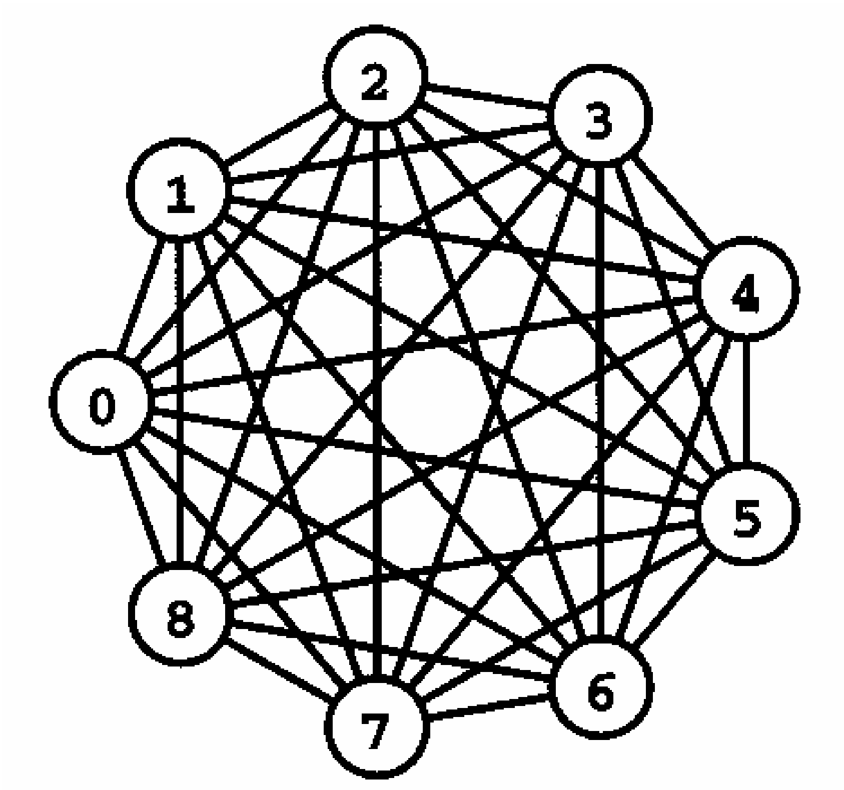
圖形結構



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一、詞彙：

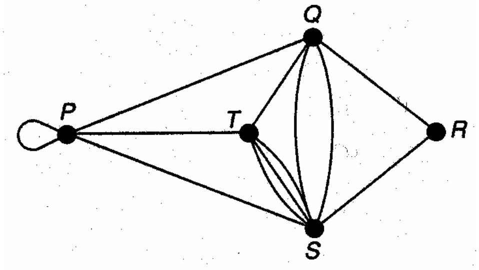
A **graph** is a pair of sets G = (V, E), where V is a set of **vertices**, and E is a set of pairs of vertices called **edges**. A **self-loop** is an edge whose endpoints are equal. **Multiple edges** (or **parallel edges**) are edges having the same pair of endpoints. Graphs that have parallel edges or self-loops are called **multigraphs**; graphs that have no parallel edges and self-loops are referred to as **simple graphs**.

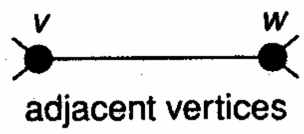
A vertex u is **adjacent to** a vertex v if (u, v) is an edge, i.e., (u, v)$\in$E. The set of vertices adjacent to v is Adj(v). An edge e = (u, v) is **incident with** the vertices u and v, which are the ends of e. Similarly, two distinct edges e and f are adjacent if they have a vertex in common. The **degree** of a vertex u is the number of edges incident with the vertex u.

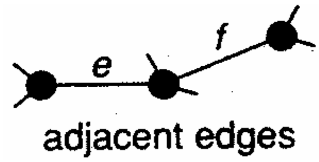
A **complete graph** on n vertices is a graph in which each vertex is adjacent to every other vertex. We use K_n to denote such a graph. A graph $\overline{G}$ is called the **complement** of graph G =(V, E) if $\overline{G}$=(V, F), where, F=E($K_{|V|}$)-E.

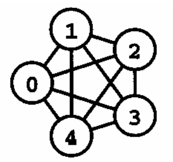
An **isomorphism** from G to H is a bijection f that maps V(G) to V(H) and E(G) to E(H) such that each edge of G with endpoints u and v is mapped to an edge with endpoints f(u) and f(v). A graph is **self-complementary** (or **self-isomorphic**) if it is isomorphic to its complement.

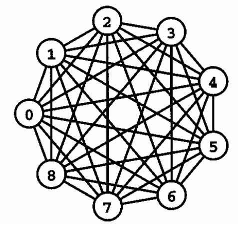
A graph G' = (V', E') is a **subgraph** of a graph G if and only if V'$\subseteq$V and E'$\subseteq$E. If E'= {(u, v) | (u, v) $\in$ E and u, v $\subseteq$ V' } then G' is a **vertex induced subgraph** of G. Unless otherwise stated, by subgraph we mean vertex induced subgraph. A complete subgraph of a graph is called a **clique**. We can obtain subgraphs of a



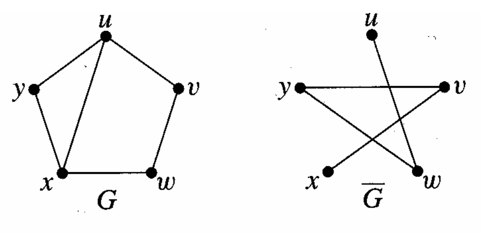




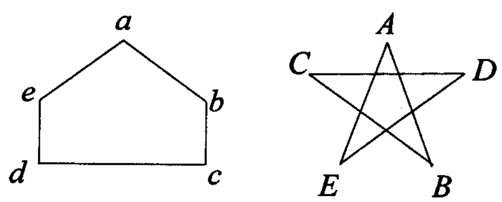




K_5 K_9



complement



self-isomorphism

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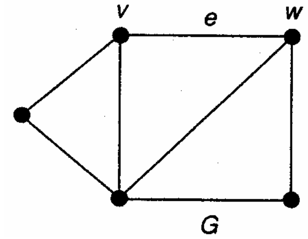
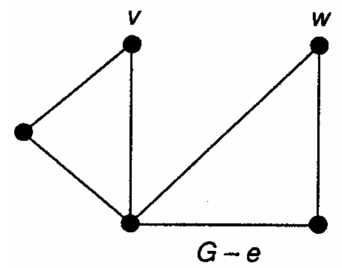
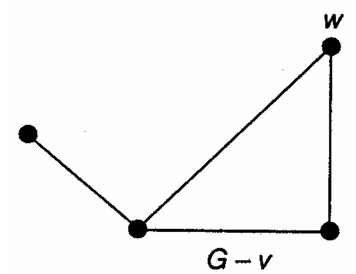
<td>graph by deleting edges and vertices. If e is an edge of a graph G, we denote by G-e the graph obtained from G by deleting the edge e. More generally, if F is any set of edges in G, we denote by G-F the graph obtained by deleting the edges in F. Similarly, if v is a vertex of G, we denote by G-v the graph obtained from G by deleting the vertex v together with the edges incident with v. More generally, if S is any set of vertices in G, we denote by G-S the graph obtained by deleting the vertices in S and all edges incident with any of them.

A **walk** of a graph G is defined as a finite alternating sequence P= v_0, e_1, …, v_{k-1}, e_k, v_k of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it.
A **tour** is a walk in which all edges are distinct. A walk is called an **open walk** if the terminal vertices are distinct. A **path** is an open walk in which no vertex appears more than once. The **length** of a path is the number of edges in it. A path is a **(u, v) path** if v_0= u and v_k=v. A **cycle** is a path of length k, k > 2 where v_0= v_k. A cycle is called **odd** if its length k is odd, otherwise it is an **even cycle**.</td>
<td>

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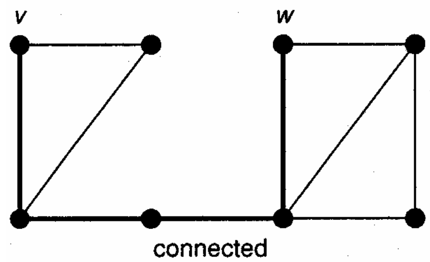
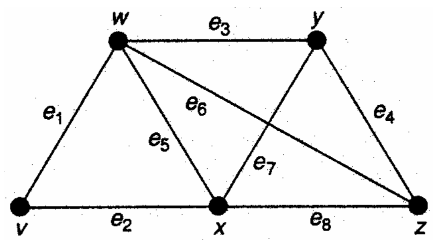
<td>Two vertices u and v in G are **connected** if G has a (u, v) path. A graph is connected if all pairs of vertices are connected. The **distance** from u to v is the length of the shortest path from u to v. A **connected component** of G is a maximal connected subgraph of G. A **disconnecting set** in a connected graph G is a set of edges whose removal disconnects G. For example, in the right graph, the sets {e_1, e_2, e_5} and {e_3, e_6, e_7, e_8} are both disconnecting sets of G. A **cutset** is defined to be a disconnecting set, no proper subset of which is a disconnecting set. In the above example,</td>
<td>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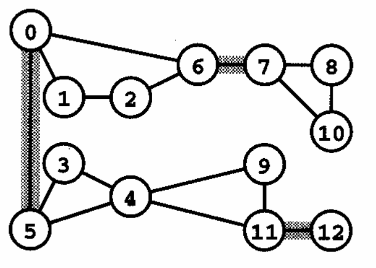
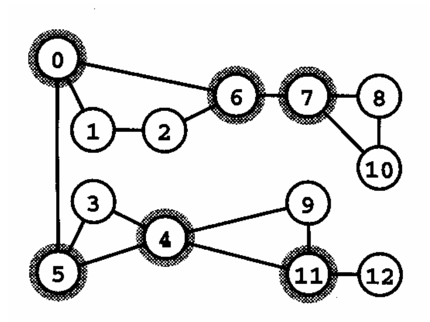
only the second disconnecting set is a cutset. Note that the removal of the edges in a cutset always leaves a graph with exactly two components. If a cutset has only one edge e, we call e a **bridge**. These definitions can be extended to disconnected graph. If G is connected, its **edge connectivity** λ(G) is the size of the smallest cutset in G. We also say that G is **k-edge connected** if $\lambda(G) \geq k$. A graph that has no bridges is said to be **edge-connected**. A graph that is not edge-connected is called as an **edge-separable** graph.

A **separating set** in a connected graph G is a set of vertices whose deletion disconnects G; recall that when we delete a vertex, we also remove its incident edges. If a separating set contains only one vertex v, we call v a **cut-vertex** (or **articulation point**, or **separation vertex**). A graph is **biconnected** if and only if it has no separation vertices.



An edge-separable graph



Articulation points

An acyclic connected graph is called a **tree** (**free tree).** A graph is tree if it satisfies any of the following four conditions:

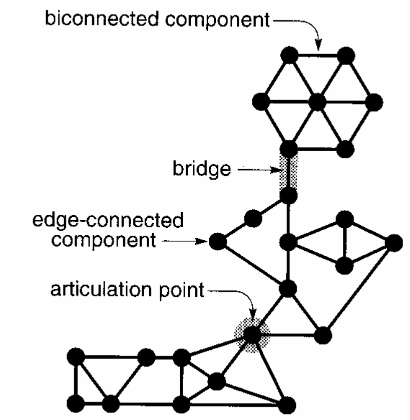
(1) G has $|E|-1$ edges and no cycles.

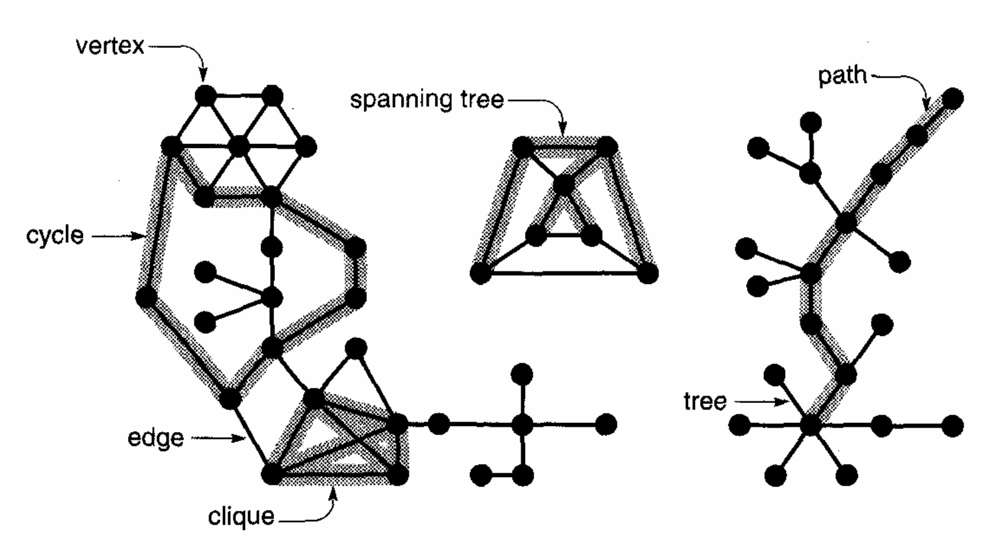
(2) G has $|E|-1$ edges and is connected.

(3) Exactly one simple path connects each pair of vertices in G.

(4) G is connected, but does not remain connected if any edge is removed.

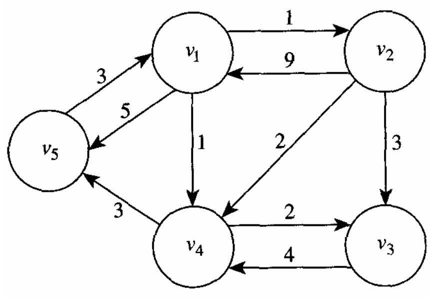
A set of tree is called a **forest**. A **spanning tree** of a connected graph is a subgraph that contains all of that graph's vertices and is a single tree. A **spanning forest** of a graph is a subgraph that contains all of that graph's vertices and is a forest.



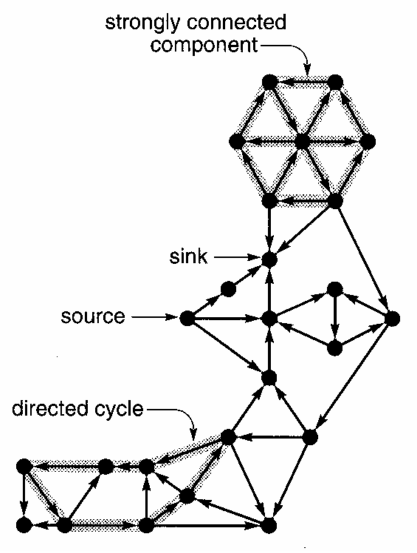


A **directed graph** (**digraph**) is a pair of sets $(V, \vec{E})$, where V is a set of vertices and $\vec{E}$ is a set of ordered pairs of distinct vertices, called **directed edges**. We use the notation $\vec{G}$ for a directed graph, unless it is clear from the context. A directed edge $\vec{e}$ = (u, v) is incident with u and v, the vertices u and v are the **head** and **tail** of $\vec{e}$, respectively; $\vec{e}$ is an **in-edge** of v and an **out-edge** of u. The **in-degree** of u denoted by $d^-(u)$ is equal to the number of in-edges of u, similarly the **out-degree** of u denoted by $d^+(u)$ is equal to the number of out-edges of u. An **orientation** for a graph G = (V,E) is an assignment of direction for each edge. An orientation is called **transitive** if, for each pair of edges (u, v) and (v, w), there exists an edge (u, w). If such a transitive orientation exists for a graph G, then G is called a **transitively orientable graph**. Definitions of subgraph, path, and walk are easily extended to directed graphs.

A **directed acyclic graph** (DAG) is a directed graph with no directed cycles. A vertex u is an **ancestor** of v (and v is a **descendent** of u) if there is a (u, v) directed path in G. A **rooted tree** (or **directed tree**) is a directed acyclic graph in which all vertices have in-degree 1 except the root, which has in-degree 0. The

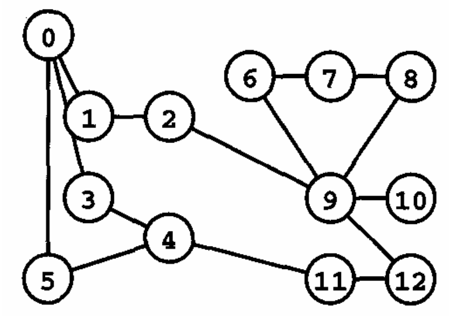


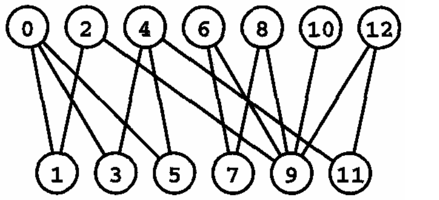
A weighted, directed graph.

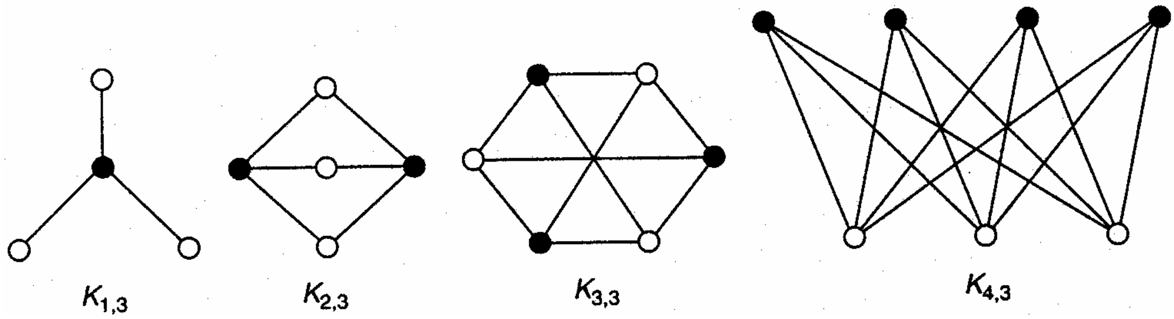


root of a rooted tree T is denoted root(T). The **subtree** of tree T rooted at y is the subtree of T induced by the descendents of u. A **leaf** is a vertex in a directed acyclic graph with no descendents.

A **bipartite** graph is a graph G whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X,Y) is called bipartition of the graph. A **complete bipartite graph** is a bipartite graph with bipartition (X,Y) in which each vertex of X is adjacent to each vertex of Y; if |X|=m and lY|=n, such a graph is denoted by $\mathbf{K_{m,n}}$. An important characterization of bipartite graphs is in terms of odd cycles. A graph is bipartite if and only if it does not contain an odd cycle. Any subgraph of a bipartite graph is bipartite.

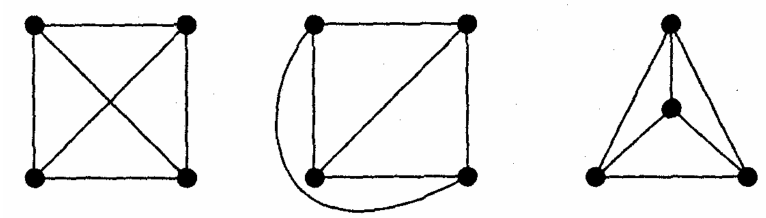




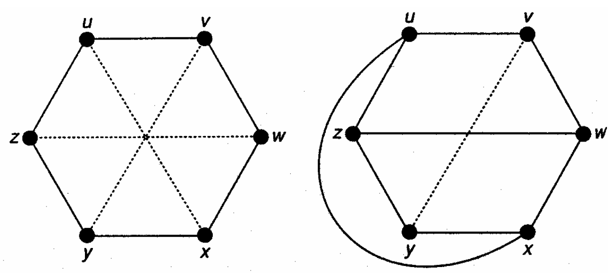


Complete bipartite graph

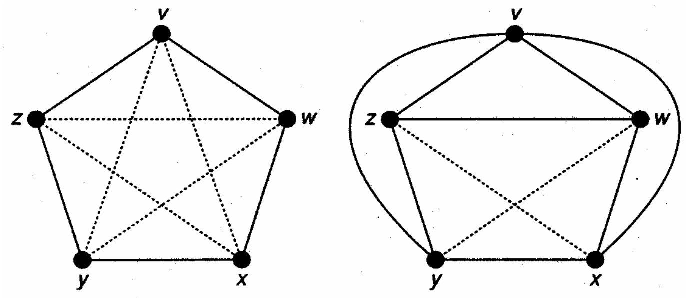
A graph is called **planar** if it can be drawn in the plane without any two edges crossing. For example, K_4 is a planar graph, while K_5 and $K_{3,3}$ is non-planar.



K_4

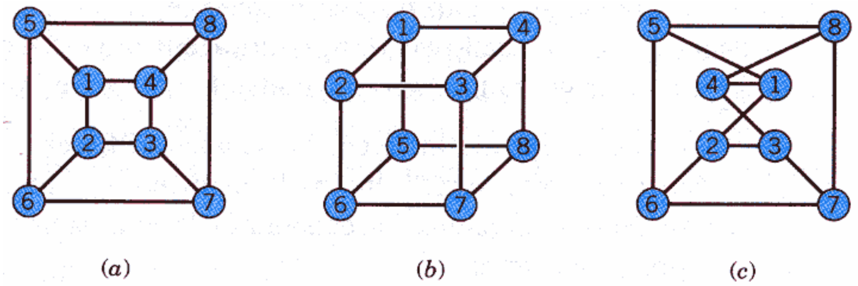


$K_{3,3}$



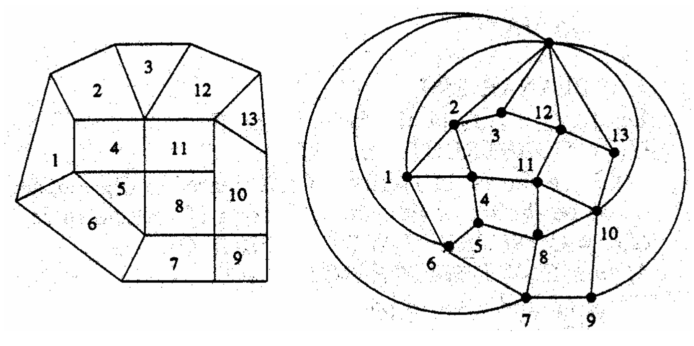
K_5

Notice that there are many different ways of ‘drawing’ a planar graph.



Different sketches of the same graph

A drawing may be obtained by mapping a vertex to a point in the plane and mapping edges to paths in the plane. Each such drawing is called an **embedding** of G. An embedding divides the plane into finite number of regions. The edges which bound a region define a **face**. The unbounded region is called the external or outside face. A face is called an **odd face** if it has odd number of edges. Similarly a face with even number of edges is called an **even face**. The **dual** of a planar embedding T is a graph $G_T = (V_T, E_T)$, such the $V_T = \{v \text{ l } v \text{ is a face in T}\}$ and two vertices share an edge if their corresponding faces share an edge in T.



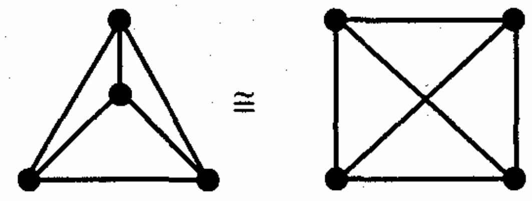
Most graphs that we encounter in practice have relatively few of the possible edges present. To quantify this concept, we define the **density** of a graph to be the average vertex degree, or 2E/V. A **dense graph** is a graph whose average vertex degree is proportional to V; a **sparse graph** is a graph whose complement is dense. In other words, we consider a graph to be dense if E is proportional to V^2 and sparse otherwise.

A **hypergraph** is a pair (V, E), where V is a set of vertices and E is a family of sets of vertices. A hypergraph is a sequence P= v_0, e_1, …, v_{k-1}, e_k, v_k of distinct vertices and distinct edges, such that vertices v_{i-1} and v_i are elements of the edge e_i, 1≤ i ≤ k. Two vertices u and v are connected in a hypergraph if the hypergraph has a (u, v) hyperpath. A hypergraph is **connected** if every pair of vertices is connected.

二、同構 (Isomorphic)

設 $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$為二個無向圖
若存在一函數 $f: V_1 \to V_2$ 滿足：
(1) f 為一對一且映成函數
(2) $\forall a, b \in V_1, \{a, b\} \in E_1 \Leftrightarrow \{f(a), f(b)\} \in E_2$
則稱 f 為一同構函數(isomorphism)且稱 G_1 與 G_2 同構(isomorphic)，記作
$G_2 \cong G_2$

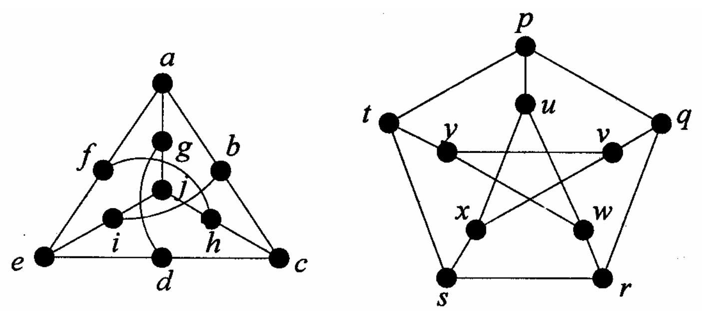
例3：下兩圖同構



(2) Isomorphic graph 的判斷 ：

1. 端點數、邊數、分支度是否相等。
2. 是否為平面圖、區域的個數是否相等。
3. 含有相同長度的簡單路徑個數是否相等。

例4：下兩圖是否同構？

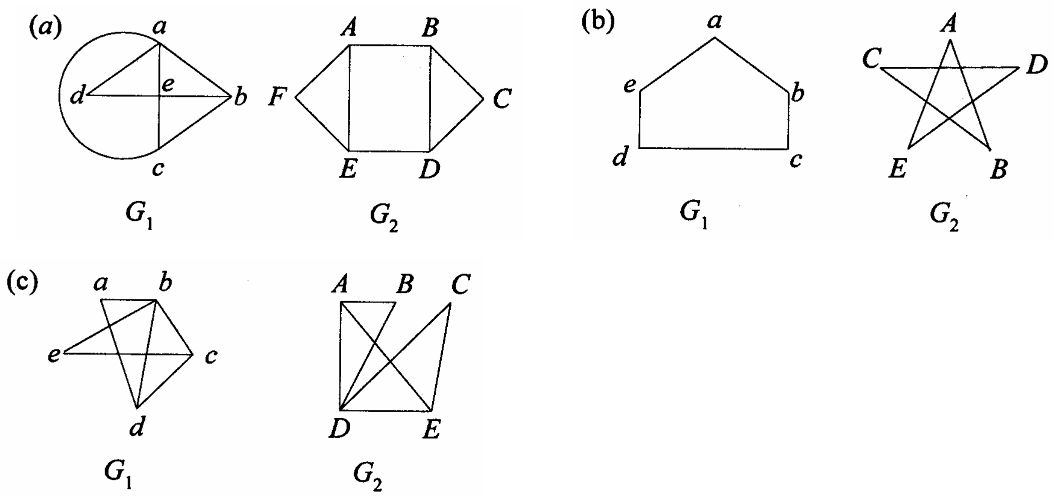


其點的對應如下：

$a \to p, b \to t, c \to y, d \to w, e \to r, f \to q, g \to u, h \to v, i \to s, j \to x$

則很容易可以驗證如此對應保留所有連結性，即此二圖同構

例5：下列圖是否同構？



【解】

(a) G_1 與 G_2 不同構：

因為 G_1 有 5 個點而 G_2 有 6 個點二者不可能同構

(b) G_1 與 G_2 同構：

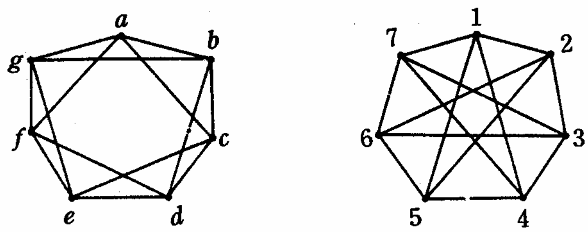
f 定義為 $f(a)=A, f(b)=B, f(c)=C, f(d)=D, f(e)=E$

G_1 與 G_2 皆為 5 個點的環路，故二者同構

(c) G_1 與 G_2 同構：

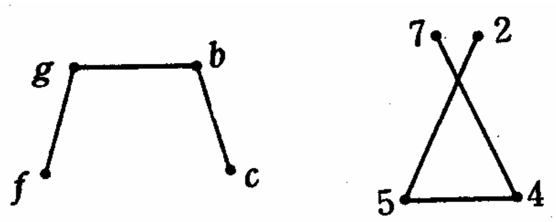
f 定義為 $f(a)=C, f(b)=D, f(c)=A, f(d)=E, f(e)=B$

例6： 下兩圖是否同構？



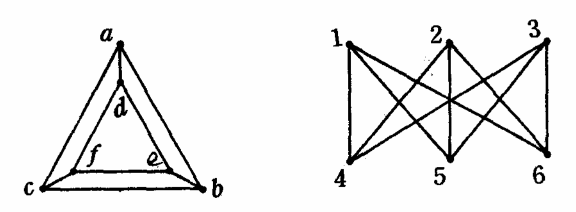
[解]

兩個圖形都有7個頂點14個邊，每一頂點的次數都為4，又且，兩個圖形都有對稱的性質。先對應a-1，接著考慮與a相鄰的點及與1相鄰的點，所形成的子圖如下。



兩個子圖都有一條路徑，一為 f-g-b-c，另一為7-4-5-2，故可對應f-7，g-4，b-5，c-2。(以上對應2亦可)。還剩下d，e與3，6需對應，因g與e相鄰不與d相鄰，4與3相鄰不與6相鄰，故對應為e-3，d-6。故若兩個圖形同構，則同構必為a-1，b-5，c-2，d-6，e-3，f-7，g-4，再一一檢查對應的邊，發現確為同構(若這種配對不為同構，則這兩個圖形不可能同構)。

例7：下兩圖是否同構？

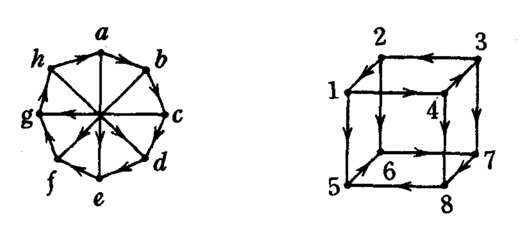


[解]

兩個圖形都有6個頂點9個邊，每一頂點的次數皆為3，又兩個圖形都有對稱性，故可考慮a對應1(換另一種對應亦可)，與a相鄰的點為b, c, d 其中b, c相鄰。而與1相鄰的點為4, 5, 6，這三點互不相鄰，因此不同構。

可換另一種方法證明不同構，觀察左邊圖形含K_3子圖(三角形)，但右邊圖形不含K_3子圖，故知不同構。

例8： 下兩圖是否同構？



[解] 每一圖形都有8個頂點，12個邊。每一圖形有4個頂點含進次數2及出次數1，另4個頂點含進次數1及出次數2。考慮左邊圖形，從任一頂點開始，都可找出一條順時針的循環路徑經過所有頂點，但右邊圖形中，若將頂點分成兩個集合，V_1={1,2,3,4},V_2={5,6,7,8}，可看出從V_2中任一頂點開始，都無法找到一條路徑走到V_1的頂點，故這兩個圖形不同構。

三、路徑問題：

1. Euler circuit and Euler trail

設G=(V, E)為不含孤立點的一無向簡單或多重圖

(1) 若存在一迴路經過G中每一邊恰一次，則稱G有Euler circuit.

(2) 若存在一路徑經過G中每一邊恰一次，則稱G有Euler trail.

2. G有Euler circuit ⇔ G為連通圖且$\forall x \in V$, deg(x)為偶數。

3. G有Euler trail ⇔ G為連通圖且沒有或恰有兩個奇自由度的節點，其餘皆為偶自由度的節點。

4. Hamiltonian cycle and Hamiltonian path

設G = (V, E)為不含孤立點的一無向簡單或多重圖

(1) 若存在一環路經過G中每一點恰一次，則稱G有Hamiltonian cycle.

(2) 若存在一路徑經過G中每一點恰一次，則稱G有Hamiltonian path.

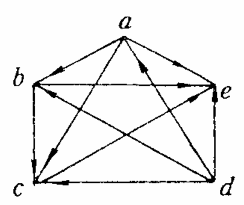
5. Euler問題有一般性的解法；Hamilton問題無一般性的解法。

6. 有向圖中 Hamilton path的畫法：

1. source(sink)當起(終)點。

2. 出分支多、入分支少的先走。

例9：求下圖之Hamilton path

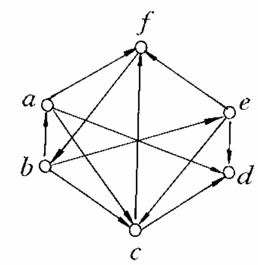


	a	b	c	d	e
indegree	1	2	3	0	4
outdegree	3	2	1	4	0

∴ 此路徑 $d \to a \to b \to c \to e$

例10：求右圖之Hamilton path

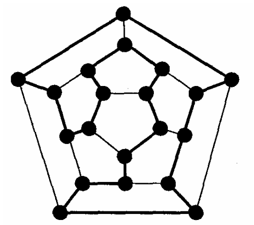
Sol: e→f→b→a→c→d



名詞	意義
Euler trail	無向圖中，包含所有的邊的simple path。
Euler circuit	起點與終點為同一點之Euler trail
Euler graph	存在Euler circuit之graph

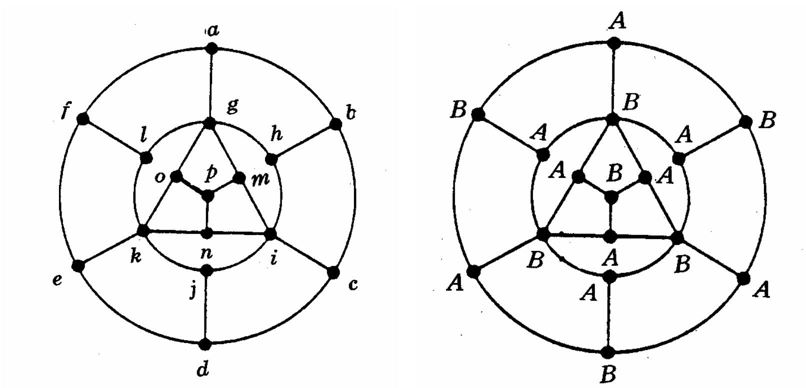
Hamilton path	無向圖中，包含所有的點的elementary path
Hamilton cycle	起點與終點為同一點之Hamilton path
Hamilton graph	存在Hamilton cycle之graph

例11：



1859愛爾蘭數學家Hamilton將20個知名的大都市平面化對應到上圖，欲環遊這20個都市且各都市恰走過一次，是否存在一種走法?此問題相當於問是否上圖中存在一漢米爾頓環路，圖中粗線部份即代表一漢米爾頓環路(並不唯一)。

例12： 證明下圖不存在Hamilton path。



[解]

將頂點a標識A，次將與A相鄰的頂點標識B，再將與B相鄰的頂點標識A，又將與A相鄰的頂點標識B，一直到所有的頂點都標識為A與B，如圖所示。若圖形有漢彌路徑，此路徑必交互經過A,B，但圖形中有9個頂點為A，7個頂點為B，故知不存在漢彌頓路徑。

四、雙分圖：G所有的邊自成一切集。

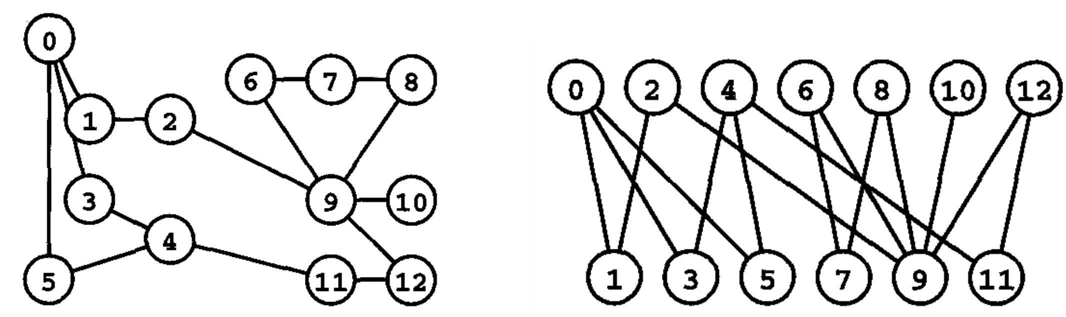
(1) 判斷法：節點可以交替的以0、1編號。

(2) 雙分圖中，若節點0的個數不等於節點1的，則不存在Hamilton cycle。

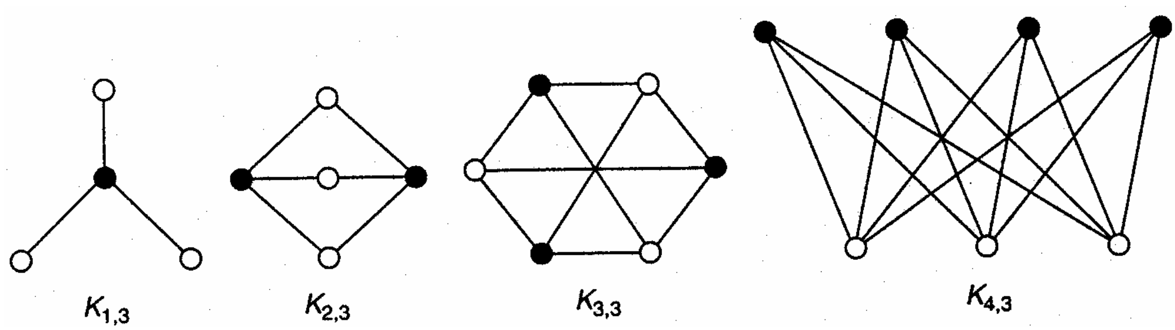
(3) 若節點0的個數與節點1的相差一個，則存在Hamilton path。

(4) 完全雙分圖：在雙分圖中，若V_1中每一點與V_2中每一點均有邊相連接，

則稱為完全雙分圖，以$K_{m,n}$表示(規定$m \le n$)



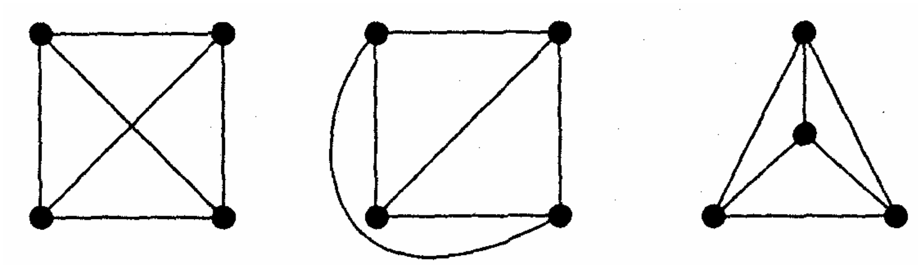
Bipartite graph



Complete bipartite graph

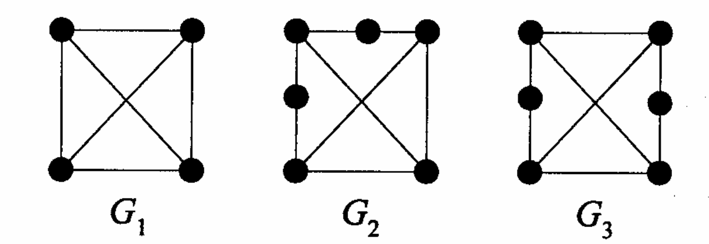
五、平面圖：可將圖形映射至平面上，使得邊不相交。

例13：K_4



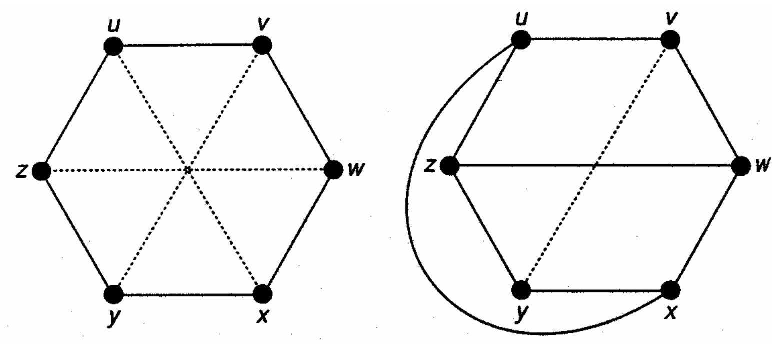
(1) **Homeomorphic** (同胚)：兩圖形若忽視分支度=2的節點即為同構，稱之。

例14：

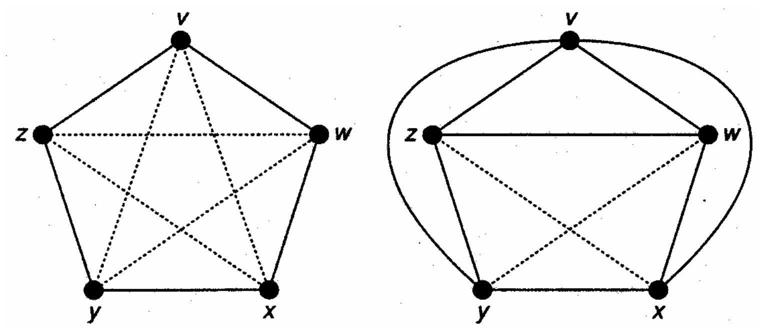


上圖中, G_1, G_2, G_3 皆同胚

(2) 平面圖不含$K_{3,3}$及K_5的同胚子圖，反之亦然。



$K_{3,3}$



K_5

(3) 區域與邊界：

(i) 平面圖的一個區域(region 或face)指以圖的邊為界圍成的一塊面積且內部不含任何頂點或邊。

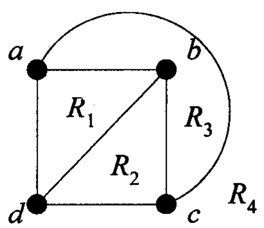
(ii) Finite region指面積為有限的區域。

(iii) Infinite region指面積為無限的區域。

(vi) 一個區域的contour指包圍此區域的邊。

(v) 二個區域至少有一個共同的邊時，稱此兩區域相鄰(adjacent)。

例15：



上圖 K_4 為一平面圖

(1)有四個區域 R_1, R_2, R_3 及 R_4, 其中 R_1, R_2, R_3 為有限區域, R_4 為無限區域

(2) R_1 的邊界為$\{a, b\}, \{b, d\}, \{a, d\}$, R_4 的邊界為$\{a, c\}, \{c, d\}, \{a, d\}$

(3) R_2 與 R_1, R_3, R_4 相鄰

(4) 尤拉公式：G為無向連通圖，v, e, r分別為節點數、邊數和區域個數。若G為平面圖，則v-e+r = 2。

例16： 一個平面圖有10個頂點，每個頂點的分支度皆為3，則此平面圖有幾個區域？

解： $2e = 10\times 3 \Rightarrow e=15 \Rightarrow r = e-v+2 = 15-10+2 = 7$

(5) G為無向連通圖，若G為平面圖，證明$1.5r \le e \le 3(v-2)$

【證明】

令 N 表所有區域所圍成的邊數總和(含無限區域及重複邊)

因為每個區域至少含 3 邊，故 $N \ge 3r$

因為每個邊至多落在 2 個區域邊界上，故 $N \le 2e$

所以 $3r \le N \le 2e$

$\Rightarrow \frac{3}{2}r \le e$ 或 $r \le \frac{2}{3}e$

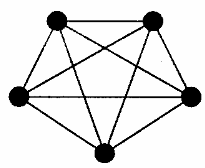
因為 G 為連通平面圖滿足尤拉公式 $v-e+r=2$

$\Rightarrow 2 = v-e+r \le v-e+\frac{2}{3}e = v-\frac{1}{3}e$

$\Rightarrow e \le 3v-6$

例17：證明K_5不是平面圖。

【證明】



K_5 的點數 $v=5$，邊數 $e=10$, $10=e>3v-6=9$

所以 K_5 不為平面圖

(6) G為簡單平面圖，G至少有一節點，其分支度≤5。

證

假設 $\forall v \in V \quad \deg(v) \geq 6$

$\Rightarrow \sum_{i=1}^{n} \deg(v_i) \geq 6n$

$\Rightarrow 2e \geq 6n$

$\Rightarrow e \geq 3n$

但 $e \leq 3n - 6$ 代入

$\Rightarrow 3n - 6 \geq 3n$

$\Rightarrow -6 \geq 0$，矛盾

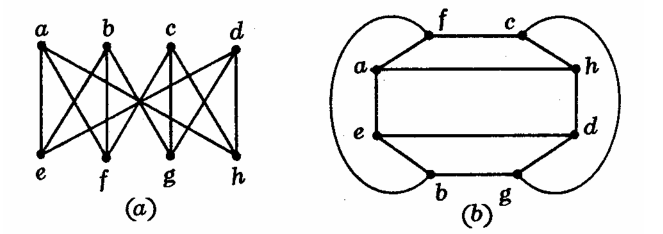
$\therefore$ 必定至少有一頂點 $\deg(v) \leq 5$

(7) 平面圖的判斷法：

1. 繪圖法：首先繪出最長的迴路；接著將剩下的邊繪於迴路之內或外。

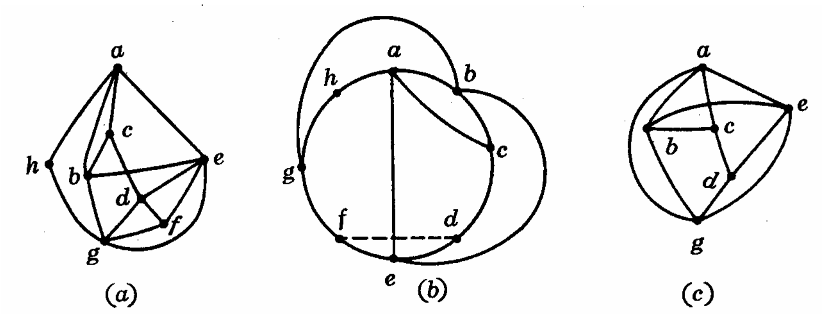
2. 平面圖的收縮：將平面圖的一邊收縮成一點，所得之圖仍為平面圖；收縮過程應避免出現複圖。

例16：下圖是否為平面圖？



【解】 先盡可能找一最長的迴路（循環路徑），我們找出 $a-f-c-h-d-g-b-e-a$，接著考慮剩下的 4 個邊 $\{a, h\}$，$\{b, f\}$，$\{c, g\}$，$\{d, e\}$ 如圖8.24 (b)，先將 $\{a, h\}$ 放在迴路內，則 $\{b, f\}$，$\{c, g\}$ 需在迴路外，剩下 $\{e, d\}$ 可放在迴路內，故此圖形爲平面。

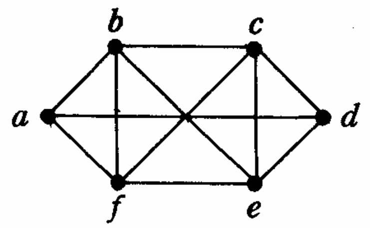
例17：下圖是否為平面圖？



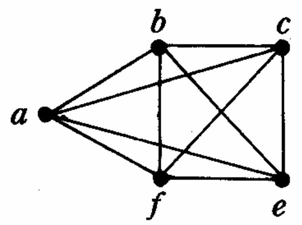
【解】 首先找出最長的迴路 $a-b-c-d-e-f-g-h-a$，如圖8.25 (b)，剩下的邊可先將 $\{a, c\}$ 放在迴路內，則 $\{b, e\}$，$\{b, g\}$ 需在迴路外，因此 $\{a, e\}$ 需在迴路內，現在考慮 $\{d, f\}$，則無法繪出平面圖。爲確定非平面性，我們需找出與 K_5 或 $K_{3,3}$ 同胚的子圖。對 K_5 而言，需要 5 個次數爲 4 的頂點，圖形中恰有 5 個頂點 a，b，d，e，g，其次數至少爲 4，但對 d 而言，沒有 $\{a, d$

}，{ b ，d } 的邊，故不可能有K_5 。現在考慮$K_{3,3}$ ，因其 6 個頂點次數皆爲 3，而在原圖形中，h 的次數爲 2，故不必考慮，f 和 d 很相似，故刪去 f ，剩下的 6 個點 a ，b ，c ，d ，e ，g 所生成的子圖如圖 8.25 (c)，其中只有 c ，d 次數爲 3，故在 a ，b ，e ，g 中需移去 2 個邊，（使得每個頂點的次數爲 3 ），對照圖 8.23 之$K_{3,3}$圖形可找出適當的配對，例如 c — 1，故可得同構 a — 4，b — 5 ，c — 1 ，d — 6，e — 2，g — 3，而 { a ，b }，{ e ，g }需移去。

例18： 下圖是否為平面圖？



G



G_1

Sol：在原圖形中，V = 6, E = 11 ≤ 3V-6 =18- 6 = 12，現收縮(a，d)而將頂點a，d併為一頂點，改變後的圖形G_1中，V_1 = 5, E_1 =10 > 3V_1-6 =9，故G_1不為平面圖，因此G亦不為平面圖。