Polarization of EM Waves

The polarization of a uniform plane wave is a measure of how its E-field vector varies with time. It describes the **time-varying** behavior of E-field vector **at a given point** in space.

$$\underline{\mathbf{E}}(\underline{\mathbf{r}},t)=\mathbf{Re}\left\{\underline{\hat{\mathbf{E}}}(x,y,z)e^{j\omega t}\right\}$$
$$=\mathbf{Re}\left\{\underline{\hat{\mathbf{E}}_0}e^{-j(\underline{k}\cdot\underline{r}-\omega t)}\right\}$$

$$\underline{\hat{E}_0}=\hat{E}_{0x}\underline{a_x}+\hat{E}_{0y}\underline{a_y}$$
$$=\left|\hat{E}_{0x}\right|e^{j\theta_x}\underline{a_x}+\left|\hat{E}_{0y}\right|e^{j\theta_y}\underline{a_y}$$

a uniform plane wave propagating in the z direction

linear polarization

$$\underline{E}(z,t)=Re\left\{\underline{\hat{E}}(z)e^{j\omega t}\right\}$$
$$=Re\left\{\left(\hat{E}_{0x}e^{-jkz}\underline{a_x}+\hat{E}_{0y}e^{-jkz}\underline{a_y}\right)e^{j\omega t}\right\}$$
$$=Re\left\{\left(\left|\hat{E}_{0x}\right|e^{j\theta_x}\underline{a_x}+\left|\hat{E}_{0y}\right|e^{j\theta_y}\underline{a_y}\right)e^{j(\omega t-kz)}\right\}$$
$$=Re\left\{\left(\left|\hat{E}_{0x}\right|\underline{a_x}\pm\left|\hat{E}_{0y}\right|\underline{a_y}\right)e^{j(\omega t-kz+\theta_x)}\right\}$$
$$=Re\left\{\left(\left|\hat{E}_{0x}\right|\underline{a_x}\pm\left|\hat{E}_{0y}\right|\underline{a_y}\right)e^{j(\omega t-kz+\theta_x)}\right\}\qquad \theta_y-\theta_x=\pm m\pi, m=0,1,2\cdots$$
$$=\left(\left|\hat{E}_{0x}\right|\underline{a_x}\pm\left|\hat{E}_{0y}\right|\underline{a_y}\right)cos(\omega t-kz+\theta_x)$$
$$=E_x\underline{a_x}\pm E_y\underline{a_y}$$

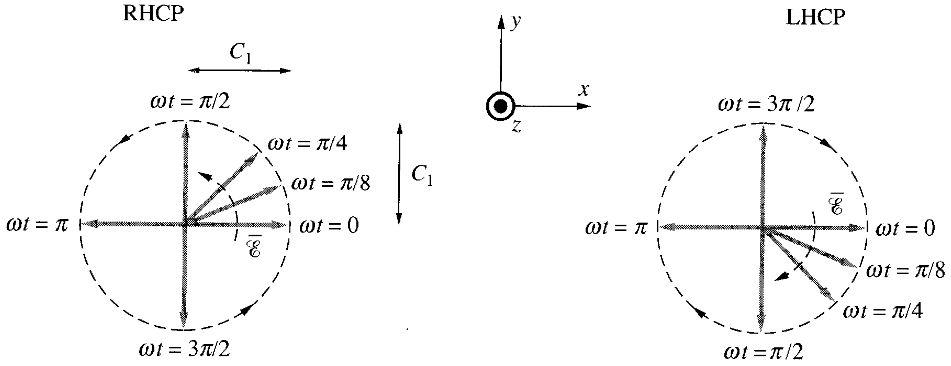
$$\begin{cases}E_x(z,t)=\left|\hat{E}_{0x}\right|\cos(\omega t-kz+\theta_x)\\ E_y(z,t)=\left|\hat{E}_{0y}\right|\cos(\omega t-kz+\theta_x)\end{cases}$$

circular polarization

$$\underline{E}(z,t)=Re\left\{\underline{\hat{E}}(z)e^{j\omega t}\right\}$$
$$=Re\left\{\left(\left|\hat{E}_{0x}\right|e^{j\theta_x}\underline{a_x}+\left|\hat{E}_{0y}\right|e^{j\theta_y}\underline{a_y}\right)e^{j(\omega t-kz)}\right\}$$
$$=Re\left\{\left(\left|\hat{E}_{0x}\right|\underline{a_x}+\left|\hat{E}_{0y}\right|e^{j(\theta_y-\theta_x)}\underline{a_y}\right)e^{j(\omega t-kz+\theta_x)}\right\}\qquad \left|\hat{E}_{0x}\right|=\left|\hat{E}_{0y}\right|$$
$$=Re\left\{\left(\left|\hat{E}_{0x}\right|\underline{a_x}+e^{\pm j\pi/2}\left|\hat{E}_{0x}\right|\underline{a_y}\right)e^{j(\omega t-kz+\theta_x)}\right\}\qquad \theta_y-\theta_x=\pm\pi/2+2m\pi,$$
$$=Re\left\{\left(\left|\hat{E}_{0x}\right|\underline{a_x}\pm j\left|\hat{E}_{0x}\right|\underline{a_y}\right)e^{j(\omega t-kz+\theta_x)}\right\}\qquad m=0,\pm1,\pm2\cdots$$
$$=\left|\hat{E}_{0x}\right|cos(\omega t-kz+\theta_x)\underline{a_x}\mp\left|\hat{E}_{0x}\right|sin(\omega t-kz+\theta_x)\underline{a_y}$$
$$=E_x\underline{a_x}\mp E_y\underline{a_y}$$

$$\begin{cases} E_x(z,t) = \left|\hat{E}_{0x}\right| \cos(\omega t - kz + \theta_x) \\ E_y(z,t) = \left|\hat{E}_{0x}\right| \sin(\omega t - kz + \theta_x) \end{cases}$$

$$\left|\underline{E}\right|^2 = E_x^{\ 2} + E_y^{\ 2} = \left|\hat{E}_{0x}\right|^2 \qquad \text{equation for a circle}$$



$\theta_y - \theta_x = -\pi/2$ right-hand circularly polarized wave for forwarding waves

$$\underline{E}(z,t) = \left|\hat{E}_{0x}\right| \cos(\omega t - kz + \theta_x)\underline{a}_x + \left|\hat{E}_{0y}\right| \sin(\omega t - kz + \theta_x)\underline{a}_y$$

$\theta_y - \theta_x = \pi/2$ left-hand circularly polarized wave for forwarding waves

$$\underline{E}(z,t) = \left|\hat{E}_{0x}\right| \cos(\omega t - kz + \theta_x)\underline{a}_x - \left|\hat{E}_{0y}\right| \sin(\omega t - kz + \theta_x)\underline{a}_y$$

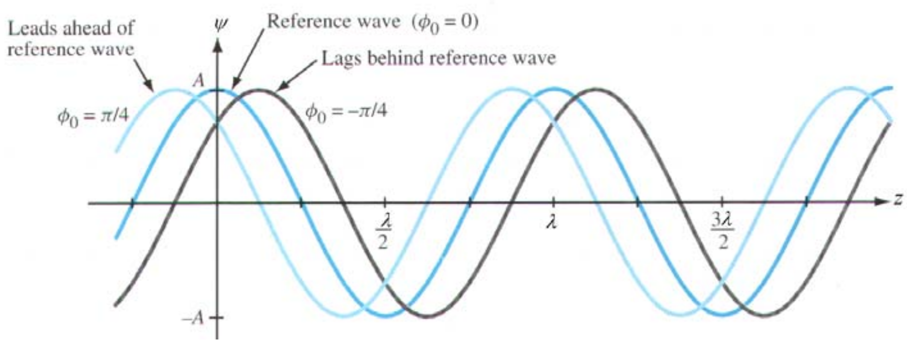
let $z = 0, \theta_x = 0$ $\underline{E}(z,t) = \left|\hat{E}_{0x}\right| \cos(\omega t)\underline{a}_x - \left|\hat{E}_{0y}\right| \sin(\omega t)\underline{a}_y$

[Comment] IEEE convention, thumb pointing in the direction of propagation

$e^{j\omega t}$

Optics convention, thumb pointing to where the wave is coming from

$e^{-j\omega t}$



Purely comparing cosωt function to sinωt function, cosωt function is leading sinωt function by π/2;

cos(ωt+θ) is leading cosωt by θ. For backwarding wave, whenever $\theta_y - \theta_x = \pi/2$, there is a right-hand circularly polarized wave; there is a left-hand circularly polarized wave when $\theta_y - \theta_x = -\pi/2$.

elliptical polarization

$$\underline{E}(z,t) = Re\{\underline{\hat{E}}(z)e^{j\omega t}\}$$

$$= Re\{(|\hat{E}_{0x}|e^{j\theta_x}\underline{a_x} + |\hat{E}_{0y}|e^{j\theta_y}\underline{a_y})e^{j(\omega t - kz)}\}$$

$$= |\hat{E}_{0x}|cos(\omega t - kz + \theta_x)\underline{a_x} \pm |\hat{E}_{0y}|cos(\omega t - kz + \theta_y)\underline{a_y}$$

$$= E_x\underline{a_x} \pm E_y\underline{a_y}$$

$$\begin{cases} \dfrac{E_x}{|\hat{E}_{0x}|} = cos\left(\underbrace{\omega t - kz}_{=\tau} + \theta_x\right) = cos\,\tau\,cos\,\theta_x - sin\,\vartheta\,sin\,\theta_x \\ \dfrac{E_y}{|\hat{E}_{0y}|} = cos(\omega t - kz + \theta_y) = cos\,\tau\,cos\,\theta_y - sin\,\tau\,sin\,\theta_y \end{cases}$$

$$\begin{cases} \dfrac{E_x}{|\hat{E}_{0x}|}sin\,\theta_y - \dfrac{E_y}{|\hat{E}_{0y}|}sin\,\theta_x = cos\,\tau\,sin\left(\underbrace{\theta_y - \theta_x}_{=\theta}\right) = cos\,\tau\,sin\,\theta \\ \dfrac{E_x}{|\hat{E}_{0x}|}cos\,\theta_y - \dfrac{E_y}{|\hat{E}_{0y}|}cos\,\theta_x = sin\,\tau\,sin(\theta_y - \theta_x) = sin\,\tau\,sin\,\vartheta \end{cases}$$

(A) $$\left(\frac{E_x}{|\hat{E}_{0x}|}\right)^2 + \left(\frac{E_y}{|\hat{E}_{0y}|}\right)^2 - 2\frac{E_x}{|\hat{E}_{0x}|}\frac{E_y}{|\hat{E}_{0y}|}cos\,\theta = sin^2\,\theta, \qquad \theta = \theta_y - \theta_x$$

$$\psi = \frac{1}{2}tan^{-1}[tan(2\alpha)cos\,\theta], \qquad tan\,\alpha = \frac{|\hat{E}_{0y}|}{|\hat{E}_{0x}|}$$

$$\begin{cases} E_{x'} = E_x\,cos\,\psi + E_y\,sin\,\psi \\ E_{y'} = -E_x\,sin\,\psi + E_y\,cos\,\psi \end{cases}$$

$$\Rightarrow \left(\frac{E_{x'}}{A_{x'}}\right)^2 + \left(\frac{E_{y'}}{A_{y'}}\right)^2 = 1$$

$0 < \theta_y - \theta_x < \pi$ left-hand elliptically polarized wave for forwarding waves

$\pi < \theta_y - \theta_x < 2\pi$ right-hand elliptically polarized wave for forwarding waves

$$\begin{cases} E_{x'} = E_x \cos\psi + E_y \sin\psi \\ E_{y'} = -E_x \sin\psi + E_y \cos\psi \end{cases}$$

$$\left(\frac{E_{x'}}{A_{x'}}\right)^2 + \left(\frac{E_{y'}}{A_{y'}}\right)^2 = 1 \quad \Rightarrow \quad \begin{cases} E_{x'} = A_{x'} \cos(\omega t - kz + \theta') = A_{x'} \cos(\tau + \theta') \\ E_{y'} = A_{y'} \cos(\omega t - kz + \theta') = A_{y'} \cos(\tau + \theta') \end{cases}$$

$$\left(\frac{E_{x'}}{A_{x'}}\right)^2 + \left(\frac{E_{y'}}{A_{y'}}\right)^2 = 1$$

$$\Rightarrow \left(\frac{E_x \cos\psi + E_y \sin\psi}{A_{x'}}\right)^2 + \left(\frac{-E_x \sin\psi + E_y \cos\psi}{A_{y'}}\right)^2 = 1$$

$$\Rightarrow \left(\frac{\cos^2\psi}{A_{x'}{}^2} + \frac{\sin^2\psi}{A_{y'}^2}\right)E_x^2 + \left(\frac{\sin^2\psi}{A_{x'}{}^2} + \frac{\cos^2\psi}{A_{y'}^2}\right)E_y^2 - 2E_xE_y \sin\psi\cos\psi\left(\frac{1}{A_{y'}^2} - \frac{1}{A_{x'}{}^2}\right) = 1$$

Comparing with (A)

$$\Rightarrow \begin{cases} \dfrac{\cos^2\psi}{A_{x'}{}^2} + \dfrac{\sin^2\psi}{A_{y'}^2} = \dfrac{1}{\left|\hat{E}_{0x}\right|^2 \sin^2\theta} & (a) \\ \dfrac{\sin^2\psi}{A_{x'}{}^2} + \dfrac{\cos^2\psi}{A_{y'}^2} = \dfrac{1}{\left|\hat{E}_{0y}\right|^2 \sin^2\theta} & (b) \\ \sin\psi\cos\psi\left(\dfrac{1}{A_{y'}^2} - \dfrac{1}{A_{x'}{}^2}\right) = \dfrac{\cos\theta}{\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right| \sin^2\theta} & (c) \end{cases}$$

$\because$ conservation of light intensity (energy) $\Rightarrow \quad A_{x'}^2 + A_{y'}^2 = \left|\hat{E}_{0x}\right|^2 + \left|\hat{E}_{0y}\right|^2$

$$(a)+(b) \quad \Rightarrow \quad \frac{A_{x'}{}^2 + A_{y'}^2}{A_{x'}{}^2 A_{y'}^2} = \frac{\left|\hat{E}_{0x}\right|^2 + \left|\hat{E}_{0y}\right|^2}{\left|\hat{E}_{0x}\right|^2\left|\hat{E}_{0y}\right|^2 \sin^2\vartheta} = \frac{A_{x'}{}^2 + A_{y'}^2}{\left|\hat{E}_{0x}\right|^2\left|\hat{E}_{0y}\right|^2 \sin^2\vartheta}$$

$$\Rightarrow \quad A_{x'}{}^2 A_{y'}^2 = \left|\hat{E}_{0x}\right|^2\left|\hat{E}_{0y}\right|^2 \sin^2\vartheta$$

$$\Rightarrow A_{x'}A_{y'} = \left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right| \sin\vartheta$$

$$\begin{cases} E_x = E_{x'} \cos\psi - E_{y'} \sin\psi \\ E_y = E_{x'} \sin\psi + E_{y'} \cos\psi \end{cases}$$

$$\left(\frac{E_x}{\left|\hat{E}_{0x}\right|}\right)^2 + \left(\frac{E_y}{\left|\hat{E}_{0y}\right|}\right)^2 - 2\frac{E_x}{\left|\hat{E}_{0x}\right|}\frac{E_y}{\left|\hat{E}_{0y}\right|}\cos\theta = \sin^2\theta$$

$$\Rightarrow \left(\frac{E_{x'}\cos\psi - E_{y'}\sin\psi}{\left|\hat{E}_{0x}\right|}\right)^2 + \left(\frac{E_{x'}\sin\psi + E_{y'}\cos\psi}{\left|\hat{E}_{0y}\right|}\right)^2 -$$

$$2\frac{E_{x'}\cos\psi - E_{y'}\sin\psi}{\left|\hat{E}_{0x}\right|}\frac{E_{x'}\sin\psi + E_{y'}\cos\psi}{\left|\hat{E}_{0y}\right|}\cos\theta = \sin^2\theta$$

$$\Rightarrow \left(\underbrace{\frac{\cos^2\psi}{\left|\hat{E}_{0x}\right|^2} + \frac{\sin^2\psi}{\left|\hat{E}_{0y}\right|^2} - \frac{2\sin\psi\cos\psi\cos\vartheta}{\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right|}}_{=\sin^2\vartheta / A_{x'}^2}\right)E_{x'}^2 + \left(\underbrace{\frac{\sin^2\psi}{\left|\hat{E}_{0x}\right|^2} + \frac{\cos^2\psi}{\left|\hat{E}_{0y}\right|^2} + \frac{2\sin\psi\cos\psi\cos\vartheta}{\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right|}}_{=\sin^2\vartheta / A_{y'}^2}\right)E_{y'}^2$$

$$+2\left(\underbrace{\frac{-\sin\psi\cos\psi}{\left|\hat{E}_{0x}\right|^2} + \frac{\sin\psi\cos\psi}{\left|\hat{E}_{0y}\right|^2} - \frac{\left(\sin^2\psi - \cos^2\psi\right)\cos\vartheta}{\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right|}}_{=0}\right)E_{x'}E_{y'} = \sin^2\theta$$

$$\Rightarrow \left(\underbrace{\frac{A_{y'}^2}{\left|\hat{E}_{0x}\right|^2\left|\hat{E}_{0y}\right|^2}}_{=\sin^2\vartheta / A_{x'}^2}\right)E_{x'}^2 + \left(\underbrace{\frac{A_{x'}^2}{\left|\hat{E}_{0x}\right|^2\left|\hat{E}_{0y}\right|^2}}_{=\sin^2\vartheta / A_{y'}^2}\right)E_{y'}^2 = \sin^2\vartheta$$

$$\Rightarrow \left(\frac{E_{x'}}{A_{x'}}\right)^2 + \left(\frac{E_{y'}}{A_{y'}}\right)^2 = \frac{\left|\hat{E}_{0x}\right|^2\left|\hat{E}_{0y}\right|^2\sin^2\vartheta}{A_{x'}^2 A_{y'}^2} = 1$$

$$\therefore \sin\psi\cos\psi\left(\frac{\left|\hat{E}_{0x}\right|^2 - \left|\hat{E}_{0y}\right|^2}{\left|\hat{E}_{0x}\right|^2\left|\hat{E}_{0y}\right|^2}\right) = \frac{\left(\cos^2\psi - \sin^2\psi\right)\cos\theta}{\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right|}$$

$$\Rightarrow \frac{1}{2}\sin 2\psi\left(\frac{\left|\hat{E}_{0x}\right|^2 - \left|\hat{E}_{0y}\right|^2}{\left|\hat{E}_{0x}\right|^2\left|\hat{E}_{0y}\right|^2}\right) = \frac{\cos 2\psi\cos\theta}{\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right|}$$

$$\Rightarrow \tan 2\psi = \frac{2\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right|\cos\theta}{\left|\hat{E}_{0x}\right|^2 - \left|\hat{E}_{0y}\right|^2} \quad \Rightarrow \quad \tan 2\psi = 2\tan 2\alpha\cos\theta$$

$$\therefore \psi = \frac{1}{2}\tan^{-1}\left[\tan(2\alpha)\cos\theta\right]$$

$$tan(x+y)=\frac{tan\,x+tan\,y}{1-tan\,x\,tan\,y} \quad \Rightarrow \quad tan\,2\alpha=\frac{2\,tan\,\alpha}{1-tan^2\,\alpha}=\frac{2\frac{\left|\hat{E}_{0y}\right|}{\left|\hat{E}_{0x}\right|}}{1-\left(\frac{\left|\hat{E}_{0y}\right|}{\left|\hat{E}_{0x}\right|}\right)^2}=\frac{2\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right|}{\left|\hat{E}_{0x}\right|^2-\left|\hat{E}_{0y}\right|^2}$$

$$\begin{cases} E_{x'}=A_{x'}\,cos(\tau+\theta')=E_x\,cos\,\psi+E_y\,sin\,\psi \\ E_{y'}=A_{y'}\,cos(\tau+\theta')=-E_x\,sin\,\psi+E_y\,cos\,\psi \end{cases}$$

$$\Rightarrow \begin{cases} A_{x'}(cos\,\tau\,cos\,\theta'-sin\,\tau\,sin\,\theta')=\left|\hat{E}_{0x}\right|(cos\,\tau\,cos\,\theta_x-sin\,\vartheta\,sin\,\theta_x)cos\,\psi+ \\ \qquad \left|\hat{E}_{0y}\right|(cos\,\tau\,cos\,\theta_y-sin\,\vartheta\,sin\,\theta_y)sin\,\psi \\ A_{y'}(cos\,\tau\,cos\,\theta'-sin\,\tau\,sin\,\theta')=-\left|\hat{E}_{0x}\right|(cos\,\tau\,cos\,\theta_x-sin\,\vartheta\,sin\,\theta_x)sin\,\psi+ \\ \qquad \left|\hat{E}_{0y}\right|(cos\,\tau\,cos\,\theta_y-sin\,\vartheta\,sin\,\theta_y)cos\,\psi \end{cases}$$

$$\Rightarrow \begin{cases} A_{x'}\,cos\,\theta'\,cos\,\tau-A_{x'}\,sin\,\theta'\,sin\,\tau=\left(\left|\hat{E}_{0x}\right|cos\,\theta_x\,cos\,\psi+\left|\hat{E}_{0y}\right|cos\,\theta_y\,sin\,\psi\right)cos\,\tau- \\ \qquad \left(\left|\hat{E}_{0x}\right|sin\,\theta_x\,cos\,\psi+\left|\hat{E}_{0y}\right|sin\,\theta_y\,sin\,\psi\right)sin\,\tau \\ A_{y'}\,cos\,\theta'\,cos\,\tau-A_{y'}\,sin\,\theta'\,sin\,\tau=\left(\left|\hat{E}_{0x}\right|sin\,\theta_x\,sin\,\psi-\left|\hat{E}_{0y}\right|sin\,\theta_y\,cos\,\psi\right)cos\,\tau- \\ \qquad \left(\left|\hat{E}_{0x}\right|cos\,\theta_x\,sin\,\psi-\left|\hat{E}_{0y}\right|cos\,\theta_y\,cos\,\psi\right)sin\,\tau \end{cases}$$

$$\Rightarrow \begin{cases} A_{x'}\,cos\,\theta'=\left|\hat{E}_{0x}\right|cos\,\theta_x\,cos\,\psi+\left|\hat{E}_{0y}\right|cos\,\theta_y\,sin\,\psi & (1) \\ A_{x'}\,sin\,\theta'=\left|\hat{E}_{0x}\right|sin\,\theta_x\,cos\,\psi+\left|\hat{E}_{0y}\right|sin\,\theta_y\,sin\,\psi & (2) \end{cases} \quad \begin{cases} A_{y'}\,cos\,\theta'=\left|\hat{E}_{0x}\right|sin\,\theta_x\,sin\,\psi-\left|\hat{E}_{0y}\right|sin\,\theta_y\,cos\,\psi & (3) \\ A_{y'}\,sin\,\theta'=\left|\hat{E}_{0x}\right|cos\,\theta_x\,sin\,\psi-\left|\hat{E}_{0y}\right|cos\,\theta_y\,cos\,\psi & (4) \end{cases}$$

$$\begin{matrix}(1)^2+(2)^2 \\ (3)^2+(4)^2\end{matrix} \Rightarrow \begin{cases} A_{x'}^2=\left|\hat{E}_{0x}\right|^2 cos^2\,\psi+\left|\hat{E}_{0y}\right|^2 sin^2\,\psi+2\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right|sin\,\psi\,cos\,\psi\,cos\,\vartheta \\ A_{y'}^2=\left|\hat{E}_{0x}\right|^2 sin^2\,\psi+\left|\hat{E}_{0y}\right|^2 cos^2\,\psi-2\left|\hat{E}_{0x}\right|\left|\hat{E}_{0y}\right|sin\,\psi\,cos\,\psi\,cos\,\vartheta \end{cases}$$

$$\Rightarrow \quad A_{x'}^2+A_{y'}^2=\left|\hat{E}_{0x}\right|^2+\left|\hat{E}_{0y}\right|^2$$