optimal control

Ch.1 Introduction

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The main objective of optimal control is to determine control signals that will cause a process (plant) to satisfy some physical constraints and at the same time extremize (maximize or minimize) a chosen performance criterion (performance index or cost function).

We are interested in finding the optimal control $\mathbf{u}^*(t)$ (* indicates optimal condition) that will drive the plant ***P*** from initial state to final state with some constraints on controls and states and at the same time extremizing the given performance index *J*.

Performance Index may take several forms

1. **Performance Index for Time-Optimal Control System:**
   We try to transfer a system from an arbitrary initial state $\mathbf{x}(t_0)$ to a specified final state $\mathbf{x}(t_f)$ in minimum time. The corresponding performance index (PI) is

   $J = \int_{t_0}^{t_f} dt = t_f - t_0 = t^*$.

2. **Performance Index for Fuel-Optimal Control System:**
   Consider a spacecraft that the magnitude $u(t)$ be the thrust of a rocket engine and assume that the magnitude $|u(t)|$ of the thrust is proportional to the rate of fuel consumption. In order to minimize the total expenditure of fuel, we may formulate the performance index as $J = \int_{t_0}^{t_f} |u(t)| dt$.

   For several controls, we may write it as $J = \int_{t_0}^{t_f} \sum_{i=1}^{m} R_i |u_i(t)| dt$

   Where R_i is a weighting factor for the *i*-th control.

3. **Performance Index for Minimum-Energy Control System:**
   Consider $u_i(t)$ as the current in the *i*-th loop of an electric network. Then $\sum_{i=1}^{m} u_i^2(t) r_i$ (where r_i is the resistance of the *i*-th loop) is the total power or the total rate of energy expenditure of the network. Then, minimization of total expended energy, we have a performance criterion as $J = \int_{t_0}^{t_f} \sum_{i=1}^{m} u_i^2(t) r_i dt$

or in general, $J = \int_{t_0}^{t_f} \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)dt$

where **R** is a positive definite[1] (p.d.) matrix and the superscript "T" denotes transpose here.

4. **Performance Index for Tracking Control System:**
   Similar to the minimum-energy control system, we can think of minimization of the integral of the squared error of a tracking system. Assume $\mathbf{x}_d(t)$ is the desired value, $\mathbf{x}_a(t)$ is the actual value, and $\mathbf{x}(t) = \mathbf{x}_a(t) - \mathbf{x}_d(t)$, is the error. We then have $J = \int_{t_0}^{t_f} \mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t)dt$

   where **Q** is a weighting matrix, which is positive semi-definite[1] (p.s.d.).

5. **Performance Index for Terminal State Control System:**
   In a terminal target problem, we are interested in minimizing the error between the desired target position $\mathbf{x}_d(t_f)$ and the actual target position $\mathbf{x}_a(t_f)$ at the end of the maneuver or at the final time t_f. The terminal (final) error is $\mathbf{x}(t_f) = \mathbf{x}_a(t_f) - \mathbf{x}_d(t_f)$. Taking care of positive and negative values of error and weighting factors, we structure the cost function as $J = \mathbf{x}^{\mathrm{T}}(t_f)\mathbf{F}\mathbf{x}(t_f)$

   which is also called the terminal cost function. Here, **F** is a positive semi-definite matrix.

6. **Performance Index for General Optimal Control System:**
   Combining the above formulations, we have a performance index in general form which causes a linear time-invariant system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ to give the trajectory $\mathbf{x}^*(t)$ that optimizes or extremizes (minimizes or maximizes) a performance index

   $$J = \mathbf{x}^{\mathrm{T}}(t_f)\mathbf{F}\mathbf{x}(t_f) + \int_{t_0}^{t_f} \left[\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)\right]dt$$

   or which causes the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$

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[1] Positive definite = p.d. ⇔ negative definite ; n.d.
Positive semi-definite = p.s.d.⇔ negative semi-definite ; n.s.d.

$$\begin{cases} \text{If } \mathbf{A}_{n\times n} \text{ is p.d., then } \mathrm{X}^{\mathrm{T}}\mathbf{A}\mathbf{X} > 0, \forall \mathbf{X} \neq 0. \\ \text{If } \mathbf{A}_{n\times n} \text{ is p.s.d., then } \mathrm{X}^{\mathrm{T}}\mathbf{A}\mathbf{X} \geq 0, \forall \mathbf{X} \neq 0. \\ \text{If } \mathbf{A}_{n\times n} \text{ is n.d., then } \mathrm{X}^{\mathrm{T}}\mathbf{A}\mathbf{X} < 0, \forall \mathbf{X} \neq 0. \\ \text{If } \mathbf{A}_{n\times n} \text{ is n.s.d., then } \mathrm{X}^{\mathrm{T}}\mathbf{A}\mathbf{X} \leq 0, \forall \mathbf{X} \neq 0. \end{cases}$$

Refer to Appendix A.

to give the state $\mathbf{x}^*(t)$ that optimizes the general performance index

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

where **R** is a positive definite matrix, and **Q** and **F** are positive semi-definite matrices, respectively. Note that the matrices **Q** and **R** may be time varying. The particular form of performance index with **Q** and **R** is called quadratic (in terms of the states and controls, which are 2nd order) form.
The problems arising in optimal control are classified based on the structure of the performance index J.

Mayer problem: $J = S(\mathbf{x}(t_f), t_f)$

Lagrange problem: $J = \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$

Bolza problem: $J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$

There are many other forms of cost functions depending on our performance specifications. However, the afore-mentioned performance indices (with quadratic forms) lead to some very elegant results in optimal control systems.

Furthermore, there might be constraints (either lower limit or upper limit) imposed on the control inputs and state variables.
Constraints

$u_{i,\min} \le u_i \le u_{i,\max}$, $i = 1,2,\cdots m$ or $\mathbf{u}_{\min} \le \mathbf{u} \le \mathbf{u}_{\max}$

$x_{j,\min} \le x_j \le x_{j,\max}$, $j = 1,2,\cdots n$ or $\mathbf{x}_{\min} \le \mathbf{x} \le \mathbf{x}_{\max}$

For example,
$u_1 \le 50$, upper limit
$u_2 \in R$
$u_3 \ge -2$, lower limit

Appendix A. Ruth-Hurwitz criterion of stability

$$\mathbf{A}_{n\times n} = \left[a_{ij} \right]_{n\times n} \quad , \quad a_{ij} > 0$$

$$D_k = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \qquad \det(D) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix} > 0$$

Conditions

k=1,2,3….→ satisfied → $\mathbf{A}_{n\times n}$ is a p.d. matrix →the system is stable!