Time-harmonic uniform plane waves in lossless medium

$$\underline{E}(x,y,z,t)=\underline{E}(\underline{r},t)=Re\{\underline{\hat{E}}(\underline{r})e^{j\omega t}\}=\frac{1}{2}\left(\underline{\hat{E}}(\underline{r})e^{j\omega t}+\underline{\hat{E}}^*(\underline{r})e^{-j\omega t}\right)$$

$$\underline{H}(x,y,z,t)=\underline{H}(\underline{r},t)=Re\{\underline{\hat{H}}(\underline{r})e^{jwt}\}=\frac{1}{2}\left(\underline{\hat{H}}(\underline{r})e^{j\omega t}+\underline{\hat{H}}^*(\underline{r})e^{-j\omega t}\right)$$

$$\underline{J}(x,y,z,t)=\underline{J}(\underline{r},t)=Re\{\underline{\hat{J}}(\underline{r})e^{jwt}\}$$

$$\rho(x,y,z,t)=\rho(\underline{r},t)=Re\{\hat{\rho}(\underline{r})e^{jwt}\}$$

$$\frac{\partial}{\partial t}\rightarrow j\omega$$

$$\nabla\times\underline{\hat{E}}=-j\omega\mu\underline{\hat{H}} \qquad \nabla\times\underline{\hat{H}}=j\omega\varepsilon\underline{\hat{E}}$$

$$\nabla\cdot\underline{\hat{D}}=\rho \qquad \nabla\cdot\underline{\hat{B}}=0$$

$$\nabla\times\nabla\times\underline{\hat{E}}=-j\omega\mu\nabla\times\underline{\hat{H}}$$

$$\nabla\left(\nabla\cdot\underline{\hat{E}}\right)-\nabla^2\underline{\hat{E}}=-j\omega\mu\left(-j\omega\varepsilon\underline{\hat{E}}\right)$$

$$\nabla^2\underline{\hat{E}}+\omega^2\mu\varepsilon\underline{\hat{E}}=0, \qquad \nabla^2\underline{\hat{E}}+k^2\underline{\hat{E}}=0$$

$k=\omega\sqrt{\mu\varepsilon}$ phase constant; wave number; propagation constant

$$\nabla\times\nabla\times\underline{\hat{H}}=j\omega\varepsilon\nabla\times\underline{\hat{E}}$$

$$\nabla\left(\nabla\cdot\underline{\hat{H}}\right)-\nabla^2\underline{\hat{H}}=j\omega\varepsilon\left(-j\omega\mu\underline{\hat{H}}\right)$$

$$\nabla^2\underline{H}+\omega^2\mu\varepsilon\underline{\hat{H}}=0, \qquad \nabla^2\underline{H}+k^2\underline{\hat{H}}=0$$

$$\underline{\hat{E}}=\hat{E}_x\underline{a_x}+\hat{E}_y\underline{a_y}+\hat{E}_z\underline{a_z}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\hat{E}_x+k^2\hat{E}_x=0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\hat{E}_y+k^2\hat{E}_y=0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\hat{E}_z+k^2\hat{E}_z=0$$

Considering $\hat{\boldsymbol{E}}_x$ only and let $\hat{E}_x(x,y,z)=f(x)g(y)h(z)$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\hat{E}_x+k^2\hat{E}_x=0$$

$$\Rightarrow \quad f''gh+fg''h+fgh''+k^2fgh=0$$

$$\Rightarrow \quad \frac{f''}{f}+\frac{g''}{g}+\frac{h''}{h}+k^2=0$$

$$\Rightarrow \quad \begin{cases} \dfrac{f''}{f}=-k_x{}^2 & f''+k_x{}^2 f=0 \\ \dfrac{g''}{g}=-k_y{}^2 & g''+k_y{}^2 g=0 \\ \dfrac{h''}{h}=-k_z{}^2 & h''+k_z{}^2 h=0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} f=Ae^{-jk_x x}+Be^{jk_x x} \\ g=Ce^{-jk_y y}+De^{jk_y y} \\ h=Ee^{-jk_z z}+Fe^{jk_z z} \end{cases}$$

Assuming only $e^{-jk_x x}$ term exists

$$\hat{E}_x(x,y,z)=f(x)g(y)h(z)=ACEe^{-j(k_x x+k_y y+k_z z)}$$

$$\Rightarrow \quad \begin{cases} \hat{E}_x(x,y,z)=\hat{E}_{0x}e^{-j(k_x x+k_y y+k_z z)} \\ \hat{E}_y(x,y,z)=\hat{E}_{0y}e^{-j(k_x x+k_y y+k_z z)} \\ \hat{E}_z(x,y,z)=\hat{E}_{0z}e^{-j(k_x x+k_y y+k_z z)} \end{cases} \qquad \boldsymbol{\hat{E}_{0x}, \hat{E}_{0y}, \hat{E}_{0z}} \text{ are constant}$$

$$\underline{k}=k_x\underline{i}+k_y\underline{j}+k_z\underline{k}=k\underline{a}_n, \qquad k=\sqrt{k_x{}^2+k_y{}^2+k_z{}^2}$$

$$\underline{r}=x\underline{i}+y\underline{j}+z\underline{k}$$

$$\underline{k}\cdot\underline{r}=k_x x+k_y y+k_z z$$

$$\underline{\hat{E}}(x,y,z)=\hat{E}_x\underline{i}+\hat{E}_y\underline{j}+\hat{E}_z\underline{k}$$

$$=\left[\hat{E}_{0x}\underline{i}+\hat{E}_{0y}\underline{j}+\hat{E}_{0z}\underline{k}\right]e^{-j(k_x x+k_y y+k_z z)} \qquad \underline{\boldsymbol{\hat{E}_0}} \text{ is a constant vector}$$

$$=\underline{\hat{E}_0}e^{-j\underline{k}\cdot\underline{r}}$$

$$\underline{E}(\underline{r},t)=Re\left\{\underline{\hat{E}}(x,y,z)e^{j\omega t}\right\}$$

$$=Re\left\{\underline{\hat{E}_0}e^{j(\omega t-\underline{k}\cdot\underline{r})}\right\}$$

Generally speaking, for $\nabla^2\underline{\hat{E}}+k^2\underline{\hat{E}}=0$

$$\underline{\hat{E}}(\underline{r})=\underline{\hat{E}}(x,y,z)=\underline{\hat{E}_0^+}e^{-j\underline{k}\cdot\underline{r}}+\underline{\hat{E}_0^-}e^{j\underline{k}\cdot\underline{r}}$$

$\underline{\hat{E}}(x,y,z)=\hat{E}_x(z)\underline{a}_x$ of a uniform plane wave is uniform over the *xy* plane

$\Rightarrow$ independent of the *x* and *y* coordinates

$$\Rightarrow \quad \frac{\partial \hat{E}_x}{\partial x} = \frac{\partial \hat{E}_x}{\partial y} = 0$$

$$\therefore \quad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\hat{E}_x + k^2 \hat{E}_x = 0 \quad \Rightarrow \quad \frac{\partial^2 \hat{E}_x}{\partial z^2} + k^2 \hat{E}_x = 0$$

$$\hat{E}_x(z) = \underbrace{\hat{E}^+_{x0} e^{-jkz}}_{\hat{E}^+_x(z)} + \underbrace{\hat{E}^-_{x0} e^{jkz}}_{\hat{E}^-_x(z)}$$

$$E_x(z,t) = Re\left\{\left(\hat{E}^+_{x0} e^{-jkz} + \hat{E}^-_{x0} e^{jkz}\right) e^{j\omega t}\right\}$$

Relation between $\underline{E}$ and $\underline{H}$ in a uniform plane wave

$$\nabla \cdot \underline{\hat{E}} = 0 = \nabla \cdot \underline{\hat{E}_0} e^{-j\underline{k}\cdot\underline{r}}$$

uniform plane wave $\nabla \cdot \underline{\hat{E}_0} = 0$

$$= e^{-j\underline{k}\cdot\underline{r}} \nabla \cdot \underline{\hat{E}_0} + \underline{\hat{E}_0} \cdot \nabla e^{-j\underline{k}\cdot\underline{r}}$$

$$= \underline{\hat{E}_0} \cdot \nabla e^{-j\underline{k}\cdot\underline{r}}$$

$$= \underline{\hat{E}_0} \cdot [-j\underline{k} e^{-j\underline{k}\cdot\underline{r}}]$$

$$= -j\left(\underline{k} \cdot \underline{\hat{E}_0} e^{-j\underline{k}\cdot\underline{r}}\right)$$

$$= -j\left(\underline{k} \cdot \underline{\hat{E}}\right)$$

$$\underline{k} \cdot \underline{\hat{E}} = 0 \quad \Rightarrow \quad \underline{k} \perp \underline{\hat{E}} \quad \Rightarrow \quad \underline{k} \perp \underline{E}$$

$$\nabla e^{-j\underline{k}\cdot\underline{r}} = \left(\frac{\partial}{\partial x}\underline{a_x} + \frac{\partial}{\partial y}\underline{a_y} + \frac{\partial}{\partial z}\underline{a_z}\right) e^{-j(k_x x + k_y y + k_z z)}$$

$$= -j\left(k_x \underline{a_x} + k_y \underline{a_y} + k_z \underline{a_z}\right) e^{-j(k_x x + k_y y + k_z zt)}$$

$$= -j\underline{k} e^{-j(k_x x + k_y y + k_z zt)}$$

$$= -j\underline{k} e^{-j(\underline{k}\cdot\underline{r}t)} \qquad \nabla \rightarrow \pm j\underline{k}$$

$$\left(\nabla \cdot \varphi \underline{A} = \varphi \nabla \cdot \underline{A} + \underline{A} \cdot \nabla \varphi\right)$$

$$-j\omega\mu_0 \underline{\hat{H}} = \nabla \times \underline{\hat{E}} = \nabla \times \underline{\hat{E}_0} e^{-j\underline{k}\cdot\underline{r}}$$

$$\nabla \times \varphi \underline{A} = \varphi \nabla \times \underline{A} + \underline{A} \times \nabla \varphi$$

$$-j\omega\mu_0 \underline{\hat{H}} = -\underline{\hat{E}_0} \times \nabla e^{-j\underline{k}\cdot\underline{r}}$$

$$\underline{\hat{H}} = -\frac{1}{-j\omega\mu_0} \underline{\hat{E}_0} \times -j\underline{k} e^{-j\underline{k}\cdot\underline{r}}$$

$$= \frac{\underline{k} \times \underline{\hat{E}_0} e^{-j\underline{k}\cdot\underline{r}}}{\omega\mu_0} = \frac{\underline{k} \times \underline{\hat{E}}}{\omega\mu_0}$$

$$= \frac{k}{\omega\mu_0} \underline{a_n} \times \underline{\hat{E}_0} e^{-j\underline{k}\cdot\underline{r}}$$

$$= \frac{1}{\eta_0} \underline{a_n} \times \underline{\hat{E}_0} e^{-j\underline{k}\cdot\underline{r}} = \frac{1}{\eta_0} \underline{a_n} \times \underline{\hat{E}}$$

$$\frac{k}{\omega\mu_0} = \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\mu_0} = \sqrt{\frac{\varepsilon_0}{\mu_0}} = \frac{1}{\eta_0}$$

$$= \underline{\hat{H}_0} e^{-j\underline{k}\cdot\underline{r}}$$

Complex Poynting vector

$$\underline{S} = \underline{E} \times \underline{H} = \frac{1}{2}\left[\underline{\hat{E}}(\underline{r})e^{j\omega t} + \underline{\hat{E}}^*(\underline{r})e^{-j\omega t}\right] \times \frac{1}{2}\left[\underline{\hat{H}}(\underline{r})e^{j\omega t} + \underline{\hat{H}}^*(\underline{r})e^{-j\omega t}\right]$$

$$= \frac{1}{4}\left[\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}^*(\underline{r}) + \underline{\hat{E}}^*(\underline{r}) \times \underline{\hat{H}}(\underline{r})\right] + \frac{1}{4}\left[\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}(\underline{r})e^{j2\omega t} + \underline{\hat{E}}^*(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})e^{-j2\omega t}\right]$$

since $\left[\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})\right]^* = \underline{\hat{E}}^*(\underline{r}) \times \underline{\hat{H}}(\underline{r}),\quad \left[\underline{\hat{E}}^*(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})e^{-j2\omega t}\right]^* = \underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}(\underline{r})e^{j2\omega t}$

$$\underline{S} = \frac{1}{2}Re\left\{\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})\right\} + \frac{1}{2}Re\left\{\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}(\underline{r})e^{j2\omega t}\right\}$$

actual measurements for light $\Rightarrow$ time-average Poynting vector

$$\langle \underline{S} \rangle = \frac{1}{T}\int_0^T \underline{S}dt = \frac{1}{2}\mathrm{Re}\left\{\underline{\hat{E}}(\underline{r}) \times \underline{\hat{H}}^*(\underline{r})\right\}$$

Instantaneous Poynting vector

$$\underline{P}(z,t) = \mathrm{Re}\left\{\underline{\hat{E}}(z)e^{j\omega t}\right\} \times \mathrm{Re}\left\{\underline{\hat{H}}(z)e^{j\omega t}\right\}$$

$$= \frac{1}{2}\left[\underline{\hat{E}}(z)e^{j\omega t} + \underline{\hat{E}}^*(z)e^{-j\omega t}\right] \times \frac{1}{2}\left[\underline{\hat{H}}(z)e^{j\omega t} + \underline{\hat{H}}^*(z)e^{-j\omega t}\right]$$

$$= \frac{1}{4}\left\{\left[\underline{\hat{E}}(z) \times \underline{\hat{H}}^*(z) + \underline{\hat{E}}^*(z) \times \underline{\hat{H}}(z)\right] + \left[\underline{\hat{E}}(z) \times \underline{\hat{H}}(z)e^{j2\omega t} + \underline{\hat{E}}^*(z) \times \underline{\hat{H}}^*(z)e^{-j2\omega t}\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\underline{\hat{E}}(z) \times \underline{\hat{H}}^*(z) + \underline{\hat{E}}(z) \times \underline{\hat{H}}(z)e^{j2\omega t}\right\}$$

$$\therefore \langle \underline{S} \rangle = \frac{1}{2}\mathrm{Re}\left\{\underline{\hat{E}}_1 \times \underline{\hat{H}}_1^*\right\}$$

Complex Poynting vector $\underline{\hat{S}} = \underline{\hat{E}} \times \underline{\hat{H}}^*$

$$\underline{\hat{S}} = \underline{\hat{E}} \times \underline{\hat{H}}^* = \underline{\hat{E}_0}e^{-j\underline{k}\cdot\underline{r}} \times \left(\frac{1}{\eta_0}\underline{a_n} \times \underline{\hat{E}_0}e^{-j\underline{k}\cdot\underline{r}}\right)^*$$

$$= \underline{\hat{E}_0} \times \frac{1}{\eta_0}\left(\underline{a_n} \times \underline{\hat{E}_0}\right)^* = \frac{1}{\eta_0}\left(\underline{\hat{E}_0} \cdot \underline{\hat{E}_0}^*\right)\underline{a_n} - \frac{1}{\eta_0}\left(\underbrace{\underline{\hat{E}_0} \cdot \underline{a_n}}_{=0}\right)\underline{\hat{E}_0}^*$$

$$= \frac{1}{\eta_0}\left|\underline{\hat{E}_0}\right|^2 \underline{a_n}$$

irradiance :

$$I = \left|\langle \underline{S} \rangle\right|$$

Reflection, transmission, and refraction of waves at planar interfaces

Electromagnetic boundary conditions

1. Faraday's law

$$\oint_c \underline{E} \cdot \underline{dl} = -\frac{\partial}{\partial t}\int_s \underline{B} \cdot \underline{ds} \qquad \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\Rightarrow \quad \underline{a}_n \times \left[\underline{E}_2 - \underline{E}_1\right] = 0 \quad \rightarrow \quad E_{1t} = E_{2t}$$

2. Ampere's law

$$\oint_c \underline{H} \cdot \underline{dl} = \int_s \underline{J} \cdot \underline{ds} + \frac{d}{dt}\int_s \underline{D} \cdot \underline{ds} \qquad \nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

in geneal $J_S = 0 \;\Rightarrow\; \underline{a}_n \times \left[\underline{H}_2 - \underline{H}_1\right] = 0 \;\rightarrow\; H_{2t} = H_{1t}$

a perfect conductor $(\sigma = \infty)$

$$\Rightarrow \quad \underline{a}_n \times \underline{H}_1 = \underline{J}_s$$

3. Gauss's law for electric fields

$$\int_s \underline{D} \cdot \underline{ds} = \int_v \rho dv \qquad \nabla \cdot \underline{D} = \rho$$

$$\Rightarrow \quad \underline{a}_n \cdot \left(\underline{D}_2 - \underline{D}_1\right) = \rho_S \quad \rightarrow \quad D_{2n} - D_{1n} = \rho_S$$

4. Gauss's law for magnetic fields

$$\int_s \underline{B} \cdot \underline{ds} = 0 \qquad \nabla \cdot \underline{B} = 0$$

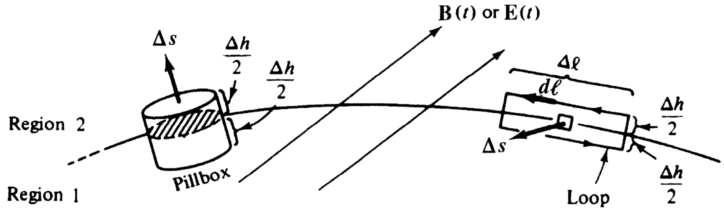
$$\Rightarrow \quad \underline{a}_n \cdot \left(\underline{B}_2 - \underline{B}_1\right) = 0 \quad \rightarrow \quad B_{1n} = B_{2n}$$

5. equation of continuity

$$\int_s \underline{J} \cdot \underline{ds} = -\frac{\partial}{\partial t}\int_v \rho dv \qquad \nabla \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \quad \underline{a}_n \cdot \left(\underline{J}_{2n} - \underline{J}_{1n}\right) = -\frac{\partial \rho_S}{\partial t} \quad \rightarrow \quad \underline{J}_{2n} - \underline{J}_{1n} = -\frac{\partial \rho_S}{\partial t}$$

Normal condition



$$\nabla \cdot \underline{D} = \rho \qquad \nabla \cdot \underline{B} = 0$$

$$\int_v \nabla \cdot \underline{D} dv = \int_v \rho dv \qquad \int_v \nabla \cdot \underline{B} dv = 0$$

$$\oint_s \underline{D} \cdot \underline{ds} = \int_v \rho dv \qquad \int_s \underline{B} \cdot \underline{ds} = 0$$

$$\lim_{\delta \to 0} \oint_s \underline{D} \cdot \underline{ds} = \lim_{\delta \to 0} \int_v \rho dv \cong \int_s \rho_s ds \qquad \lim_{\delta \to 0} \int_s \underline{B} \cdot \underline{ds} = 0$$

$$-\underline{D_1} \cdot \underline{a_n} dS + \underline{D_2} \cdot \underline{a_n} dS = \rho_s ds \qquad -\underline{B_1} \cdot \underline{a_n} dS + \underline{B_2} \cdot \underline{a_n} dS = 0$$

$$\underline{a_n} \cdot \left(-\underline{D_1} + \underline{D_2} \right) = \rho_s \qquad \underline{a_n} \cdot \left(-\underline{B_1} + \underline{B_2} \right) = 0$$

$$-D_{1n} + D_{2n} = \rho_s \qquad -B_{1n} + B_{2n} = 0$$

$$D_{2n} - D_{1n} = \rho_s \qquad B_{1n} = B_{2n}$$

$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \rho_s \qquad \mu_1 H_{1n} = \mu_2 H_{2n}$$

Tangential condition

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

$$\int_s \nabla \times \underline{E} \cdot \underline{ds} = -\frac{\partial}{\partial t} \int_s \underline{B} \cdot \underline{ds} \qquad \int_s \nabla \times \underline{H} \cdot \underline{ds} = \int_s \underline{J} \cdot \underline{ds} + \frac{\partial}{\partial t} \int_s \underline{D} \cdot \underline{ds}$$

$$\oint_c \underline{E} \cdot \underline{dl} = -\frac{\partial}{\partial t} \int_s \underline{B} \cdot \underline{ds} \qquad \oint_c \underline{H} \cdot \underline{dl} = \int_s \underline{J} \cdot \underline{ds} + \frac{\partial}{\partial t} \int_s \underline{D} \cdot \underline{ds}$$

$$\lim_{\delta \to 0} \oint_c \underline{E} \cdot \underline{dl} = -\frac{\partial}{\partial t} \left(\underbrace{\lim_{\delta \to 0} \int_s \underline{B} \cdot \underline{ds}}_{\to 0} \right) \qquad \lim_{\delta \to 0} \oint_c \underline{H} \cdot \underline{dl} = \lim_{\delta \to 0} \int_s \underline{J} \cdot \underline{ds} + \frac{\partial}{\partial t} \left(\underbrace{\lim_{\delta \to 0} \int_s \underline{D} \cdot \underline{ds}}_{\to 0} \right)$$

$\because$ area $\to$ 0 $\Rightarrow$ flux $\to$ 0 $\qquad$ $\because$ area $\to$ 0 $\Rightarrow$ flux $\to$ 0

$$\underline{E_1} \cdot \underline{dl_1} + \underline{E_2} \cdot \underline{dl_2} = 0 \qquad \underline{H_1} \cdot \underline{dl_1} + \underline{H_2} \cdot \underline{-\underline{a_t} \times \underline{a_n}} = \left(\underline{J_S} \cdot \underline{a_t} \right) dl, \qquad \underline{J_S}$$

: surface current density

$$-\underline{E_1} \cdot \left(\underline{a_t} \times \underline{a_n} \right) dl + \underline{E_2} \cdot \left(\underline{a_t} \times \underline{a_n} \right) dl = 0, \qquad -\underline{H_1} \cdot \underline{a_t} \times \underline{a_n} dl + \underline{H_2} \cdot \underline{a_t} \times \underline{a_n} dl = \left(\underline{J_S} \cdot \underline{a_t} \right) dl,$$

$$\underline{dl_1} = -\underline{a_t} \times \underline{a_n} dl \qquad \underline{dl_1} = -\underline{a_t} \times \underline{a_n} dl$$

$$\underline{dl_2} = \underline{a_t} \times \underline{a_n} dl \qquad \underline{dl_2} = \underline{a_t} \times \underline{a_n} dl$$

$$\underline{a_t} \times \underline{a_n} \cdot \left(\underline{E_2} - \underline{E_1} \right) = 0 \qquad \underline{a_t} \times \underline{a_n} \cdot \left(\underline{H_2} - \underline{H_1} \right) = \underline{J_S} \cdot \underline{a_t}$$

$$\underline{a_t} \cdot \underline{a_n} \times \left(\underline{E_2} - \underline{E_1} \right) = 0 \qquad \underline{a_t} \cdot \underline{a_n} \times \left(\underline{H_2} - \underline{H_1} \right) = \underline{a_t} \cdot \underline{J_S}$$

$$\underline{a_n} \times \left(\underline{E_2} - \underline{E_1} \right) = 0 \qquad \underline{a_n} \times \left(\underline{H_2} - \underline{H_1} \right) = \underline{J_S}$$

$$E_{2t} = E_{1t} \qquad H_{2t} - H_{1t} = J_S$$

Laws of reflection and refraction

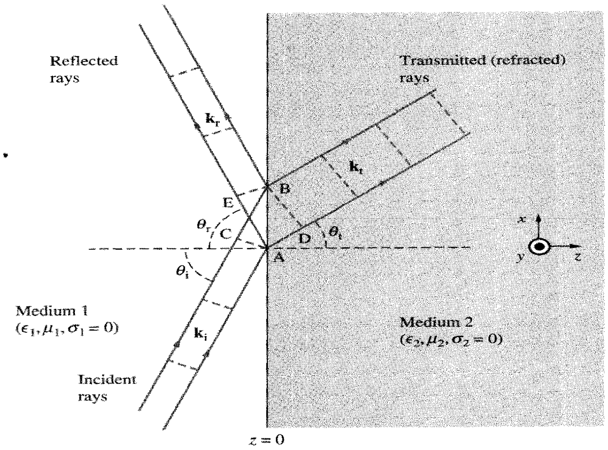
Law of reflection

$$\underbrace{CB}_{v_{p1}t} = \underbrace{AE}_{v_{p1}t}$$

$$\Rightarrow \quad \underbrace{AB\sin\theta_i}_{CB} = \underbrace{AB\sin\theta_r}_{AE}$$

$$\Rightarrow \quad \sin\theta_i = \sin\theta_r$$

$$\Rightarrow \quad \boxed{\theta_i = \theta_r \qquad \text{Snell's reflection law}}$$



law of refraction

$$\text{since} \quad \sin\theta_i = \frac{BC}{AB} = \frac{v_{p1}t}{AB}, \qquad \sin\theta_t = \frac{AD}{AB} = \frac{v_{p2}t}{AB}$$

$$\Rightarrow \quad \frac{\sin\theta_i}{\sin\theta_t} = \frac{v_{p1}}{v_{p2}} = \sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

for nonmagnetic media $\mu_1 = \mu_2 = \mu_0$

$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\frac{\varepsilon_{2r}\varepsilon_0}{\varepsilon_{1r}\varepsilon_0}} = \sqrt{\frac{\varepsilon_{2r}}{\varepsilon_{1r}}}$$

$$\Rightarrow \quad \frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_{2r}}{\varepsilon_{1r}}}, \qquad \sqrt{\varepsilon_{1r}}\,\sin\theta_i = \sqrt{\varepsilon_{2r}}\,\sin\theta_t$$

refractive index $n = c/v_p = \sqrt{(\mu\varepsilon)/(\mu_0\varepsilon_0)} = \sqrt{(\mu_0\varepsilon_0\varepsilon_r)/(\mu_0\varepsilon_0)} = \sqrt{\varepsilon_r}$

$$\Rightarrow \quad \frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1} \quad \Rightarrow \quad \boxed{n_1\sin\theta_i = n_2\sin\theta_t \qquad \text{Snell's law of refraction}}$$

- **Waves at interface**

Incident wave	Reflected wave	Refracted wave
$\underline{\hat{E}}_i = \underline{\hat{E}}_{0i}e^{-j\underline{k}_i\cdot\underline{r}}$	$\underline{\hat{E}}_r = \underline{\hat{E}}_{0r}e^{-j\underline{k}_r\cdot\underline{r}}$	$\underline{\hat{E}}_t = \underline{\hat{E}}_{0t}e^{-j\underline{k}_t\cdot\underline{r}}$
$\underline{\hat{H}}_i = \underline{\hat{H}}_{0i}e^{-j\underline{k}_i\cdot\underline{r}}$	$\underline{\hat{H}}_r = \underline{\hat{H}}_{0r}e^{-j\underline{k}_r\cdot\underline{r}}$	$\underline{\hat{H}}_t = \underline{\hat{H}}_{0t}e^{-j\underline{k}_t\cdot\underline{r}}$

choosing $\underline{k}_i = k_{xi}\underline{a}_x + k_{zi}\underline{a}_z$

The plane of incidence is defined as the plane containing the normal to the interface and the k-vector indicating the direction of propagation of the incident wave (i.e. the *xz* plane).

$$\underline{k}_r = k_{xr}\underline{a}_x + k_{yr}\underline{a}_y + k_{zr}\underline{a}_z, \qquad \underline{k}_t = k_{xt}\underline{a}_x + k_{yt}\underline{a}_y + k_{zt}\underline{a}_z$$

at the interface $z = 0$,

$$\underline{a}_z \times [\underline{E}_2 - \underline{E}_1] = 0$$

$\underline{a}_z$ is the unit vector normal to the interface

$$\underline{a}_z \times \hat{\underline{E}}_i\big|_{z=0} + \underline{a}_z \times \hat{\underline{E}}_r\big|_{z=0} = \underline{a}_z \times \hat{\underline{E}}_t\big|_{z=0}$$

$$\Rightarrow \quad \underline{a}_z \times \hat{\underline{E}}_{0i} e^{-j\underline{k}_i \cdot \underline{r}}\big|_{z=0} + \underline{a}_z \times \hat{\underline{E}}_{0r} e^{-j\underline{k}_r \cdot \underline{r}} = \underline{a}_z \times \hat{\underline{E}}_{0t} e^{-j\underline{k}_t \cdot \underline{r}}$$

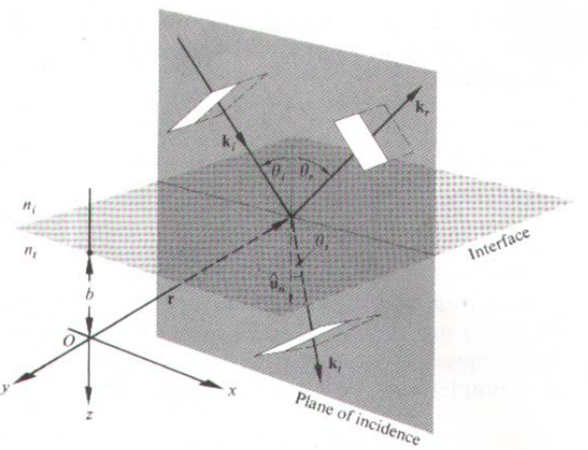
$$\Rightarrow \quad \begin{cases} \underline{a}_z \times \hat{\underline{E}}_{0i}\big|_{z=0} + \underline{a}_z \times \hat{\underline{E}}_{0r}\big|_{z=0} = \underline{a}_z \times \hat{\underline{E}}_{0t}\big|_{z=0} \\ \underline{k}_i \cdot \underline{r}\big|_{z=0} = \underline{k}_r \cdot \underline{r}\big|_{z=0} = \underline{k}_t \cdot \underline{r}\big|_{z=0} \end{cases}$$

$$\underline{k}_i \cdot \underline{r}\big|_{z=0} = \underline{k}_r \cdot \underline{r}\big|_{z=0} = \underline{k}_t \cdot \underline{r}\big|_{z=0}$$

$$\Rightarrow \quad k_{xi} x + k_{zi} z\big|_{z=0} = k_{xr} x + k_{yr} y + k_{zr} z\big|_{z=0} = k_{xt} x + k_{yt} y + k_{zt} z\big|_{z=0}$$

$$\Rightarrow \quad k_{xi} x = k_{xr} x + k_{yr} y = k_{xt} x + k_{yt} y$$

$$\Rightarrow \quad \begin{cases} k_{xi} = k_{xr} = k_{xt} \\ k_{yr} = k_{yt} \end{cases}$$



$$k_{xi} = k_{xr}$$

$$\Rightarrow \quad k_i \sin\theta_i = k_r \sin\theta_r$$

$\because$ k_i and k_r are all in medium 1

$$\therefore \quad k_i = k_r = \omega\sqrt{\mu_1 \varepsilon_1}$$

$$\Rightarrow \quad \sin\theta_i = \sin\theta_r$$

$$\Rightarrow \quad \boxed{\theta_i = \theta_r \qquad \text{law of reflection}}$$

$$k_{xi} = k_{xt}$$

$$\Rightarrow \quad k_i \sin\theta_i = k_t \sin\theta_t$$

$$\Rightarrow \quad \omega\sqrt{\mu_1 \varepsilon_1} \sin\theta_i = \omega\sqrt{\mu_2 \varepsilon_2} \sin\theta_t$$

$$\Rightarrow \quad \sqrt{\varepsilon_1} \sin\theta_i = \sqrt{\varepsilon_2} \sin\theta_t$$

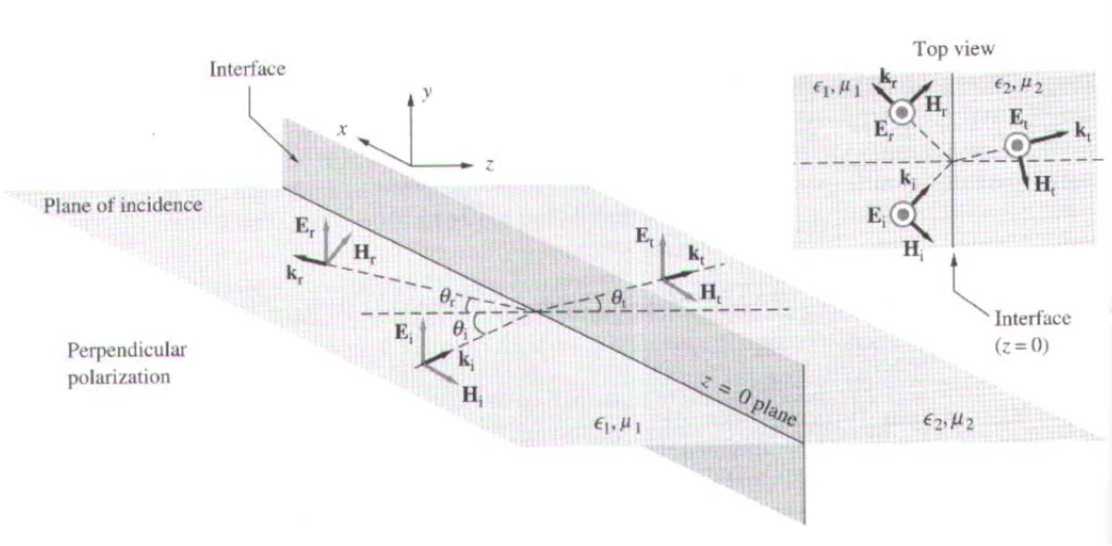
$$\Rightarrow \quad \boxed{n_i \sin\theta_i = n_t \sin\theta_t \qquad \text{Snell's law of refraction}}$$

Perpendicular polarization (Transverse E, horizontal, E-wave, senkrecht-wave) : E-field is perpendicular to the plane of incidence,

parallel polarization (TM, vertical, H-wave, p-wave) : E-filed is parallel to the plane of incidence

- **Perpendicular polarization**

  Oblique incidence at a dielectric boundary



Incident wave	Reflected wave	Refracted wave

$\hat{\underline{E}}_i = \hat{E}_{0i} e^{-j\underline{k}_i \cdot \underline{r}} \underline{a}_y$ $\quad \hat{\underline{E}}_r = \hat{E}_{0r} e^{-j\underline{k}_r \cdot \underline{r}} \underline{a}_y$ $\quad \hat{\underline{E}}_t = \hat{E}_{0t} e^{-j\underline{k}_t \cdot \underline{r}} \underline{a}_y$

$\hat{\underline{H}}_i = \dfrac{\hat{E}_{0i} e^{-j\underline{k}_i \cdot \underline{r}}}{\eta_1} \left(\underline{a}_{ni} \times \underline{a}_y \right)$ $\quad \hat{\underline{H}}_r = \dfrac{\hat{E}_{0r} e^{-j\underline{k}_i \cdot \underline{r}}}{\eta_1} \left(\underline{a}_{nr} \times \underline{a}_y \right)$ $\quad \hat{\underline{H}}_t = \dfrac{\hat{E}_{0t} e^{-j\underline{k}_i \cdot \underline{r}}}{\eta_2} \left(\underline{a}_{nt} \times \underline{a}_y \right)$

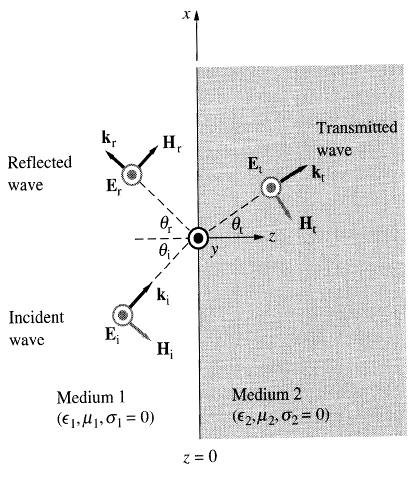
$\underline{a}_{ni} = \sin\theta_i \underline{a}_x + \cos\theta_i \underline{a}_z$ $\quad \underline{a}_{nr} = \sin\theta_r \underline{a}_x - \cos\theta_r \underline{a}_z$

$\underline{k}_i = k_i \sin\theta_i \underline{a}_x + k_i \cos\theta_i \underline{a}_z$ $\quad \underline{k}_r = k_r \sin\theta_r \underline{a}_x - k_r \cos\theta_r \underline{a}_z$ $\quad \underline{k}_t = k_t \sin\theta_t \underline{a}_x + k_t \cos\theta_t \underline{a}_z$

$\underline{r} = x\underline{a}_x + y\underline{a}_y + z\underline{a}_z$

$$\begin{cases} \hat{\underline{E}}_i = \hat{E}_{0i} e^{-j(k_i \sin\theta_i x + k_i \cos\theta_i z)} \underline{a}_y \\ \hat{\underline{E}}_r = \hat{E}_{0r} e^{-j(k_r \sin\theta_r x - k_r \cos\theta_r z)} \underline{a}_y \\ \hat{\underline{E}}_t = \hat{E}_{0t} e^{-j(k_t \sin\theta_t x + k_t \cos\theta_t z)} \underline{a}_y \end{cases}$$

$$\begin{cases} \hat{\underline{H}}_i = \dfrac{\hat{E}_{0i}}{\eta_1} \left(-\cos\theta_i \underline{a}_x + \sin\theta_i \underline{a}_z \right) e^{-j(k_i \sin\theta_i x + k_i \cos\theta_i z)} \\ \hat{\underline{H}}_r = \dfrac{\hat{E}_{0r}}{\eta_1} \left(\cos\theta_r \underline{a}_x + \sin\theta_r \underline{a}_z \right) e^{-j(k_r \sin\theta_r x - k_r \cos\theta_r z)} \\ \hat{\underline{H}}_t = \dfrac{\hat{E}_{0t}}{\eta_2} \left(-\cos\theta_t \underline{a}_x + \sin\theta_t \underline{a}_z \right) e^{-j(k_t \sin\theta_t x + k_t \cos\theta_t z)} \end{cases}$$



$\hat{\underline{E}}, \hat{\underline{H}}$ tangential component continues at $z = 0$

$$\Rightarrow \quad \begin{cases} \underline{a}_y \cdot \hat{\underline{E}}_i \big|_{z=0} + \underline{a}_y \cdot \hat{\underline{E}}_r \big|_{z=0} = \underline{a}_y \cdot \hat{\underline{E}}_t \big|_{z=0} \\ \underline{a}_x \cdot \hat{\underline{H}}_i \big|_{z=0} + \underline{a}_x \cdot \hat{\underline{H}}_r \big|_{z=0} = \underline{a}_x \cdot \hat{\underline{H}}_t \big|_{z=0} \end{cases}$$

$$\Rightarrow \begin{cases} \hat{E}_{0i}e^{-jk_i \sin\theta_i x} + \hat{E}_{0r}e^{-j(-k_r \cos\theta_r z)} = \hat{E}_{0t}e^{-jk_t \sin\theta_t x} \\ -\dfrac{\hat{E}_{0i}}{\eta_1}\cos\theta_i e^{-jk_i \sin\theta_i x} + \dfrac{\hat{E}_{0r}}{\eta_1}\cos\theta_r e^{-j(-k_r \cos\theta_r z)} = -\dfrac{\hat{E}_{0t}}{\eta_2}\cos\theta_t e^{-jk_t \sin\theta_t x} \end{cases}$$

$$\Rightarrow \begin{cases} \hat{E}_{0i} + \hat{E}_{0r} = \hat{E}_{0t}, \qquad k_i \sin\theta_i x = k_i \sin\theta_r x = k_t \sin\theta_t x \\ -\dfrac{\hat{E}_{0i}}{\eta_1}\cos\theta_i + \dfrac{\hat{E}_{0r}}{\eta_1}\cos\theta_r = -\dfrac{\hat{E}_{0t}}{\eta_2}\cos\theta_t \end{cases}$$

$$\hat{E}_{0i} + \hat{E}_{0r} = \hat{E}_{0t} \qquad \Rightarrow \qquad 1 + \frac{\hat{E}_{0r}}{\hat{E}_{0i}} = \frac{\hat{E}_{0t}}{\hat{E}_{0i}}$$

$$-\frac{\hat{E}_{0i}}{\eta_1}\cos\theta_i + \frac{\hat{E}_{0r}}{\eta_1}\cos\theta_r = -\frac{\hat{E}_{0t}}{\eta_2}\cos\theta_t$$

$$\Rightarrow \quad -\sqrt{\varepsilon_{1r}}\,\hat{E}_{0i}\cos\theta_i + \sqrt{\varepsilon_{1r}}\,\hat{E}_{0r}\cos\theta_r = -\sqrt{\varepsilon_{2r}}\,\hat{E}_{0t}\cos\theta_t$$

$$\Rightarrow \quad -n_1\hat{E}_{0i}\cos\theta_i + n_1\hat{E}_{0r}\cos\theta_r = -n_2\hat{E}_{0t}\cos\theta_t$$

$$\Rightarrow \quad -n_1\cos\theta_i + n_1\frac{\hat{E}_{0r}}{\hat{E}_{0i}}\cos\theta_r = -n_2\frac{\hat{E}_{0t}}{\hat{E}_{0i}}\cos\theta_t = -n_2\left(1 + \frac{\hat{E}_{0r}}{\hat{E}_{0i}}\right)\cos\theta_t$$

$$\Rightarrow \quad \frac{\hat{E}_{0r}}{\hat{E}_{0i}} = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}$$

amplitude reflection coefficient

$$\boxed{r_\perp \equiv \frac{\hat{E}_{0r}}{\hat{E}_{0i}} = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}}$$

or

$$r_\perp = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} = \frac{\cos\theta_i - (n_2/n_1)\cos\theta_t}{\cos\theta_i + (n_2/n_1)\cos\theta_t} = \frac{\cos\theta_i - (n_2/n_1)\sqrt{1-\sin^2\theta_t}}{\cos\theta_i + (n_2/n_1)\sqrt{1-\sin^2\theta_t}}$$

$$= \frac{\cos\theta_i - (n_2/n_1)\sqrt{1-(n_1^2/n_2^2)\sin^2\theta_i}}{\cos\theta_i + (n_2/n_1)\sqrt{1-(n_1^2/n_2^2)\sin^2\theta_i}} = \frac{\cos\theta_i - \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}$$

$$\Rightarrow \quad \boxed{r_\perp = \frac{\cos\theta_i - \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}}$$

or

$$r_\perp = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} = \frac{(n_2\sin\theta_t/\sin\theta_i)\cos\theta_i - n_2\cos\theta_t}{(n_2\sin\theta_t/\sin\theta_i)\cos\theta_i + n_2\cos\theta_t}$$

$$= \frac{\sin\theta_t\cos\theta_i - \sin\theta_i\cos\theta_t}{\sin\theta_t\cos\theta_i + \sin\theta_i\cos\theta_t} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$\Rightarrow \quad \boxed{r_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}}$$

amplitude transmission coefficient

$$t_{\perp} \equiv \frac{\hat{E}_{0t}}{\hat{E}_{0i}} = 1 + \frac{\hat{E}_{0r}}{\hat{E}_{0i}} = 1 + r_{\perp} \quad \Rightarrow \quad \boxed{t_{\perp} = 1 + r_{\perp}}$$

$$t_{\perp} = 1 + r_{\perp} = 1 + \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} = \frac{2 n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

$$\Rightarrow \boxed{t_{\perp} = \frac{2 n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}}$$

or

$$t_{\perp} = 1 + r_{\perp} = 1 + \frac{\cos\theta_i - \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}} = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}$$

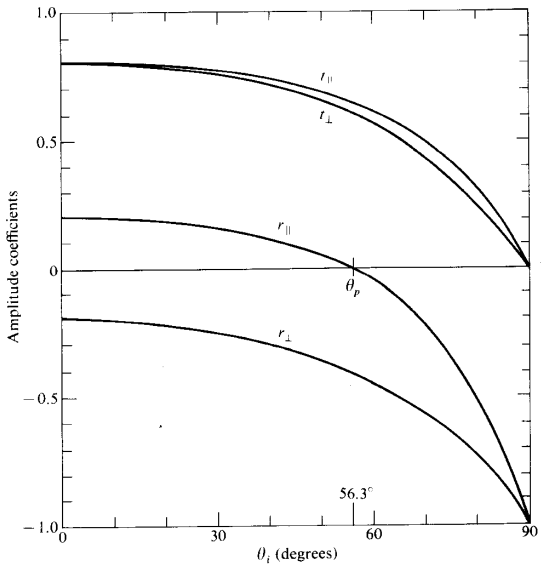
$$\Rightarrow \boxed{t_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}}$$

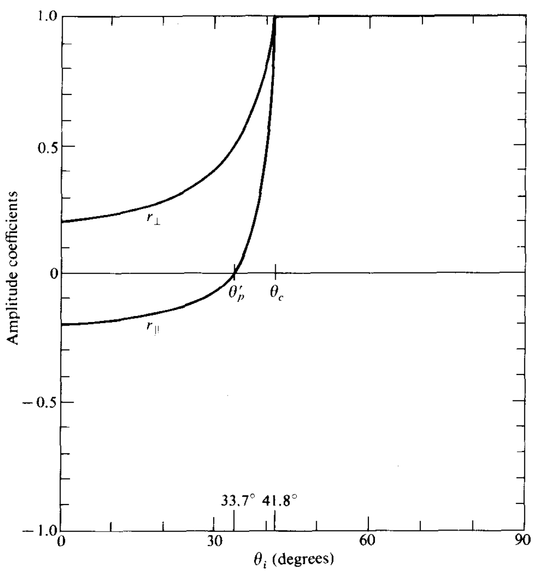
or

$$t_{\perp} = 1 + r_{\perp} = 1 - \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = 1 - \frac{\sin\theta_i \cos\theta_t - \sin\theta_t \cos\theta_i}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t} = \frac{2\sin\theta_t \cos\theta_i}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t}$$

$$= \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)}$$

$$\Rightarrow \boxed{t_{\perp} = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)}}$$





Considering the power flowing across the interface

Incident wave :

Time-average Poynting vector

$$\langle \underline{S}_i \rangle = \frac{1}{2}\mathrm{Re}\left\{\underline{\hat{E}}_i \times \underline{\hat{H}}_i^*\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left[\hat{E}_{0i} e^{-j(k_i \sin\theta_i x + k_i \cos\theta_i z)}\,\underline{a}_y\right]\times\left[\frac{\hat{E}_{0i}^*}{\eta_1}\left(-\cos\theta_i\,\underline{a}_x + \sin\theta_i\,\underline{a}_z\right)e^{j(k_i \sin\theta_i x + k_i \cos\theta_i z)}\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{\left|\hat{E}_{0i}\right|^2}{\eta_1}\cos\theta_i\,\underline{a}_z + \frac{\left|\hat{E}_{0i}\right|^2}{\eta_1}\sin\theta_i\,\underline{a}_x\right\} = \frac{\left|\hat{E}_{0i}\right|^2}{2\eta_1}\left\{\sin\theta_i\,\underline{a}_x + \cos\theta_i\,\underline{a}_z\right\}$$

$$\langle \underline{S}_i \rangle\cdot\underline{a}_z = \left\langle\left|\underline{S}_i\right|\right\rangle\cos\theta_i = \frac{\left|\hat{E}_{i0}\right|^2}{2\eta_1}\cos\theta_i$$

reflected wave :

$$\langle \underline{S}_r \rangle = \frac{1}{2}\mathrm{Re}\left\{\underline{\hat{E}}_r \times \underline{\hat{H}}_r^*\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left[\hat{E}_{0r} e^{-j(k_r \sin\theta_r x - k_r \cos\theta_r z)}\,\underline{a}_y\right]\times\left[\frac{\hat{E}_{0r}^*}{\eta_1}\left(\cos\theta_r\,\underline{a}_x + \sin\theta_r\,\underline{a}_z\right)e^{j(k_r \sin\theta_r x - k_r \cos\theta_r z)}\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{-\frac{\left|\hat{E}_{0r}\right|^2}{\eta_1}\cos\theta_r\,\underline{a}_z + \frac{\left|\hat{E}_{0r}\right|^2}{\eta_1}\sin\theta_r\,\underline{a}_x\right\} = \frac{\left|\hat{E}_{0r}\right|^2}{2\eta_1}\left\{\sin\theta_r\,\underline{a}_x - \cos\theta_r\,\underline{a}_z\right\}$$

$$\langle \underline{S}_r \rangle\cdot\left(-\underline{a}_z\right) = \left\langle\left|\underline{S}_r\right|\right\rangle\cos\theta_r = \frac{\left|\hat{E}_{r0}\right|^2}{2\eta_1}\cos\theta_r$$

refracted wave:

$$\langle \underline{S}_t \rangle = \frac{1}{2}\mathrm{Re}\left\{\underline{\hat{E}}_t \times \underline{\hat{H}}_t^*\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left[\hat{E}_{0t} e^{-j(k_t \sin\theta_t x + k_t \cos\theta_t z)}\,\underline{a}_y\right]\times\left[\frac{\hat{E}_{0t}^*}{\eta_2}\left(-\cos\theta_t\,\underline{a}_x + \sin\theta_t\,\underline{a}_z\right)e^{j(k_t \sin\theta_t x + k_t \cos\theta_t z)}\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{\left|\hat{E}_{0t}\right|^2}{\eta_2}\cos\theta_t\,\underline{a}_z + \frac{\left|\hat{E}_{0t}\right|^2}{\eta_2}\sin\theta_t\,\underline{a}_x\right\} = \frac{\left|\hat{E}_{0t}\right|^2}{2\eta_2}\left\{\sin\theta_y\,\underline{a}_x + \cos\theta_t\,\underline{a}_z\right\}$$

$$\langle \underline{S}_t \rangle\cdot\underline{a}_z = \left\langle\left|\underline{S}_t\right|\right\rangle\cos\theta_t = \frac{\left|\hat{E}_{t0}\right|^2}{2\eta_t}\cos\theta_t$$

reflectance

$$R_{\perp} = \frac{\langle|\underline{S}_r|\rangle\cos\theta_r}{\langle|\underline{S}_i|\rangle\cos\theta_i} = \frac{\dfrac{|\hat{E}_{r0}|^2}{2\eta_1}\cos\theta_r}{\dfrac{|\hat{E}_{i0}|^2}{2\eta_1}\cos\theta_i} = \frac{|\hat{E}_{r0}|^2}{|\hat{E}_{i0}|^2} = r_{\perp}^2$$

$$\therefore \quad \boxed{R_{\perp} = \frac{|\hat{E}_{r0}|^2}{|\hat{E}_{i0}|^2} = r_{\perp}^2}$$

transmittance

$$T_{\perp} = \frac{\langle|\underline{S}_t|\rangle\cos\theta_t}{\langle|\underline{S}_i|\rangle\cos\theta_i} = \frac{\dfrac{|\hat{E}_{t0}|^2}{2\eta_2}\cos\theta_t}{\dfrac{|\hat{E}_{i0}|^2}{2\eta_1}\cos\theta_i} = \frac{\eta_1|\hat{E}_{t0}|^2\cos\theta_t}{\eta_2|\hat{E}_{i0}|^2\cos\theta_i} = \frac{n_2|\hat{E}_{t0}|^2\cos\theta_t}{n_1|\hat{E}_{i0}|^2\cos\theta_i} = \left(\frac{n_2}{n_1}\right)\left(\frac{\cos\theta_t}{\cos\theta_i}\right)t_{\perp}^2$$

$$\therefore \quad \boxed{T_{\perp} = \frac{n_2|\hat{E}_{t0}|^2\cos\theta_t}{n_1|\hat{E}_{i0}|^2\cos\theta_i} = \left(\frac{n_2}{n_1}\right)\left(\frac{\cos\theta_t}{\cos\theta_i}\right)t_{\perp}^2}$$

Conservation of power

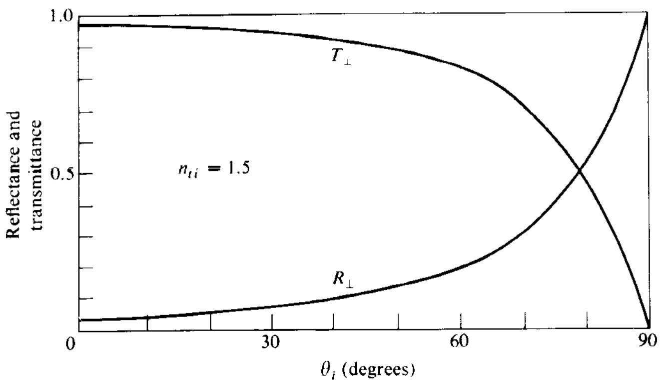
$$\langle|\underline{S}_i|\rangle\cos\theta_i = \langle|\underline{S}_r|\rangle\cos\theta_r + \langle|\underline{S}_t|\rangle\cos\theta_t$$

$$\Rightarrow \frac{|\hat{E}_{i0}|^2}{2\eta_1}\cos\theta_i = \frac{|\hat{E}_{r0}|^2}{2\eta_1}\cos\theta_r + \frac{|\hat{E}_{t0}|^2}{2\eta_2}\cos\theta_t \quad \Rightarrow \quad \frac{\cos\theta_i}{\cos\theta_r} = \frac{|\hat{E}_{r0}|^2}{|\hat{E}_{i0}|^2} + \frac{\eta_1|\hat{E}_{t0}|^2}{\eta_2|\hat{E}_{i0}|^2}\frac{\cos\theta_t}{\cos\theta_r}$$

$$\Rightarrow 1 = \frac{|\hat{E}_{r0}|^2}{|\hat{E}_{i0}|^2} + \frac{\eta_1|\hat{E}_{t0}|^2}{\eta_2|\hat{E}_{i0}|^2}\frac{\cos\theta_t}{\cos\theta_i} \quad \Rightarrow \quad \frac{|\hat{E}_{r0}|^2}{|\hat{E}_{i0}|^2} = 1 - \frac{\eta_1|\hat{E}_{t0}|^2\cos\theta_t}{\eta_2|\hat{E}_{i0}|^2\cos\theta_i}$$

$$\Rightarrow \quad R_{\perp} = 1 - T_{\perp}$$

$$\therefore \quad \boxed{R_{\perp} + T_{\perp} = 1}$$



(a)

Normal incidence on a dielectric boundary

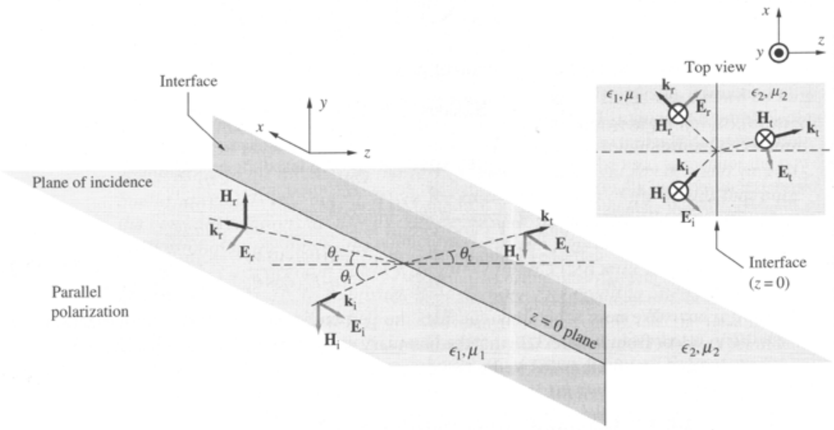
$$\theta_i = \theta_t = 0$$

amplitude reflection coefficient $r_{\perp} \equiv \frac{\hat{E}_{0r}}{\hat{E}_{0i}} = \frac{n_1 - n_2}{n_2 + n_1}$

amplitude transmission coefficient $t_{\perp} = \frac{\hat{E}_{0t}}{\hat{E}_{0i}} = \frac{2n_1}{n_1 + n_2}$

- **Parallel polarization**

  Oblique incidence at a dielectric boundary



Incident wave	Reflected wave	Refracted wave

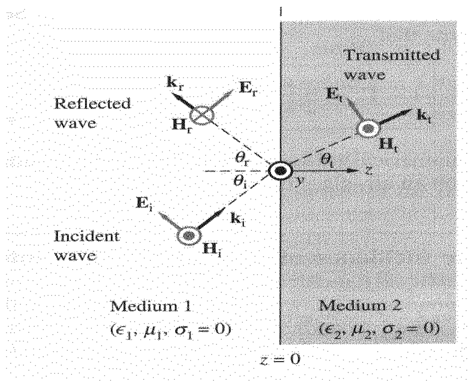
$$\underline{\hat{E}}_i = \hat{E}_{0i} e^{-j\underline{k}_i \cdot \underline{r}} \left(\underline{a}_y \times \underline{a}_{ni} \right) \qquad \underline{\hat{E}}_r = \hat{E}_{0r} e^{-j\underline{k}_r \cdot \underline{r}} \left(\underline{a}_y \times \underline{a}_{nr} \right) \qquad \underline{\hat{E}}_t = \hat{E}_{0t} e^{-j\underline{k}_t \cdot \underline{r}} \left(\underline{a}_y \times \underline{a}_{nt} \right)$$

$$\underline{\hat{H}}_i = \frac{\hat{E}_{0i} e^{-j\underline{k}_i \cdot \underline{r}}}{\eta_1} \underline{a}_y \qquad \underline{\hat{H}}_r = \frac{\hat{E}_{0r} e^{-j\underline{k}_r \cdot \underline{r}}}{\eta_1} \underline{a}_y \qquad \underline{\hat{H}}_t = \frac{\hat{E}_{0t} e^{-j\underline{k}_t \cdot \underline{r}}}{\eta_2} \underline{a}_y$$

$$\underline{a}_{ni} = \sin\theta_i \underline{a}_x + \cos\theta_i \underline{a}_z \qquad \underline{a}_{nr} = \sin\theta_r \underline{a}_x - \cos\theta_r \underline{a}_z$$

$$\underline{k}_i = k_i \sin\theta_i \underline{a}_x + k_i \cos\theta_i \underline{a}_z \qquad \underline{k}_r = k_r \sin\theta_r \underline{a}_x - k_r \cos\theta_r \underline{a}_z \qquad \underline{k}_t = k_t \sin\theta_t \underline{a}_x + k_t \cos\theta_t \underline{a}_z$$

$$\Rightarrow \begin{cases} \underline{\hat{E}}_i = \hat{E}_{0i} \left(\cos\theta_i \underline{a}_x - \sin\theta_i \underline{a}_z \right) e^{-j(k_i \sin\theta_i x + k_i \cos\theta_i z)} \\ \underline{\hat{E}}_r = \hat{E}_{0r} \left(\cos\theta_r \underline{a}_x + \sin\theta_r \underline{a}_z \right) e^{-j(k_r \sin\theta_r x - k_r \cos\theta_r z)} \\ \underline{\hat{E}}_t = \hat{E}_{0t} \left(\cos\theta_t \underline{a}_x - \sin\theta_t \underline{a}_z \right) e^{-j(k_t \sin\theta_t x + k_t \cos\theta_t z)} \end{cases}$$



$$\Rightarrow \begin{cases} \underline{\hat{H}}_i = \frac{\hat{E}_{0i}}{\eta_1} e^{-j(k_i \sin\theta_i x + k_i \cos\theta_i z)} \underline{a}_y \\ \underline{\hat{H}}_r = -\frac{\hat{E}_{0r}}{\eta_1} e^{-j(k_r \sin\theta_r x - k_r \cos\theta_r z)} \underline{a}_y \\ \underline{\hat{H}}_t = \frac{\hat{E}_{0t}}{\eta_2} e^{-j(k_t \sin\theta_t x + k_t \cos\theta_t z)} \underline{a}_y \end{cases}$$

Apply B.C., at $z = 0$,

$$\Rightarrow \quad \begin{cases} \underline{a}_x \cdot \underline{\hat{E}}_i \big|_{z=0} + \underline{a}_x \cdot \underline{\hat{E}}_r \big|_{z=0} = \underline{a}_x \cdot \underline{\hat{E}}_t \big|_{z=0} \\ \underline{a}_y \cdot \underline{\hat{H}}_i \big|_{z=0} + \underline{a}_y \cdot \underline{\hat{H}}_r \big|_{z=0} = \underline{a}_y \cdot \underline{\hat{H}}_t \big|_{z=0} \end{cases}$$

$$\Rightarrow \quad \begin{cases} \left(\hat{E}_{0i} + \hat{E}_{0r}\right)\cos\theta_i = \hat{E}_{0t}\cos\theta_t \\ \dfrac{\hat{E}_{0i}}{\eta_1} - \dfrac{\hat{E}_{0r}}{\eta_1} = \dfrac{\hat{E}_{0t}}{\eta_2} \end{cases}$$

$$\left(\hat{E}_{0i} + \hat{E}_{0r}\right)\cos\theta_i = \hat{E}_{0t}\cos\theta_t \quad \Rightarrow \quad \frac{\hat{E}_{0t}}{\hat{E}_{i0}} = \left(1 + \frac{\hat{E}_{0r}}{\hat{E}_{0i}}\right)\frac{\cos\theta_i}{\cos\theta_t}$$

$$\frac{1}{\eta_1} - \frac{1}{\eta_1}\left(\frac{\hat{E}_{0r}}{\hat{E}_{0i}}\right) = \frac{1}{\eta_2}\left(\frac{\hat{E}_{0t}}{\hat{E}_{0i}}\right) \quad \Rightarrow \quad n_1 - n_1\left(\frac{\hat{E}_{0r}}{\hat{E}_{0i}}\right) = n_2\left(\frac{\hat{E}_{0t}}{\hat{E}_{0i}}\right)$$

$$\Rightarrow \quad n_1 - n_1\left(\frac{\hat{E}_{0r}}{\hat{E}_{0i}}\right) = n_2\left(1 + \frac{\hat{E}_{0r}}{\hat{E}_{0i}}\right)\frac{\cos\theta_i}{\cos\theta_t}$$

$$\Rightarrow \quad \frac{\hat{E}_{0r}}{\hat{E}_{0i}} = \frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}$$

amplitude reflection coefficient

$$\boxed{r_{//} \equiv \frac{\hat{E}_{0r}}{\hat{E}_{0i}} = \frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}}$$

or

$$r_{//} = \frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i} = \frac{\cos\theta_t - (n_2/n_1)\cos\theta_i}{\cos\theta_t + (n_2/n_1)\cos\theta_i} = \frac{\sqrt{1-\sin^2\theta_t} - (n_2/n_1)\cos\theta_i}{\sqrt{1-\sin^2\theta_t} + (n_2/n_1)\cos\theta_i}$$

$$= \frac{\sqrt{1-(n_1/n_2)^2\sin^2\theta_i} - (n_2/n_1)\cos\theta_i}{\sqrt{1-(n_1/n_2)^2\sin^2\theta_i} + (n_2/n_1)\cos\theta_i} = \frac{\sqrt{(n_2/n_1)^2 - \sin^2\theta_i} - (n_2/n_1)^2\cos\theta_i}{\sqrt{(n_2/n_1)^2 - \sin^2\theta_i} + (n_2/n_1)^2\cos\theta_i}$$

$$\Rightarrow \quad \boxed{r_{//} = \frac{\sqrt{(n_2/n_1)^2 - \sin^2\theta_i} - (n_2/n_1)^2\cos\theta_i}{\sqrt{(n_2/n_1)^2 - \sin^2\theta_i} + (n_2/n_1)^2\cos\theta_i}}$$

or

$$\boxed{r_{//} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}}$$

amplitude transmission coefficient

$$t_{//} \equiv \frac{\hat{E}_{0t}}{\hat{E}_{0i}} = \left(1 + \frac{\hat{E}_{0r}}{\hat{E}_{0i}}\right)\frac{\cos\theta_i}{\cos\theta_t} = \left(1 + r_{//}\right)\frac{\cos\theta_i}{\cos\theta_t}$$

$$\Rightarrow \quad t_{//} = (1+r_{//})\frac{\cos\theta_i}{\cos\theta_t} = \left(1+\frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}\right)\frac{\cos\theta_i}{\cos\theta_t} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}$$

$$\Rightarrow \quad \boxed{t_{//} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}}$$

or

$$t_{//} = (1+r_{//})\frac{\cos\theta_i}{\cos\theta_t} = \left[1+\frac{\sqrt{(n_2/n_1)^2-\sin^2\theta_i} - (n_2/n_1)^2\cos\theta_i}{\sqrt{(n_2/n_1)^2-\sin^2\theta_i} + (n_2/n_1)^2\cos\theta_i}\right]\frac{\cos\theta_i}{\sqrt{1-(n_1/n_2)^2\sin^2\theta_i}}$$

$$= \left[\frac{2\sqrt{(n_2/n_1)^2-\sin^2\theta_i}}{\sqrt{(n_2/n_1)^2-\sin^2\theta_i} + (n_2/n_1)^2\cos\theta_i}\right]\frac{(n_2/n_1)\cos\theta_i}{\sqrt{(n_2/n_1)^2-\sin^2\theta_i}}$$

$$= \frac{2(n_2/n_1)\cos\theta_i}{\sqrt{(n_2/n_1)^2-\sin^2\theta_i} + (n_2/n_1)^2\cos\theta_i}$$

$$\Rightarrow \quad \boxed{t_{//} = \frac{2(n_2/n_1)\cos\theta_i}{\sqrt{(n_2/n_1)^2-\sin^2\theta_i} + (n_2/n_1)^2\cos\theta_i}}$$

or

$$\boxed{t_{//} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)}}$$

$$n_1 - n_1\left(\frac{\hat{E}_{0r}}{\hat{E}_{0i}}\right) = n_2\left(\frac{\hat{E}_{0t}}{\hat{E}_{0i}}\right) \quad \Rightarrow \quad 1-\left(\frac{\hat{E}_{0r}}{\hat{E}_{0i}}\right) = \frac{n_2}{n_1}\left(\frac{\hat{E}_{0t}}{\hat{E}_{0i}}\right) \quad \Rightarrow \quad 1-r_{//} = \left(\frac{n_2}{n_1}\right)t_{//}$$

$$\Rightarrow \quad \boxed{1 = r_{//} + \left(\frac{n_2}{n_1}\right)t_{//}}$$

Incident wave :

$$\langle \underline{S}_i \rangle = \frac{1}{2}\mathrm{Re}\left\{\underline{\hat{E}}_i \times \underline{\hat{H}}_i^*\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left[\hat{E}_{0i}(\cos\theta_i\,\underline{a}_x - \sin\theta_i\,\underline{a}_z)e^{-j(k_i\sin\theta_i x + k_i\cos\theta_i z)}\right]\times\left[\frac{\hat{E}_{0i}^*}{\eta_1}e^{j(k_i\sin\theta_i x + k_i\cos\theta_i z)}\,\underline{a}_y\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{\left|\hat{E}_{0i}\right|^2}{\eta_1}\cos\theta_i\,\underline{a}_z + \frac{\left|\hat{E}_{0i}\right|^2}{\eta_1}\sin\theta_i\,\underline{a}_x\right\} = \frac{\left|\hat{E}_{0i}\right|^2}{2\eta_1}\left\{\sin\theta_i\,\underline{a}_x + \cos\theta_i\,\underline{a}_z\right\}$$

$$\langle \underline{S}_i \rangle \cdot \underline{a}_z = \langle |\underline{S}_i| \rangle \cos\theta_i = \frac{\left|\hat{E}_{i0}\right|^2}{2\eta_1}\cos\theta_i$$

reflected wave :

$$\langle \underline{S}_r \rangle = \frac{1}{2}\mathrm{Re}\left\{\underline{\hat{E}}_r \times \underline{\hat{H}}_r^*\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left[\underline{\hat{E}}_{0r}\left(\cos\theta_r \underline{a}_x + \sin\theta_r \underline{a}_z\right)e^{-j(k_r \sin\theta_r x - k_r \cos\theta_r z)}\right]\times\left[-\frac{\hat{E}_{0r}^*}{\eta_1}e^{j(k_r \sin\theta_r x - k_r\cos\theta_r z)}\underline{a}_y\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{-\frac{\left|\hat{E}_{0r}\right|^2}{\eta_1}\cos\theta_r \underline{a}_z + \frac{\left|\hat{E}_{0r}\right|^2}{\eta_1}\sin\theta_r \underline{a}_x\right\} = \frac{\left|\hat{E}_{0r}\right|^2}{2\eta_1}\left\{\sin\theta_r \underline{a}_x - \cos\theta_r \underline{a}_z\right\}$$

$$\langle \underline{S}_r \rangle \cdot (-\underline{a}_z) = \langle |\underline{S}_r| \rangle \cos\theta_r = \frac{\left|\hat{E}_{r0}\right|^2}{2\eta_1}\cos\theta_r$$

refracted wave:

$$\langle \underline{S}_t \rangle = \frac{1}{2}\mathrm{Re}\left\{\underline{\hat{E}}_t \times \underline{\hat{H}}_t^*\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\left[\underline{\hat{E}}_{0t}\left(\cos\theta_t \underline{a}_x - \sin\theta_t \underline{a}_z\right)e^{-j(k_t \sin\theta_t x + k_t \cos\theta_t z)}\right]\times\left[\frac{\hat{E}_{0t}^*}{\eta_2}e^{-j(k_t \sin\theta_t x + k_t\cos\theta_t z)}\underline{a}_y\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{\left|\hat{E}_{0t}\right|^2}{\eta_2}\cos\theta_t \underline{a}_z + \frac{\left|\hat{E}_{0t}\right|^2}{\eta_2}\sin\theta_t \underline{a}_x\right\} = \frac{\left|\hat{E}_{0t}\right|^2}{2\eta_2}\left\{\sin\theta_y \underline{a}_x + \cos\theta_t \underline{a}_z\right\}$$

$$\langle \underline{S}_t \rangle \cdot \underline{a}_z = \langle |\underline{S}_t| \rangle \cos\theta_t = \frac{\left|\hat{E}_{t0}\right|^2}{2\eta_t}\cos\theta_t$$

reflectance

$$R_{//} = \frac{\langle |\underline{S}_r| \rangle \cos\theta_r}{\langle |\underline{S}_i| \rangle \cos\theta_i} = \frac{\frac{\left|\hat{E}_{r0}\right|^2}{2\eta_1}\cos\theta_r}{\frac{\left|\hat{E}_{i0}\right|^2}{2\eta_1}\cos\theta_i} = \frac{\left|\hat{E}_{r0}\right|^2}{\left|\hat{E}_{i0}\right|^2} = r_{//}^2$$

$$\therefore \quad \boxed{R_{//} = \frac{\left|\hat{E}_{r0}\right|^2}{\left|\hat{E}_{i0}\right|^2} = r_{//}^2}$$

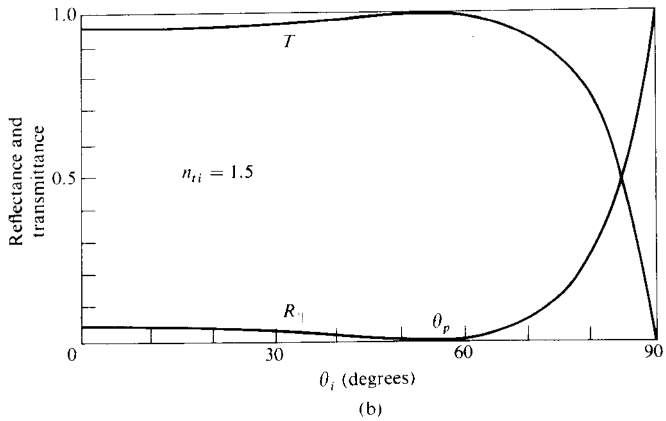
transmittance

$$T_{//} = \frac{\langle |\underline{S}_t| \rangle \cos\theta_t}{\langle |\underline{S}_i| \rangle \cos\theta_i} = \frac{\frac{\left|\hat{E}_{t0}\right|^2}{2\eta_2}\cos\theta_t}{\frac{\left|\hat{E}_{i0}\right|^2}{2\eta_1}\cos\theta_i} = \frac{\eta_1\left|\hat{E}_{t0}\right|^2\cos\theta_t}{\eta_2\left|\hat{E}_{i0}\right|^2\cos\theta_i} = \frac{n_2\left|\hat{E}_{t0}\right|^2\cos\theta_t}{n_1\left|\hat{E}_{i0}\right|^2\cos\theta_i} = \left(\frac{n_2}{n_1}\right)\left(\frac{\cos\theta_t}{\cos\theta_i}\right)t_{//}^2$$

$$\therefore \quad \boxed{T_{//} = \left(\frac{n_2}{n_1}\right)\left(\frac{\cos\theta_t}{\cos\theta_i}\right) t_{//}^2}$$

Conservation of power

$$\therefore \quad \boxed{R_{//} + T_{//} = 1}$$



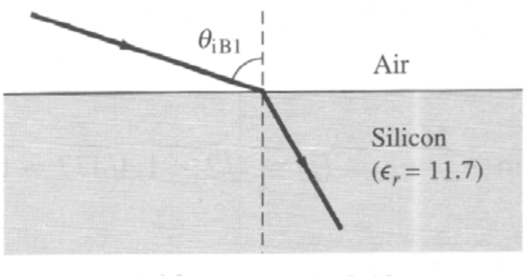
Brewster's angle

If $r_{//} = 0$ (total transmission), then $n_1 \cos\theta_t - n_2 \cos\theta_i = 0$

$$\Rightarrow \quad \cos\theta_t = (n_2/n_1)\cos\theta_i$$

By Snell's law of refraction, $n_1 \sin\theta_i = n_2 \sin\theta_t$

$$\Rightarrow \quad \sin\theta_t = (n_1/n_2)\sin\theta_i$$



$$\therefore \quad \sin^2\theta_t + \cos^2\theta_t$$

$$= \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i + \left(\frac{n_2}{n_1}\right)^2 \cos^2\theta_i$$

$$= 1 = \sin^2\theta_i + \cos^2\theta_i$$

$$\left[1 - \left(\frac{n_1}{n_2}\right)^2\right]\sin^2\theta_i = \left[\left(\frac{n_2}{n_1}\right)^2 - 1\right]\cos^2\theta_i$$

$$\tan^2\theta_i = \left(\frac{n_2}{n_1}\right)^2 \quad \Rightarrow \quad \theta_{iB} = \tan^{-1}\frac{n_2}{n_1}$$

for an air-silicon ($\varepsilon_r = 11.7$) interface

$$\theta_{iB} = \tan^{-1}\sqrt{\varepsilon_r} = \tan^{-1}\sqrt{11.7} \approx 73.7^0$$

Total internal reflection

$$\theta_t = 90^o \quad \Rightarrow \quad \sin\theta_{ic} = \frac{n_2}{n_1}$$

$\therefore$ **critical angle** $\theta_{ic} = \sin^{-1}\dfrac{n_2}{n_1}$