Reflection, transmission, and refraction of waves at planar interfaces

Electromagnetic boundary conditions

1. Faraday's law

$$\oint_c \underline{E}\cdot \underline{dl} = -\frac{\partial}{\partial t}\int_s \underline{B}\cdot\underline{ds} \qquad \nabla\times\underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\Rightarrow \quad \underline{a_n}\times\left\lfloor \underline{E_2}-\underline{E_1}\right\rfloor = 0 \quad \rightarrow \quad E_{1t} = E_{2t}$$

2. Ampere's law

$$\oint_c \underline{H}\cdot\underline{dl} = \int_s \underline{J}\cdot\underline{ds} + \frac{d}{dt}\int_s \underline{D}\cdot\underline{ds} \qquad \nabla\times\underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

in geneal $J_S = 0 \;\Rightarrow\; \underline{a_n}\times\left\lfloor \underline{H_2}-\underline{H_1}\right\rfloor = 0 \;\rightarrow\; H_{2t} = H_{1t}$

a perfect conductor $(\sigma = \infty)$

$$\Rightarrow \quad \underline{a_n}\times\underline{H_1} = \underline{J_s}$$

3. Gauss's law for electric fields

$$\int_s \underline{D}\cdot\underline{ds} = \int_v \rho dv \qquad \nabla\cdot\underline{D} = \rho$$

$$\Rightarrow \quad \underline{a_n}\cdot\left(\underline{D_2}-\underline{D_1}\right) = \rho_S \quad \rightarrow \quad D_{2n} - D_{1n} = \rho_S$$

4. Gauss's law for magnetic fields

$$\int_s \underline{B}\cdot\underline{ds} = 0 \qquad \nabla\cdot\underline{B} = 0$$

$$\Rightarrow \quad \underline{a_n}\cdot\left(\underline{B_2}-\underline{B_1}\right) = 0 \quad \rightarrow \quad B_{1n} = B_{2n}$$

5. equation of continuity

$$\int_s \underline{J}\cdot\underline{ds} = -\frac{\partial}{\partial t}\int_v \rho dv \qquad \nabla\cdot\underline{J} = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \quad \underline{a_n}\cdot\left(\underline{J_{2n}}-\underline{J_{1n}}\right) = -\frac{\partial \rho_S}{\partial t} \quad \rightarrow \quad \underline{J_{2n}}-\underline{J_{1n}} = -\frac{\partial \rho_S}{\partial t}$$

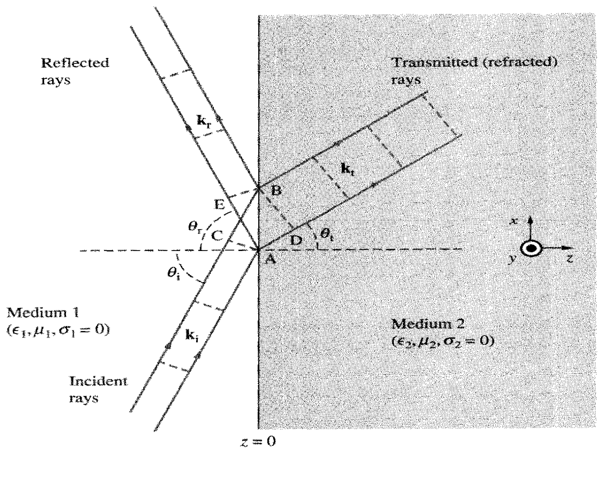
Snell's law of refraction

$$\sin\theta_i = \frac{AC}{AB} = \frac{v_{p1}t}{AB}, \qquad \sin\theta_t = \frac{AD}{AB} = \frac{v_{p2}t}{AB}$$

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{v_{p1}}{v_{p2}} = \sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}}$$

$$n = c/v_p = \sqrt{(\mu\varepsilon)/(\mu_0\varepsilon_0)}$$

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1} \quad \Rightarrow \quad n_1\sin\theta_i = n_2\sin\theta_t$$



for nonmagnetic media $\mu_1 = \mu_2 = \mu_0$

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_{2r}}{\varepsilon_{1r}}}, \qquad \sqrt{\varepsilon_{1r}}\sin\theta_i = \sqrt{\varepsilon_{2r}}\sin\theta_t$$

since $\underbrace{CB}_{v_{p1}t} = \underbrace{AE}_{v_{p1}t}$ $\Rightarrow$ $\underbrace{AB\sin\theta_i}_{CB} = \underbrace{AB\sin\theta_r}_{AE}$

Snell's reflection law $\sin\theta_i = \sin\theta_r$, $\quad \theta_i = \theta_r$

$$\langle \underline{S}_{iav} \rangle \cos\theta_i = \langle \underline{S}_{rav} \rangle \cos\theta_r + \langle \underline{S}_{tav} \rangle \cos\theta_t$$

$$\frac{\left|\hat{E}_{i0}\right|^2}{2\eta_1}\cos\theta_i = \frac{\left|\hat{E}_{r0}\right|^2}{\eta_1}\cos\theta_r + \frac{\left|\hat{E}_{t0}\right|^2}{\eta_2}\cos\theta_t$$

Fresnel's equations

$$\Gamma_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$
$$\Im_\perp = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)}$$
$$\Gamma_{//} = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$
$$\Im_{//} = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}$$

$$\Gamma_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$
$$\Im_\perp = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i - n_t\cos\theta_t}$$
$$\Gamma_{//} = \frac{n_i\cos\theta_t - n_t\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}$$
$$\Im_{//} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}$$

$$\Gamma_\perp = \frac{\cos\theta_i - \sqrt{(n_t/n_i)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(n_t/n_i)^2 - \sin^2\theta_i}}$$
$$\Im_\perp = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{(n_t/n_i)^2 - \sin^2\theta_i}}$$
$$\Gamma_{//} = \frac{-\left(n_t^2/n_i^2\right)\cos\theta_i + \sqrt{\left(n_t^2/n_i^2\right) - \sin^2\theta_i}}{\left(n_t^2/n_i^2\right)\cos\theta_i + \sqrt{\left(n_t^2/n_i^2\right) - \sin^2\theta_i}}$$
$$\Im_{//} = \frac{2(n_t/n_i)\cos\theta_i}{\left(n_t^2/n_i^2\right)\cos\theta_i + \sqrt{\left(n_t^2/n_i^2\right) - \sin^2\theta_i}}$$

Incident wave	Reflected wave	Refracted wave

$$\hat{\underline{E}}_i = \hat{\underline{E}}_{i0}e^{-j\underline{k}_i\cdot\underline{r}} \qquad \hat{\underline{E}}_r = \hat{\underline{E}}_{r0}e^{-j\underline{k}_r\cdot\underline{r}} \qquad \hat{\underline{E}}_t = \hat{\underline{E}}_{t0}e^{-j\underline{k}_t\cdot\underline{r}}$$
$$\hat{\underline{H}}_i = \hat{\underline{H}}_{i0}e^{-j\underline{k}_i\cdot\underline{r}} \qquad \hat{\underline{H}}_r = \hat{\underline{H}}_{r0}e^{-j\underline{k}_r\cdot\underline{r}} \qquad \hat{\underline{H}}_t = \hat{\underline{H}}_{t0}e^{-j\underline{k}_t\cdot\underline{r}}$$

The plane of incidence is defined as the plane containing the normal to the interface and the k-vector indicating the direction of propagation of the incident wave (i.e. the *xz* plane).

at the interface $z = 0$, $\quad \underline{a}_z \times \left[\underline{E}_2 - \underline{E}_1\right] = 0$

$\underline{a}_z$ is the unit vector normal to the interface

$$\underline{a}_z \times \underline{\hat{E}}_i\big|_{z=0} + \underline{a}_z \times \underline{\hat{E}}_r\big|_{z=0} = \underline{a}_z \times \underline{\hat{E}}_t\big|_{z=0}$$

$$\underline{a}_z \times \underline{\hat{E}}_{i0} e^{-j\underline{k}_i\cdot\underline{r}}\big|_{z=0} + \underline{a}_z \times \underline{\hat{E}}_{r0} e^{-j\underline{k}_r\cdot\underline{r}} = \underline{a}_z \times \underline{\hat{E}}_{t0} e^{-j\underline{k}_t\cdot\underline{r}}$$

$$\Rightarrow \begin{cases} \underline{a}_z \times \underline{\hat{E}}_{i0}\big|_{z=0} + \underline{a}_z \times \underline{\hat{E}}_{r0}\big|_{z=0} = \underline{a}_z \times \underline{\hat{E}}_{t0}\big|_{z=0} \\ \underline{k}_i \cdot \underline{r}\big|_{z=0} = \underline{k}_r \cdot \underline{r}\big|_{z=0} = \underline{k}_t \cdot \underline{r}\big|_{z=0} \end{cases}$$

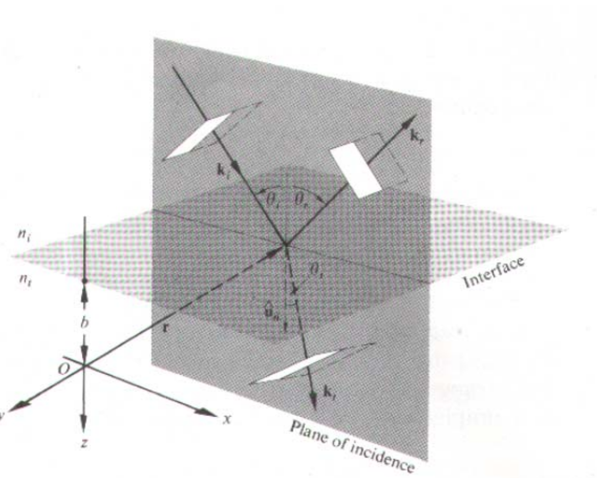
※ ($\underline{r}$ lies in the interface, plane of constant phase $(\underline{k}_i - \underline{k}_r)\cdot \underline{r}\big|_{z=0} = 0 \quad \Rightarrow \quad \underline{a}_z \times (\underline{k}_i - \underline{k}_r) = 0$)

$$\underline{k}_i \cdot \underline{r}\big|_{z=0} = \underline{k}_r \cdot \underline{r}\big|_{z=0} = \underline{k}_t \cdot \underline{r}\big|_{z=0}$$

$$\Rightarrow \quad k_{ix}x + k_{iz}z\big|_{z=0} = k_{rx}x + k_{ry}y + k_{rz}z\big|_{z=0} = k_{tx}x + k_{ty}y + k_{tz}z\big|_{z=0}$$

$$\Rightarrow \quad k_{ix}x = k_{rx}x + k_{ry}y = k_{tx}x + k_{ty}y$$

$$\Rightarrow \begin{cases} k_{ix} = k_{rx} = k_{tx} \\ k_{ry} = k_{ty} \end{cases}$$



$$k_{ix} = k_{rx}$$

$$\Rightarrow \quad k_i \sin\theta_i = k_r \sin\theta_r$$

since the incident and reflected wave are in the same medium, $\therefore k_i = k_r$

$$\left(k_i = \omega\sqrt{\mu_1\varepsilon_1}, \quad k_r = \omega\sqrt{\mu_1\varepsilon_1}\right)$$

$\Rightarrow \quad \theta_i = \theta_r$ law of reflection

$$k_{ix} = k_{tx}$$

$$k_i \sin\theta_i = k_t \sin\theta_t$$

$$\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_i = \omega\sqrt{\mu_2\varepsilon_2}\sin\theta_t$$

$$\sqrt{\varepsilon_1}\sin\theta_i = \sqrt{\varepsilon_2}\sin\theta_t$$

$\Rightarrow \sqrt{\varepsilon_{r1}}\sin\theta_i = \sqrt{\varepsilon_{r2}}\sin\theta_t \qquad (\varepsilon_1 = \varepsilon_0\varepsilon_{r1}, \quad \varepsilon_2 = \varepsilon_0\varepsilon_{r2})$

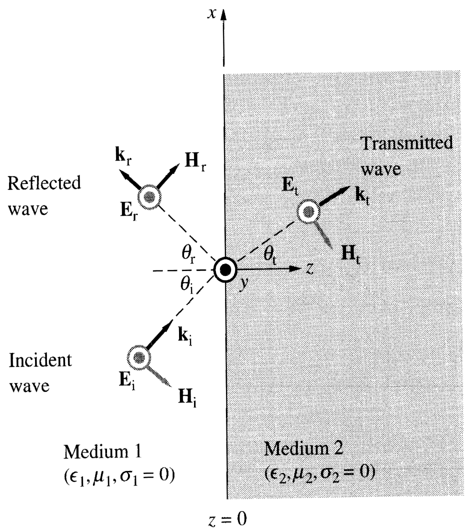
$$n = \frac{c}{v_p} = \frac{\dfrac{1}{\sqrt{\mu_0\varepsilon_0}}}{\dfrac{1}{\sqrt{\mu\varepsilon}}} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\frac{\mu_0\mu_r\varepsilon_0\varepsilon_r}{\mu_0\varepsilon_0}} = \sqrt{\varepsilon_r}$$

$n_i \sin\theta_i = n_t \sin\theta_t$ Snell's law of refraction

Perpendicular polarization (Transverse E, horizontal, E-wave, senkrecht-wave) : E-field is perpendicular to the plane of incidence, parallel polarization (TM, vertical, H-wave, p-wave) : E-filed is parallel to the plane of incidence

Perpendicular polarization—Oblique incidence at a dielectric boundary

Incident wave	Reflected wave	Refracted wave
$\underline{\hat{E}}_i = \hat{E}_{i0} e^{-j\underline{k}_i\cdot\underline{r}}\underline{a}_y$	$\underline{\hat{E}}_r = \hat{E}_{r0} e^{-j\underline{k}_r\cdot\underline{r}}\underline{a}_y$	$\underline{\hat{E}}_t = \hat{E}_{t0} e^{-j\underline{k}_i\cdot\underline{r}}\underline{a}_y$
$\underline{\hat{H}}_i = \frac{\hat{E}_{i0} e^{-j\underline{k}_i\cdot\underline{r}}}{\eta_1}\left(\underline{a}_{k_i}\times\underline{a}_y\right)$	$\underline{\hat{H}}_r = \frac{\hat{E}_{r0} e^{-j\underline{k}_r\cdot\underline{r}}}{\eta_1}\left(\underline{a}_{k_r}\times\underline{a}_y\right)$	$\underline{\hat{H}}_t = \frac{\hat{E}_{t0} e^{-j\underline{k}_i\cdot\underline{r}}}{\eta_2}\left(\underline{a}_{k_t}\times\underline{a}_y\right)$



$\underline{a}_{k_i} = \sin\theta_i \underline{a}_x + \cos\theta_i \underline{a}_z$ $\underline{a}_{k_r} = \sin\theta_r \underline{a}_x - \cos\theta_r \underline{a}_z$ $\underline{a}_{k_t} = \sin\theta_t \underline{a}_x + \cos\theta_t \underline{a}_z$

$\underline{k}_i = k_i\sin\theta_i \underline{a}_x + k_i\cos\theta_i \underline{a}_z$ $\underline{k}_r = k_r\sin\theta_r \underline{a}_x - k_r\cos\theta_r \underline{a}_z$ $\underline{k}_t = k_t\sin\theta_t \underline{a}_x + k_t\cos\theta_t \underline{a}_z$

$\underline{r} = x\underline{a}_x + y\underline{a}_y + z\underline{a}_z$

$$\begin{cases} \underline{\hat{E}}_i = \hat{E}_{i0} e^{-j(k_i\sin\theta_i x + k_i\cos\theta_i z)}\underline{a}_y \\ \underline{\hat{E}}_r = \hat{E}_{r0} e^{-j(k_r\sin\theta_r x - k_r\cos\theta_r z)}\underline{a}_y \\ \underline{\hat{E}}_t = \hat{E}_{t0} e^{-j(k_t\sin\theta_t x + k_t\cos\theta_t z)}\underline{a}_y \end{cases}$$

$$\begin{cases} \underline{\hat{H}}_i = \frac{\hat{E}_{i0}}{\eta_1}\left(-\cos\theta_i \underline{a}_x + \sin\theta_i \underline{a}_z\right) e^{-j(k_i\sin\theta_i x + k_i\cos\theta_i z)} \\ \underline{\hat{H}}_r = \frac{\hat{E}_{r0}}{\eta_1}\left(\cos\theta_r \underline{a}_x + \sin\theta_r \underline{a}_z\right) e^{-j(k_r\sin\theta_r x - k_r\cos\theta_r z)} \\ \underline{\hat{H}}_t = \frac{\hat{E}_{t0}}{\eta_2}\left(-\cos\theta_t \underline{a}_x + \sin\theta_t \underline{a}_z\right) e^{-j(k_t\sin\theta_t x + k_t\cos\theta_t z)} \end{cases}$$

at the boundary $z = 0$, $\underline{\hat{E}}, \underline{\hat{H}}$ tangential component continues $\Rightarrow$

$$\begin{cases} \underline{a}_n \times \left[\underline{\hat{E}}_2 - \underline{\hat{E}}_1\right] = 0 \\ \underline{a}_n \times \left[\underline{\hat{H}}_2 - \underline{\hat{H}}_1\right] = 0 \end{cases} \Rightarrow \begin{matrix} -\underline{a}_z \times \underline{\hat{E}}_i\big|_{z=0} - \underline{a}_z \times \underline{\hat{E}}_r\big|_{z=0} = -\underline{a}_z \times \underline{\hat{E}}_t\big|_{z=0} \\ -\underline{a}_z \times \underline{\hat{H}}_i\big|_{z=0} - \underline{a}_z \times \underline{\hat{H}}_r\big|_{z=0} = -\underline{a}_z \times \underline{\hat{H}}_t\big|_{z=0} \end{matrix}$$

$$\begin{cases} \underline{a}_y \cdot \underline{\hat{E}}_i\big|_{z=0} + \underline{a}_y \cdot \underline{\hat{E}}_r\big|_{z=0} = \underline{a}_y \cdot \underline{\hat{E}}_t\big|_{z=0} \\ \underline{a}_x \cdot \underline{\hat{H}}_i\big|_{z=0} + \underline{a}_x \cdot \underline{\hat{H}}_r\big|_{z=0} = \underline{a}_x \cdot \underline{\hat{H}}_t\big|_{z=0} \end{cases}$$

$$\hat{E}_{i0} e^{-jk_i \sin\theta_i x} \underline{a}_x + \hat{E}_{r0} e^{-jk_r \sin\theta_r x} \underline{a}_x = \hat{E}_{t0} e^{-jk_t \sin\theta_t x} \underline{a}_x$$

$$\Rightarrow \hat{E}_{i0} + \hat{E}_{r0} = \hat{E}_{t0}, \qquad k_i \sin\theta_i x = k_i \sin\theta_r x = k_t \sin\theta_t x$$

$$\frac{\hat{E}_{i0}}{\eta_1}\left(-\cos\theta_i \underline{a}_x\right) e^{-j(k_i \sin\theta_i x)} + \frac{\hat{E}_{r0}}{\eta_1}\left(\cos\theta_r \underline{a}_x\right) e^{-j(k_r \sin\theta_r x)} = \frac{\hat{E}_{t0}}{\eta_2}\left(-\cos\theta_t \underline{a}_x\right) e^{-j(k_t \sin\theta_t x)}$$

$$\Rightarrow -\frac{\hat{E}_{i0}}{\eta_1}\cos\theta_i + \frac{\hat{E}_{r0}}{\eta_1}\cos\theta_r = -\frac{\hat{E}_{t0}}{\eta_2}\cos\theta_t$$

$$1 + \frac{\hat{E}_{r0}}{\hat{E}_{i0}} = \frac{\hat{E}_{t0}}{\hat{E}_{i0}}$$

$$-\frac{\cos\theta_i}{\eta_1} + \frac{\hat{E}_{r0}}{\hat{E}_{i0}}\frac{\cos\theta_r}{\eta_1} = -\frac{\hat{E}_{t0}}{\hat{E}_{i0}}\frac{\cos\theta_t}{\eta_2} = -\left(1 + \frac{\hat{E}_{r0}}{\hat{E}_{i0}}\right)\frac{\cos\theta_t}{\eta_2}$$

$$\Rightarrow \frac{\hat{E}_{r0}}{\hat{E}_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_r + \eta_1\cos\theta_t} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

Reflection coefficient $\Gamma_\perp \equiv \frac{\hat{E}_{r0}}{\hat{E}_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{\sqrt{\varepsilon_{1r}}\cos\theta_i - \sqrt{\varepsilon_{2r}}\cos\theta_t}{\sqrt{\varepsilon_{1r}}\cos\theta_i + \sqrt{\varepsilon_{2r}}\cos\theta_t}$

$$\Rightarrow \Gamma_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

or
$$\Gamma_\perp = \frac{\cos\theta_i - \sqrt{(\varepsilon_{2r}/\varepsilon_{1r})}\cos\theta_t}{\cos\theta_i + \sqrt{(\varepsilon_{2r}/\varepsilon_{1r})}\cos\theta_t} = \frac{\cos\theta_i - \sqrt{(\varepsilon_{2r}/\varepsilon_{1r})}\sqrt{1-\sin^2\theta_t}}{\cos\theta_i + \sqrt{(\varepsilon_{2r}/\varepsilon_{1r})}\sqrt{1-\sin^2\theta_t}}$$

$$= \frac{\cos\theta_i - \sqrt{(\varepsilon_{2r}/\varepsilon_{1r})}\sqrt{1-(\varepsilon_{1r}/\varepsilon_{2r})\sin^2\theta_i}}{\cos\theta_i + \sqrt{(\varepsilon_{2r}/\varepsilon_{1r})}\sqrt{1-(\varepsilon_{1r}/\varepsilon_{2r})\sin^2\theta_i}} = \frac{\cos\theta_i - \sqrt{(\varepsilon_{2r}/\varepsilon_{1r}) - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(\varepsilon_{2r}/\varepsilon_{1r}) - \sin^2\theta_i}}$$

$$\Rightarrow \Gamma_\perp = \frac{\cos\theta_i - \sqrt{(\varepsilon_{2r}/\varepsilon_{1r}) - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(\varepsilon_{2r}/\varepsilon_{1r}) - \sin^2\theta_i}} \quad \text{or} \quad \Gamma_\perp = \frac{\cos\theta_i - \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}$$

or
$$\Gamma_\perp = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} = \frac{(n_2\sin\theta_t/\sin\theta_i)\cos\theta_i - n_2\cos\theta_t}{(n_2\sin\theta_t/\sin\theta_i)\cos\theta_i + n_2\cos\theta_t}$$

$$= \frac{\sin\theta_t \cos\theta_i - \sin\theta_i \cos\theta_t}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$\Rightarrow \quad \Gamma_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

Transmission coefficient $\Im_\perp = \frac{\hat{E}_{t0}}{\hat{E}_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \Rightarrow \Im_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i - n_t \cos\theta_t}$

or $\Im_\perp = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{(\varepsilon_{2r}/\varepsilon_{1r}) - \sin^2\theta_i}}$, $\Im_\perp = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{(n_t/n_i)^2 - \sin^2\theta_i}}$

$$\Im_\perp = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)}$$

$$1 + \Gamma_\perp = \Im_\perp$$

Normal incidence on a dielectric boundary

$$\theta_i = \theta_t = 0 = \theta_r$$

$$\underline{a}_{k_i} = \underline{a}_z \qquad \underline{a}_{k_r} = -\underline{a}_z \qquad \underline{a}_{k_t} = \underline{a}_z$$

$$\underline{k}_i = k_i \underline{a}_z = k_1 \underline{a}_z \qquad \underline{k}_r = -k_r \underline{a}_z = -k_1 \underline{a}_z \qquad \underline{k}_t = k_2 \underline{a}_z$$

$$\begin{cases} \underline{\hat{E}}_i = \hat{E}_{i0} e^{-jk_i z} \underline{a}_y = \hat{E}_{i0} e^{-jk_1 z} \underline{a}_y \\ \underline{\hat{E}}_r = \hat{E}_{r0} e^{+jk_r z} \underline{a}_y = \hat{E}_{r0} e^{+jk_1 z} \underline{a}_y \\ \underline{\hat{E}}_t = \hat{E}_{t0} e^{-jk_t z} \underline{a}_y = \hat{E}_{t0} e^{-jk_2 z} \underline{a}_y \end{cases} \qquad \begin{cases} \underline{\hat{H}}_i = -\frac{\hat{E}_{i0}}{\eta_1} e^{-jk_i z} \underline{a}_x = -\frac{\hat{E}_{i0}}{\eta_1} e^{-jk_1 z} \underline{a}_x \\ \underline{\hat{H}}_r = \frac{\hat{E}_{r0}}{\eta_1} e^{+jk_r z} \underline{a}_x = \frac{\hat{E}_{r0}}{\eta_1} e^{+jk_1 z} \underline{a}_x \\ \underline{\hat{H}}_t = -\frac{\hat{E}_{t0}}{\eta_2} e^{-jk_t z} \underline{a}_x = -\frac{\hat{E}_{t0}}{\eta_2} e^{-jk_2 z} \underline{a}_x \end{cases}$$

Reflection coefficient $\hat{\Gamma}_\perp \equiv \frac{\hat{E}_{r0}}{\hat{E}_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Transmission coefficient $\hat{\Im}_\perp = \frac{\hat{E}_{t0}}{\hat{E}_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}$

Ex.

Oblique incidence on a perfect conductor

Incident wave	Reflected wave

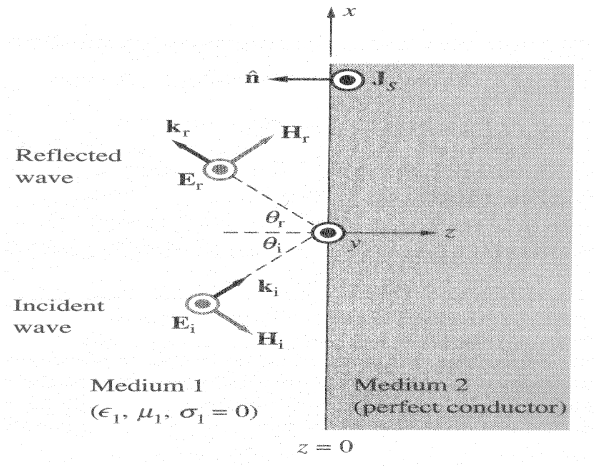
$$\underline{\hat{E}}_i = \hat{E}_{i0} e^{-j\underline{k}_i \cdot \underline{r}} \underline{a}_y \qquad \underline{\hat{E}}_r = \hat{E}_{r0} e^{-j\underline{k}_r \cdot \underline{r}} \underline{a}_y$$

$$\underline{\hat{H}}_i = \frac{\hat{E}_{i0} e^{-j\underline{k}_i \cdot \underline{r}}}{\eta_1}\left(\underline{a}_{ni} \times \underline{a}_y\right) \qquad \underline{\hat{H}}_r = \frac{\hat{E}_{r0} e^{-j\underline{k}_i \cdot \underline{r}}}{\eta_1}\left(\underline{a}_{nr} \times \underline{a}_y\right)$$

$$\underline{a}_{ni} = \sin\theta_i \underline{a}_x + \cos\theta_i \underline{a}_z \qquad \underline{a}_{nr} = \sin\theta_r \underline{a}_x - \cos\theta_r \underline{a}_z$$

$$\underline{k}_i = k_i \sin\theta_i \underline{a}_x + k_i \cos\theta_i \underline{a}_z \qquad \underline{k}_r = k_r \sin\theta_r \underline{a}_x - k_r \cos\theta_r \underline{a}_z$$

$$\begin{cases} \underline{\hat{E}}_i = \hat{E}_{i0} e^{-j(k_i \sin\theta_i x + k_i \cos\theta_i z)} \underline{a}_y \\ \underline{\hat{E}}_r = \hat{E}_{r0} e^{-j(k_r \sin\theta_r x - k_r \cos\theta_r z)} \underline{a}_y \end{cases}$$



$$\begin{cases} \underline{\hat{H}}_i = \dfrac{\hat{E}_{i0}}{\eta_1}\left(-\cos\theta_i \underline{a}_x + \sin\theta_i \underline{a}_z\right) e^{-j(k_i \sin\theta_i x + k_i \cos\theta_i z)} \\ \underline{\hat{H}}_r = \dfrac{\hat{E}_{r0}}{\eta_1}\left(\cos\theta_r \underline{a}_x + \sin\theta_r \underline{a}_z\right) e^{-j(k_r \sin\theta_r x - k_r \cos\theta_r z)} \end{cases}$$

at the boundary surface, z=0

$$\underline{a}_y \cdot \underline{\hat{E}}_i\big|_{z=0} + \underline{a}_y \cdot \underline{\hat{E}}_r\big|_{z=0} = 0$$

$$\underline{a}_x \cdot \underline{\hat{H}}_i\big|_{z=0} + \underline{a}_x \cdot \underline{\hat{H}}_r\big|_{z=0} = \hat{J}_s$$

$$\hat{E}_{i0} = -\hat{E}_{r0}, \qquad \theta_i = \theta_r$$

$$\underline{\hat{E}_1} = \underline{\hat{E}_i} + \underline{\hat{E}_r} = \hat{E}_{i0}\left[e^{-jk_i z\cos\theta_i} - e^{+jk_i z\cos\theta_i}\right] e^{-jk_i x\sin\theta_i} \underline{a_y}$$

$$= -j2\hat{E}_{i0}\sin\left(k_1 z\cos\theta_i\right) e^{-jk_i x\sin\theta_i} \underline{a_y}$$

$$\underline{\hat{H}_1} = \underline{\hat{H}_i} + \underline{\hat{H}_r}$$

$$= -\frac{2\hat{E}_{i0}}{\eta_1}\left[\cos\theta_i \cos\left(k_1 z\cos\theta_i\right)\underline{a_x} + j\sin\theta_i \sin\left(k_1 z\cos\theta_i\right)\underline{a_z}\right] e^{-jk_i \sin\theta_i x}$$

Time-average Poynting vector

$$\left\langle \underline{S_{av}} \right\rangle = \frac{1}{2} Re\left\{\underline{\hat{E}_1} \times \underline{\hat{H}_1}^*\right\}$$

$$= \frac{1}{2} Re\left\{\frac{4\hat{E}_{i0}^2}{\eta_1} sin\theta_i\, sin^2\left(k_1 z\, cos\theta_i\right)\underline{a_x} - j\frac{2\hat{E}_{i0}^2}{\eta_1} cos\theta_i\, sin\left(2k_1 z\, cos\theta_i\right)\underline{a_z}\right\}$$

$$= \frac{2\hat{E}_{i0}^2}{\eta_1} sin\theta_i\, sin^2\left(k_1 z\, cos\theta_i\right)\underline{a_x}$$

Instantaneous Poynting vector

$$\underline{P}(z,t) = \mathrm{Re}\{\underline{\hat{E}}(z)e^{j\omega t}\} \times \mathrm{Re}\{\underline{\hat{H}}(z)e^{j\omega t}\}$$

$$= \frac{1}{2}\left[\underline{\hat{E}}(z)e^{j\omega t} + \underline{\hat{E}}^*(z)e^{-j\omega t}\right] \times \frac{1}{2}\left[\underline{\hat{H}}(z)e^{j\omega t} + \underline{\hat{H}}^*(z)e^{-j\omega t}\right]$$

$$= \frac{1}{4}\left\{\left[\underline{\hat{E}}(z)\times\underline{\hat{H}}^*(z) + \underline{\hat{E}}^*(z)\times\underline{\hat{H}}(z)\right] + \left[\underline{\hat{E}}(z)\times\underline{\hat{H}}(z)e^{j2\omega t} + \underline{\hat{E}}^*(z)\times\underline{\hat{H}}^*(z)e^{-j2\omega t}\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\{\underline{\hat{E}}(z)\times\underline{\hat{H}}^*(z) + \underline{\hat{E}}(z)\times\underline{\hat{H}}(z)e^{j2\omega t}\}$$

$$\therefore \langle \underline{S_{av}} \rangle = \frac{1}{2} Re\{\underline{\hat{E}_1} \times \underline{\hat{H}_1}^*\}$$

at $z = 0$, induced current in the conductor

$$\underline{\hat{J}_s} = \underline{a_n} \times \underline{\hat{H}_1}\big|_{z=0} = [(-\underline{a_z})\times(-\underline{a_x})]\frac{2\hat{E}_{i0}}{\eta_1} cos\,\theta_i e^{-jk_i \sin\theta_i x}$$

$$= \frac{2\hat{E}_{i0}}{\eta_1} cos\,\theta_i e^{-jk_i \sin\theta_i x}\,\underline{a_y}$$

$\underline{a_n}$: the outward normal to the conductor surface

Normal incidence on a perfect conductor

$$\underline{\hat{E}_1} = -j2\hat{E}_{i0}\sin(k_1 z)\underline{a_y}$$

$$\underline{\hat{H}_1} = -\frac{2\hat{E}_{i0}}{\eta_1}\cos(k_1 z)\underline{a_x}$$

$$\underline{E_1}(z,t) = \mathrm{Re}[\underline{\hat{E}_1}(z)e^{j\omega t}] = \mathrm{Re}[-2\hat{E}_{i0}\sin(k_1 z)e^{j\omega t}e^{-j\pi/2}\underline{a_y}]$$

$$\underline{H_1}(z,t) = \mathrm{Re}[\underline{\hat{H}_1}(z)e^{j\omega t}] = \mathrm{Re}\left[-\frac{2\hat{E}_{i0}}{\eta_1}\cos(k_1 z)e^{j\omega t}\underline{a_x}\right]$$

standing waves
$$\begin{cases} \underline{E_1}(z,t) = -2\hat{E}_{i0}\sin(k_1 z)\sin(\omega t)\underline{a_y} \\ \underline{H_1}(z,t) = -\dfrac{2\hat{E}_{i0}}{\eta_1}\cos(k_1 z)\cos(\omega t)\underline{a_x} \end{cases}$$

$$\underline{S_{av}} = \frac{1}{2}\mathrm{Re}\{\underline{\hat{E}_1}\times\underline{\hat{H}_1}^*\} = \frac{1}{2}\mathrm{Re}\left\{[-j2\hat{E}_{i0}\sin(k_1 z)]\underline{a_y}\times\left[-\frac{2\hat{E}_{i0}^*}{\eta_1}\cos(k_1 z)\right]\underline{a_x}\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left[j\frac{2|\hat{E}_{i0}|^2}{\eta_1}\sin(2k_1 z)\right]\underline{a_z}\right\} = 0$$

at $z = 0$, induced current in the conductor

$$\underline{J_s} = \underline{a_n} \times \underline{H_1}\Big|_{z=0} = \left[\left(-\underline{a_z}\right) \times \left(-\underline{a_x}\right)\right] \frac{2\hat{E}_{i0}}{\eta_1}$$

$$= \frac{2\hat{E}_{i0}}{\eta_1} \underline{a_y}$$