}}^* = -j\omega\mu\underline{\hat{H}} \cdot \underline{\hat{H}}^*$$

$$\left(\nabla \times \underline{\hat{H}}^*\right)\cdot \underline{\hat{E}} = \left(\underline{\hat{J}}^* - j\omega\varepsilon\underline{\hat{E}}^*\right)\cdot \underline{\hat{E}} = \underline{\hat{J}}^* \cdot \underline{\hat{E}} - j\omega\varepsilon\underline{\hat{E}}^* \cdot \underline{\hat{E}}$$

$$\nabla \cdot \left(\underline{\hat{E}} \times \underline{\hat{H}}^*\right) = \underline{\hat{H}}^* \cdot \left(\nabla \times \underline{\hat{E}}\right) - \underline{\hat{E}} \cdot \left(\nabla \times \underline{\hat{H}}^*\right)$$

$$\nabla \cdot \left(\underline{\hat{E}} \times \underline{\hat{H}}^*\right) = \underline{\hat{H}}^* \cdot \left(-j\omega\mu\underline{\hat{H}}\right) - \underline{\hat{E}} \cdot \left(\underline{\hat{J}}^* - j\omega\varepsilon\underline{\hat{E}}^*\right)$$

$$= -\underline{\hat{J}}^* \cdot \underline{\hat{E}} + j\omega\left(\varepsilon\underline{\hat{E}}^* \cdot \underline{\hat{E}} - \mu\underline{\hat{H}} \cdot \underline{\hat{H}}^*\right)$$

$$\int_v \nabla \cdot \left(\underline{\hat{E}} \times \underline{\hat{H}}^*\right) dv = \oint_s \left(\underline{\hat{E}} \times \underline{\hat{H}}^*\right)\cdot \underline{ds} = \int_v \left[-\underline{\hat{J}}^* \cdot \underline{\hat{E}} + j\omega\left(\varepsilon\underline{\hat{E}}^* \cdot \underline{\hat{E}} - \mu\underline{\hat{H}} \cdot \underline{\hat{H}}^*\right)\right] dv$$

$$-\oint_s \left(\underline{\hat{E}} \times \underline{\hat{H}}^*\right)\cdot \underline{ds} = \int_v \underline{\hat{J}}^* \cdot \underline{\hat{E}} dv - j\omega\int_v \left(\varepsilon\underline{\hat{E}}^* \cdot \underline{\hat{E}} - \mu\underline{\hat{H}} \cdot \underline{\hat{H}}^*\right) dv$$

$$Re\{\ \} =$$

$$Im\{\ \} =$$