Plane waves in lossy media with σ_2, ε_2, μ_2.

$$\nabla\times\underline{\hat{H}}_2 = \underline{\hat{J}}_2 + j\omega\varepsilon_2\underline{\hat{E}}_2 = \sigma_2\underline{\hat{E}}_2 + j\omega\varepsilon_2\underline{\hat{E}}_2 = (\sigma_2 + j\omega\varepsilon_2)\underline{\hat{E}}_2 = j\omega\varepsilon_2\left(1 - j\frac{\sigma_2}{\omega\varepsilon_2}\right)\underline{\hat{E}}_2$$

$$= j\omega\hat{\varepsilon}_{\text{eff}}\,\underline{\hat{E}} \qquad \hat{\varepsilon}_{\text{eff}} = \varepsilon_2\left(1 - j\frac{\sigma_2}{\omega\varepsilon_2}\right)$$

$$\hat{\varepsilon}_{\text{eff}} = \varepsilon_2\left(1 - j\frac{\sigma_2}{\omega\varepsilon_2}\right)$$

$$\nabla\times\nabla\times\underline{\hat{E}}_2 = -j\omega\mu_2\nabla\times\underline{\hat{H}}_2$$
$$\nabla\left(\nabla\cdot\underline{\hat{E}}_2\right) - \nabla^2\underline{\hat{E}}_2 = -j\omega\mu_2\left(j\omega\hat{\varepsilon}_{\text{eff}}\,\underline{\hat{E}}_2\right), \qquad \nabla\cdot\underline{\hat{E}}_2 = 0 \quad \text{for source - free medium}$$
$$\nabla^2\underline{\hat{E}}_2 + \omega^2\mu_2\hat{\varepsilon}_{\text{eff}}\,\underline{\hat{E}}_2 = 0$$
$$\nabla^2\underline{\hat{E}}_2 + \hat{k}^2\underline{\hat{E}}_2 = 0$$

$$\hat{k} = \omega\sqrt{\mu\hat{\varepsilon}_{\text{eff}}} = \omega\sqrt{\mu_2\varepsilon_2\left(1 - j\frac{\sigma_2}{\omega\varepsilon_2}\right)} = \omega\sqrt{\mu_2\varepsilon_2}\left(1 - j\frac{\sigma_2}{\omega\varepsilon_2}\right)^{1/2} \qquad \left[m^{-1}\right]$$

$$\gamma = j\hat{k} = j\omega\sqrt{\mu_2\hat{\varepsilon}_{\text{eff}}} = j\omega\sqrt{\mu_2\varepsilon_2\left(1 - j\frac{\sigma_2}{\omega\varepsilon_2}\right)}$$

$$= j\omega\sqrt{\mu_2\varepsilon_2}\left(1 - j\frac{\sigma_2}{\omega\varepsilon_2}\right)^{1/2} = \alpha + j\beta$$

loss $\Rightarrow$ γ is complex $\Rightarrow$ $\gamma = \alpha + j\beta$ $\Rightarrow$ $k = \beta - j\alpha$

if uniform plane wave is propagating in the $\underline{k}$ direction

$$\underline{\hat{E}}(\underline{r}) = \underline{\hat{E}}(x,y,z) = \hat{E}_x\underline{a}_x + \hat{E}_y\underline{a}_y + \hat{E}_z\underline{a}_z$$
$$= \left[\hat{E}_{0x}\underline{a}_x + \hat{E}_{0y}\underline{a}_y + \hat{E}_{0z}\underline{a}_z\right]e^{-j(k_x x + k_y y + k_z z)} \qquad \underline{\hat{E}}_0 \text{ is a constant vector}$$
$$= \underline{\hat{E}}_0 e^{-j\underline{k}\cdot\underline{r}} = \underline{\hat{E}}_0 e^{-\underline{\gamma}\cdot\underline{r}}$$

$$\underline{\hat{E}}(\underline{r},t) = \underline{\hat{E}}(x,y,z)e^{j\omega t} = \underline{\hat{E}_0}e^{j\omega t}e^{-\underline{\gamma}\cdot\underline{r}}$$

Perpendicular polarization

Oblique incidence at a dielectric boundary

Incident wave	Reflected wave	Refracted wave

$$\underline{\hat{E}}_i = \hat{E}_{i0}e^{-j\underline{k}_i\cdot\underline{r}}\underline{a}_y \qquad \underline{\hat{E}}_r = \underline{\hat{E}}_{r0}e^{-j\underline{k}_r\cdot\underline{r}}\underline{a}_y \qquad \underline{\hat{E}}_t = \underline{\hat{E}}_{t0}e^{-j\underline{k}_i\cdot\underline{r}}\underline{a}_y = \underline{\hat{E}}_{t0}e^{-\underline{\gamma}\cdot\underline{r}}\underline{a}_y$$

$$\underline{\hat{H}}_i = \frac{\hat{E}_{i0}e^{-j\underline{k}_i\cdot\underline{r}}}{\eta_1}\left(\underline{a}_{ni}\times\underline{a}_y\right) \qquad \underline{\hat{H}}_r = \frac{\hat{E}_{r0}e^{-j\underline{k}_r\cdot\underline{r}}}{\eta_1}\left(\underline{a}_{nr}\times\underline{a}_y\right) \qquad \underline{\hat{H}}_t = \frac{\hat{E}_{t0}e^{-j\underline{k}_i\cdot\underline{r}}}{\eta_2}\left(\underline{a}_{nt}\times\underline{a}_y\right)$$

$$\underline{k}_i = k_i \sin\theta_i \underline{a}_x + k_i \cos\theta_i \underline{a}_z \qquad \underline{k}_i = \beta_1 \sin\theta_i \underline{a}_x + \beta_i \cos\theta_i \underline{a}_z ,$$
$$\underline{k}_r = k_r \sin\theta_r \underline{a}_x - k_r \cos\theta_r \underline{a}_z \quad \Rightarrow \quad \underline{k}_r = \beta_1 \sin\theta_r \underline{a}_x - \beta_1 \cos\theta_r \underline{a}_z \qquad \beta_1 = k_i = k_r$$
$$\underline{k}_t = k_t \sin\theta_t \underline{a}_x + k_t \cos\theta_t \underline{a}_z = \gamma \sin\psi \underline{a}_x + \gamma \cos\psi \underline{a}_z , \qquad \gamma_2 = jk_t = \alpha_2 + j\beta_2, \psi = \theta_t$$

$$\begin{cases} \underline{\hat{E}}_i = \hat{E}_{i0} e^{-j(\beta_1 \sin\theta_i x + \beta_1 \cos\theta_i z)} \underline{a}_y \\ \underline{\hat{E}}_r = \hat{E}_{r0} e^{-j(\beta_1 \sin\theta_r x - \beta_1 \cos\theta_r z)} \underline{a}_y \\ \underline{\hat{E}}_t = \hat{E}_{t0} e^{-(\gamma_2 \sin\psi x + \gamma_2 \cos\psi z)} \underline{a}_y \end{cases}$$

at the boundary $z = 0$,

$\underline{\hat{E}}$ tangential component continues

$\Rightarrow$

$$\underline{a}_y \cdot \underline{\hat{E}}_i \Big|_{z=0} + \underline{a}_y \cdot \underline{\hat{E}}_r \Big|_{z=0} = \underline{a}_y \cdot \underline{\hat{E}}_t \Big|_{z=0}$$

$$\hat{E}_{i0} e^{-j\beta_1 \sin\theta_i x} \underline{a}_y + \hat{E}_{r0} e^{-j\beta_1 \sin\theta_r x} \underline{a}_y$$
$$= \hat{E}_{t0} e^{-\gamma_2 \sin\psi x} \underline{a}_y$$

$$\hat{E}_{i0} + \hat{E}_{r0} = \hat{E}_{t0}$$

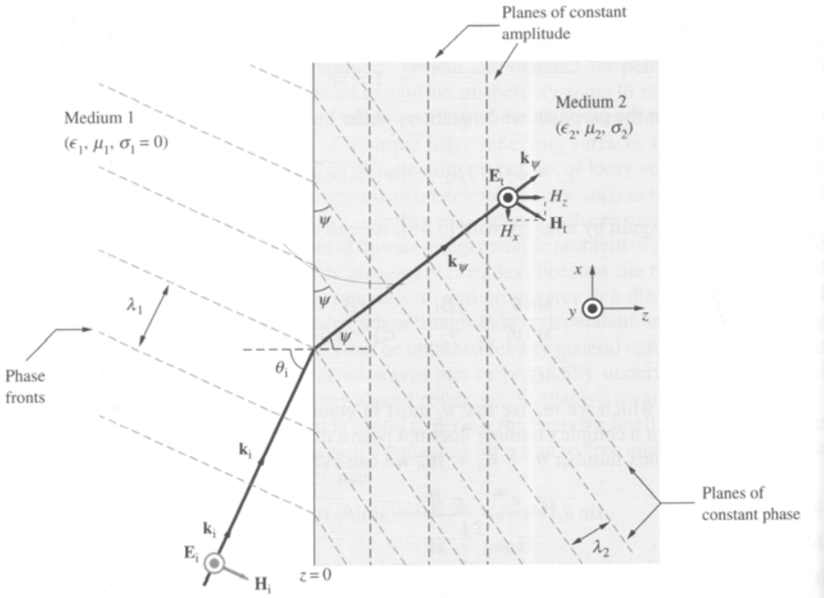

FIGURE 3.46. Oblique incidence on a lossy medium: Constant amplitude and phase fronts. The planes of constant amplitude are parallel to the interface, while the planes of constant phase are perpendicular to $\mathbf{k}_\psi$, which is at an angle ψ from the vertical. This "true" angle ψ is given by [3.59] and is also equal to the real part of the complex angle θ_t defined by [3.55].

$$\beta_1 \sin\theta_i x = \beta_1 \sin\theta_r x \qquad \theta_i = \theta_r$$
$$j\beta_1 \sin\theta_i x = \gamma_2 \sin\psi x \quad \Rightarrow \quad j\beta_1 \sin\theta_i = \gamma_2 \sin\psi$$

$$\frac{\sin\psi}{\sin\theta_i} = \frac{j\beta_1}{\gamma_2} = \frac{j\beta_1}{\alpha_2 + j\beta_2}$$

$$\gamma_2 \cos\psi = \gamma_2 \sqrt{1 - \sin^2\psi} = \sqrt{\gamma_2^2 - \gamma_2^2 \sin^2\psi} = \sqrt{\gamma_2^2 + \beta_1^2 \sin^2\theta_i} \equiv p + iq$$
$$p \equiv \mathrm{Re}\left\{\sqrt{\gamma_2^2 + \beta_1^2 \sin^2\theta_i}\right\} \qquad q \equiv \mathrm{Im}\left\{\sqrt{\gamma_2^2 + \beta_1^2 \sin^2\theta_i}\right\}$$

$$\underline{\hat{E}}_t = \hat{E}_{t0} e^{-(\gamma_2 \sin\psi x + \gamma_2 \cos\psi z)} \underline{a}_y = \hat{E}_{t0} e^{-p} e^{-j(\beta_1 \sin\theta_i x + qz)} \underline{a}_y$$

$$\beta_1 \sin\theta_i x + qz = \left(\beta_1 \sin\theta_i \underline{a}_x + q\underline{a}_z\right) \cdot \left(x\underline{a}_x + z\underline{a}_z\right) = \underline{\hat{k}}_\psi \cdot \underline{r}$$
$$\tan\psi = \frac{\beta_1 \sin\theta_i}{q}$$

$$\begin{cases} \text{attenuation constant} \ \alpha = \omega \sqrt{\dfrac{\mu\varepsilon}{2}} \left[\sqrt{1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{1/2} & [np/m] \\ \text{phase constant} \qquad \beta = \omega \sqrt{\dfrac{\mu\varepsilon}{2}} \left[\sqrt{1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{1/2} & [rad/m] \end{cases}$$

prove:

$$\hat{k} = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)} = \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta = \cos\theta(1 - j\tan\theta) = \cos\theta(1 - jx)$$

$$\begin{cases} \cos(\theta/2) = \sqrt{\dfrac{1+\cos\theta}{2}} \\ \sin(\theta/2) = \sqrt{\dfrac{1-\cos\theta}{2}} \end{cases}$$

$$x = \tan\theta \quad \Rightarrow \quad \begin{cases} \sin\theta = \dfrac{x}{\sqrt{x^2+1}} \\ \cos\theta = \dfrac{1}{\sqrt{x^2+1}} \end{cases}$$

$$1 - jx = \frac{e^{-j\theta}}{\cos\theta} = 1 - j\frac{\sigma}{\omega\varepsilon}$$

$$(1-jx)^{1/2} = \frac{e^{-j\theta/2}}{(\cos\theta)^{1/2}} = \frac{\cos(\theta/2) - j\sin(\theta/2)}{(\cos\theta)^{1/2}} = \frac{\sqrt{\dfrac{1+\cos\theta}{2}} - j\sqrt{\dfrac{1-\cos\theta}{2}}}{(\cos\theta)^{1/2}}$$

$$= \frac{\left(\dfrac{1+\dfrac{1}{\sqrt{x^2+1}}}{2}\right)^{1/2} - j\left(\dfrac{1-\dfrac{1}{\sqrt{x^2+1}}}{2}\right)^{1/2}}{\left(\dfrac{1}{\sqrt{x^2+1}}\right)^{1/2}} = \frac{1}{\sqrt{2}}\left[\left(\sqrt{x^2+1}+1\right)^{1/2} - j\left(\sqrt{x^2+1}-1\right)^{1/2}\right]$$

$$= \frac{1}{\sqrt{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} - j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2}\right\}$$

$$\hat{k} = \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} = \omega\sqrt{\mu\varepsilon}\cdot\frac{1}{\sqrt{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} - j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2}\right\}$$

$$= \omega\sqrt{\frac{\mu\varepsilon}{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} - j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2}\right\}$$

$$\gamma = j\hat{k} = j\omega\sqrt{\mu\hat{\varepsilon}} = j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$= j\omega\sqrt{\frac{\mu\varepsilon}{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} - j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2}\right\}$$

$$= \omega\sqrt{\frac{\mu\varepsilon}{2}}\left\{\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2} + j\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2}\right\} = \alpha + j\beta$$

$$\begin{cases} \alpha = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2} \\ \beta = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2} \end{cases}$$

complex intrinsic impedance of a conducting medium

$$\hat{\eta}_c = |\hat{\eta}_c|e^{j\phi_\eta} = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon\left(1-j\dfrac{\sigma}{\omega\varepsilon}\right)}} = \frac{\sqrt{\dfrac{\mu}{\varepsilon}}}{\left[1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}} e^{j(1/2)tan^{-1}[\sigma/(\omega\varepsilon)]}$$

$\sigma << \omega\varepsilon \quad \Rightarrow$ lossless media

$$\beta \approx \omega\sqrt{\mu\varepsilon}, \quad \eta_c \approx \sqrt{\mu/\varepsilon}$$

$\sigma >> \omega\varepsilon \quad \Rightarrow$ perfect conductor

$\eta_c \to 0$ as $\sigma \to \infty$

loss tangent $tan\,\delta = \dfrac{\sigma}{\omega\varepsilon}$

good conductor $tan\,\delta >> 1$

poor conductor (good insulator) $tan\,\delta << 1$

$$\hat{\varepsilon} = \varepsilon\left(1-j\frac{\sigma}{\omega\varepsilon}\right) = \varepsilon' - j\varepsilon''$$

for a good dielectric $\varepsilon'' << \varepsilon'$

loss tangent $tan\,\delta = \dfrac{\varepsilon''}{\varepsilon'} << 1$

$$\hat{k} = \omega\sqrt{\mu\hat{\varepsilon}} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')} = \omega\sqrt{\mu\varepsilon'\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)} = \omega\sqrt{\mu\varepsilon'}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{1/2}$$

$$\cong \omega\sqrt{\mu\varepsilon'}\left[1+\frac{1}{2}\left(-j\frac{\varepsilon''}{\varepsilon'}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-j\frac{\varepsilon''}{\varepsilon'}\right)^2+\cdots\right]$$

$$\cong \omega\sqrt{\mu\varepsilon'}\left[1-j\frac{\varepsilon''}{2\varepsilon'}+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2+\cdots\right]$$

$$\cong \omega\sqrt{\mu\varepsilon'}\left[1+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] - j\omega\sqrt{\mu\varepsilon'}\left(\frac{\varepsilon''}{2\varepsilon'}\right) \qquad \because \frac{\varepsilon''}{\varepsilon'} << 1$$

$$\gamma = jk = \alpha + j\beta$$

attenuation constant $\alpha \approx \omega\sqrt{\mu\varepsilon'}\left(\dfrac{\varepsilon''}{2\varepsilon'}\right) \quad [np/m]$

phase constant $\beta \approx \omega\sqrt{\mu\varepsilon'}\left[1+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$ $[rad/m]$

phase velocity $v_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon'}}\left[1-\frac{1}{8}\right]$ $[m/s]$

intrinsic impedance $\hat{\eta}_c = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon'\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$

$$\cong \sqrt{\frac{\mu}{\varepsilon'}}\left[1+j\frac{\varepsilon''}{2\varepsilon'}-\frac{3}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2+\cdots\right]$$

$$\approx \sqrt{\frac{\mu}{\varepsilon'}}\left[1-\frac{3}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]+j\sqrt{\frac{\mu}{\varepsilon'}}\left(\frac{\varepsilon''}{2\varepsilon'}\right) \qquad [\Omega]$$

for a good conductor $\sigma >> \omega\sigma$

loss tangent $\tan\delta = \frac{\sigma}{\omega\varepsilon} >> 1$

$$\hat{k} = \omega\sqrt{\mu\hat{\varepsilon}} = \omega\sqrt{\mu\varepsilon\left(1-j\frac{\sigma}{\omega\varepsilon}\right)} \cong \omega\sqrt{\mu\varepsilon\left(-j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$\cong \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}e^{j\pi/4}$$

$$= \sqrt{\omega\mu\sigma}\left(\frac{1-j}{\sqrt{2}}\right)$$

$$\gamma = j\hat{k} = (1+j)\sqrt{\pi f\mu\sigma} = \alpha + j\beta$$

attenuation constant $\alpha \cong \sqrt{\pi f\mu\sigma}$

phase constant $\beta \approx \sqrt{\pi f\mu\sigma}$

intrinsic impedance

$$\hat{\eta}_c = \sqrt{\frac{\mu}{\hat{\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon\left(1-j\frac{\sigma}{\omega\varepsilon}\right)}} \cong \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= \sqrt{\frac{\omega\mu}{\sigma}}e^{j\pi/4} = \frac{(1+j)}{\sqrt{2}}\sqrt{\frac{2\pi f\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}}$$

$$\hat{\eta}_c = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)(\sigma\delta)^{-1}$$

skin depth $\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}$

radiation pressure and electromagnetic moment