Waveguide (Clayton, CH8)

assuming perfectly conducting plate

$$\begin{cases} E_{2t} = 0 \\ B_{2n} = 0 \end{cases} \Rightarrow \begin{cases} E_{\text{tangential}} = 0 \\ H_{\text{normal}} = 0 \end{cases}$$

$$\nabla \times \underline{\hat{E}} = -j\omega\mu \underline{\hat{H}} \qquad \nabla^2 \underline{\hat{E}} + \omega^2 \mu\varepsilon \underline{\hat{E}} = 0, \qquad \nabla^2 \underline{\hat{E}} + k^2 \underline{\hat{E}} = 0$$

$$\nabla \times \underline{\hat{H}} = j\omega\varepsilon \underline{\hat{E}} \qquad \nabla^2 \underline{H} + \omega^2 \mu\varepsilon \underline{\hat{H}} = 0, \qquad \nabla^2 \underline{H} + k^2 \underline{\hat{H}} = 0$$

$$\begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \hat{E}_x & \hat{E}_y & \hat{E}_z \end{vmatrix} = -j\omega\mu\left(\hat{H}_x \underline{a_x} + \hat{H}_y \underline{a_y} + \hat{H}_z \underline{a_z}\right)$$

$$\begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \hat{H}_x & \hat{H}_y & \hat{H}_z \end{vmatrix} = j\omega\varepsilon\left(\hat{E}_x \underline{a_x} + \hat{E}_y \underline{a_y} + \hat{E}_z \underline{a_z}\right)$$

$$\frac{\partial \hat{E}_z}{\partial y} - \frac{\partial \hat{E}_y}{\partial z} = -j\omega\mu \hat{H}_x \qquad \frac{\partial \hat{H}_z}{\partial y} - \frac{\partial \hat{H}_y}{\partial z} = j\omega\varepsilon \hat{E}_x$$

$$\frac{\partial \hat{E}_x}{\partial z} - \frac{\partial \hat{E}_z}{\partial x} = -j\omega\mu \hat{H}_y \qquad \frac{\partial \hat{H}_x}{\partial z} - \frac{\partial \hat{H}_z}{\partial x} = j\omega\varepsilon \hat{E}_y$$

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = -j\omega\mu \hat{H}_z \qquad \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} = j\omega\varepsilon \hat{E}_z$$

Assuming propagation in the z direction, so that all field components vary as $e^{-\gamma z}$, $\hat{\gamma} = \alpha + j\beta$

$$\begin{aligned} \underline{E}(\underline{r},t) &= \underline{E}(x,y,z,t) = Re\left\{\underline{\hat{E}}(x,y) e^{j(\omega t - \hat{\gamma} z)}\right\} \\ \underline{H}(\underline{r},t) &= \underline{H}(x,y,z,t) = Re\left\{\underline{\hat{H}}(x,y) e^{j(\omega t - \hat{\gamma} z)}\right\} \end{aligned} \quad \Rightarrow \quad \frac{\partial}{\partial z} \to -\hat{\gamma}$$

$$Re\left\{e^{j\omega t - \gamma z}\right\} = \begin{cases} e^{-\alpha z} \cos \omega t & \text{evanescent wave} \\ \cos(\omega t - \beta z) & \text{propagating wave} \end{cases}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \underline{\hat{E}} + k^2 \underline{\hat{E}} = 0$$

$$\frac{\partial^2}{\partial z^2} e^{-\hat{\gamma} z} = \hat{\gamma}^2 e^{-\hat{\gamma} z}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \underline{\hat{H}} + k^2 \underline{\hat{H}} = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \underline{\hat{E}} + \left(\underbrace{\hat{\gamma}^2 + k^2}_{h^2}\right) \underline{\hat{E}} = 0 \qquad \nabla_t \underline{\hat{E}} + h^2 \underline{\hat{E}} = 0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\underline{\hat{H}}+\left(\underbrace{\hat{\gamma}^2+k^2}_{h^2}\right)\underline{\hat{H}}=0 \qquad \nabla_t\underline{\hat{H}}+h^2\underline{\hat{H}}=0$$

$$\underline{\hat{E}}=\hat{E}_x\underline{a_x}+\hat{E}_y\underline{a_y}+\hat{E}_z\underline{a_z} \qquad \underline{\hat{H}}=\hat{H}_x\underline{a_x}+\hat{H}_y\underline{a_y}+\hat{H}_z\underline{a_z}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\hat{E}_x+h^2\hat{E}_x=0 \qquad \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\hat{H}_x+h^2\hat{H}_x=0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\hat{E}_y+h^2\hat{E}_y=0 \qquad \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\hat{H}_y+h^2\hat{H}_y=0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\hat{E}_z+h^2\hat{E}_z=0 \qquad \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\hat{H}_z+h^2\hat{H}_z=0$$

$$\frac{\partial\hat{E}_z}{\partial y}+\hat{\gamma}\hat{E}_y=-j\omega\mu\hat{H}_x$$

$$-\hat{\gamma}\hat{E}_x-\frac{\partial\hat{E}_z}{\partial x}=-j\omega\mu\hat{H}_y$$

$$\frac{\partial\hat{E}_y}{\partial x}-\frac{\partial\hat{E}_x}{\partial y}=-j\omega\mu\hat{H}_z$$

$$\frac{\partial\hat{H}_z}{\partial y}+\hat{\gamma}\hat{H}_y=j\omega\varepsilon\hat{E}_x$$

$$-\hat{\gamma}\hat{H}_x-\frac{\partial\hat{H}_z}{\partial x}=j\omega\varepsilon\hat{E}_y$$

$$\frac{\partial\hat{H}_y}{\partial x}-\frac{\partial\hat{H}_x}{\partial y}=j\omega\varepsilon\hat{E}_z$$

$$\Rightarrow$$

$$\hat{E}_x=-\frac{1}{h^2}\left[\hat{\gamma}\frac{\partial\hat{E}_z}{\partial x}+j\omega\mu\frac{\partial\hat{H}_z}{\partial y}\right]$$

$$\hat{E}_y=-\frac{1}{h^2}\left[\hat{\gamma}\frac{\partial\hat{E}_z}{\partial y}-j\omega\mu\frac{\partial\hat{H}_z}{\partial x}\right]$$

$$\hat{H}_x=-\frac{1}{h^2}\left[-j\omega\varepsilon\frac{\partial\hat{E}_z}{\partial y}+\hat{\gamma}\frac{\partial\hat{H}_z}{\partial x}\right]$$

$$\hat{H}_y=-\frac{1}{h^2}\left[j\omega\varepsilon\frac{\partial\hat{E}_z}{\partial x}+\hat{\gamma}\frac{\partial\hat{H}_z}{\partial y}\right]$$

$$h^2=\hat{\gamma}^2+\omega^2\mu\varepsilon$$

Transverse fields are directed perpendicular to the direction of propagation (i.e. $\hat{E}_x,\hat{E}_y,\hat{H}_x,\hat{H}_y$),

whereas longitudinal fields are directed parallel to the direction of propagation (i.e. $\hat{E}_z,\hat{H}_z$)

TM modes, for which $\hat{H}_z=0,\hat{E}_z\neq 0$ TE modes, for which $\hat{E}_z=0,\hat{H}_z\neq 0$

$\hat{H}_z=0,\hat{E}_z=0$ $\rightarrow$ TEM modes

$$\hat{E}_x=-\frac{\hat{\gamma}}{h^2}\frac{\partial\hat{E}_z}{\partial x} \qquad \hat{E}_x=-\frac{j\omega\mu}{h^2}\frac{\partial\hat{H}_z}{\partial y}$$

$$\hat{E}_y=-\frac{\hat{\gamma}}{h^2}\frac{\partial\hat{E}_z}{\partial y} \qquad \hat{E}_y=\frac{j\omega\mu}{h^2}\frac{\partial\hat{H}_z}{\partial x}$$

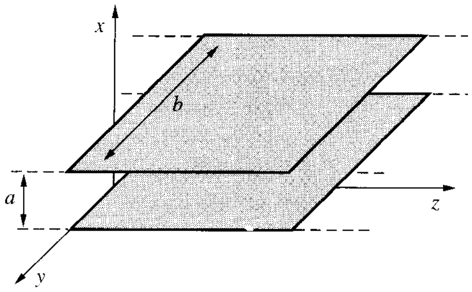
$$\hat{H}_x=\frac{j\omega\varepsilon}{h^2}\frac{\partial\hat{E}_z}{\partial y} \qquad \hat{H}_x=-\frac{\hat{\gamma}}{h^2}\frac{\partial\hat{H}_z}{\partial x}$$

$$\hat{H}_y = -\frac{j\omega\varepsilon}{h^2}\frac{\partial \hat{E}_z}{\partial x} \qquad \hat{H}_y = -\frac{\hat{\gamma}}{h^2}\frac{\partial \hat{H}_z}{\partial y}$$

※ $\hat{H}_z = 0, \hat{E}_z = 0$ seems to render all other field components zero, however, closer inspection if $h^2 = 0$

※ denominators are also zero. $\therefore$TEM is still a possible mode of propagation.

➢ **Parallel-plate waveguide**



assuming two perfectly conducting plates, waves are guided in the z direction

$$E_{\text{tangential}} = 0$$
$$H_{\text{normal}} = 0$$

$$y \quad \rightarrow \quad \pm\infty$$

for simplicity, $\frac{\partial}{\partial y}(\cdots) = \frac{\partial^2}{\partial y^2}(\cdots) = 0$

$$\Rightarrow \quad \frac{\partial^2 \hat{E}_x}{\partial x^2} + h^2 \hat{E}_x = 0 \qquad \frac{\partial^2 \hat{H}_x}{\partial x^2} + h^2 \hat{H}_x = 0$$

$$\frac{\partial^2 \hat{E}_y}{\partial x^2} + h^2 \hat{E}_y = 0 \qquad \frac{\partial^2 \hat{H}_y}{\partial x^2} + h^2 \hat{H}_y = 0$$

$$\frac{\partial^2 \hat{E}_z}{\partial x^2} + h^2 \hat{E}_z = 0 \qquad \frac{\partial^2 \hat{H}_z}{\partial x^2} + h^2 \hat{H}_z = 0$$

$$h^2 = \hat{\gamma}^2 + \omega^2 \mu\varepsilon$$

$$\frac{\partial^2 \hat{E}_z}{\partial x^2} + h^2 \hat{E}_z = 0$$
$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + h^2 \hat{H}_z = 0$$
$$\Rightarrow \quad \hat{E}_z(x) = A\sin(hx) + B\cos(hx)$$
$$\hat{H}_z(x) = A'\sin(hx) + B'\cos(hx)$$

Transverse Electric (TE) waves

$$\hat{E}_z = 0, \hat{H}_z \neq 0 \;\Rightarrow\; \hat{E}_x = \hat{H}_y = 0 \Rightarrow \quad \hat{E}_y = \frac{j\omega\mu}{h^2}\frac{\partial \hat{H}_z}{\partial x}, \quad \hat{H}_x = -\frac{\hat{\gamma}}{h^2}\frac{\partial \hat{H}_z}{\partial x}$$

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + h^2 \hat{H}_z = 0 \quad \Rightarrow \quad \hat{H}_z = X(x)e^{-\gamma z}$$

$$\Rightarrow \quad \frac{\partial^2 X}{\partial x^2} + h^2 X = 0$$

$$X(x) = A' \sin(hx) + B' \cos(hx)$$

B.C. $\begin{cases} \hat{E}_y \big|_{x=0} = 0 \\ \hat{E}_y \big|_{x=a} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial \hat{H}_z}{\partial x}\big|_{x=0} = 0 \\ \frac{\partial \hat{H}_z}{\partial x}\big|_{x=a} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial X}{\partial x}\big|_{x=0} = 0 & (1) \\ \frac{\partial X}{\partial x}\big|_{x=a} = 0 & (2) \end{cases}$

$$\frac{\partial \hat{H}_z}{\partial x} = A' h\, cos(hx) - B' h\, sin(hx)$$

Apply (1) $\rightarrow \quad \frac{\partial X}{\partial x}\big|_{x=0} = A' h = 0 \quad \Rightarrow \quad A' = 0$

$$\Rightarrow \quad X(x) = B' \cos(hx)$$

Apply (2) $\rightarrow \quad \frac{\partial X}{\partial x}\big|_{x=a} = -B' h \sin(ha) = 0 \quad \Rightarrow \quad ha = m\pi$ **for** $m = 1,2,3,\cdots$

$\rightarrow \quad h = \frac{m\pi}{a}$ **for** $m = 1,2,3,\cdots$

$$\Rightarrow \quad X(x) = B' \cos\left(\frac{m\pi}{a} x\right)$$

$$\hat{H}_z(x,z) = B' \cos\left(\frac{m\pi}{a} x\right) e^{-\gamma z}$$

$$\hat{E}_y(x,z) = \frac{j\omega\mu}{h^2} \frac{\partial \hat{H}_z}{\partial x} = -\frac{j\omega\mu a}{m\pi} B' \sin\left(\frac{m\pi}{a} x\right) e^{-\gamma z}$$

$$\hat{H}_x(x,z) = -\frac{\hat{\gamma}}{h^2} \frac{\partial \hat{H}_z}{\partial x} = \frac{\hat{\gamma} a}{m\pi} B' \sin\left(\frac{m\pi}{a} x\right) e^{-\gamma z}$$

$\hat{E}_z = 0$, $\hat{E}_x = 0$, $\hat{H}_y = 0$ (Ramo, Paul)

cutoff frequency

$h^2 = \hat{\gamma}^2 + \omega^2 \mu\varepsilon \quad \Rightarrow \quad \hat{\gamma} = \sqrt{h^2 - \omega^2 \mu\varepsilon}$

$\hat{\gamma} = 0 \quad \rightarrow \quad \omega_{c,m} \sqrt{\mu\varepsilon} = \frac{m\pi}{a} \quad \Rightarrow \quad f_{c,m} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \frac{m\pi}{a} = \frac{m}{2a\sqrt{\mu\varepsilon}} = \frac{m}{2a} v_p$

$\because$ intrinsic phase velocity $v_p = \frac{1}{\sqrt{\mu\varepsilon}} = f_{c,m} \lambda_{c,m} \quad \Rightarrow \quad \lambda_{c,m} = \frac{v_p}{f_{c,m}} = \frac{2a}{m}$

for $f > f_{c,m}$

$$\hat{\gamma} = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2 \mu\varepsilon} = \sqrt{\omega_{c,m}^2 \mu\varepsilon - \omega^2 \mu\varepsilon} = j\underbrace{\sqrt{\omega^2 \mu\varepsilon - \omega_{c,m}^2 \mu\varepsilon}}_{real}$$

$$= j\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_{c,,m}}{f}\right)^2} = j\beta_m$$

$$\lambda_m = \frac{2\pi}{\beta_m} = \frac{\lambda}{\sqrt{1-\left(\frac{f_{c,,m}}{f}\right)^2}} \qquad v_{p,m} = \frac{\omega}{\beta_m} = \frac{v_p}{\sqrt{1-\left(\frac{f_{c,,m}}{f}\right)^2}}$$

for $f < f_{c,m}$

$$\hat{\gamma} = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\varepsilon} = \sqrt{\omega_{c,m}^2\mu\varepsilon - \omega^2\mu\varepsilon} = \omega\sqrt{\mu\varepsilon}\sqrt{\left(\frac{f_{c,,m}}{f}\right)^2 - 1} = \alpha_m$$

characteristic impedance

$$Z_{TE,m} = -\frac{\hat{E}_y}{\hat{H}_x} = \frac{j\omega\mu}{\hat{\gamma}} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1-\left(\frac{f_{c,,m}}{f}\right)^2}} = \frac{\eta}{\sqrt{1-\left(\frac{f_{c,,m}}{f}\right)^2}} \qquad f > f_{c,m} \quad \Rightarrow \quad \text{power flow}$$

$$Z_{TE,m} = \frac{j\omega\mu}{\omega\sqrt{\mu\varepsilon}\sqrt{\left(\frac{f_{c,,m}}{f}\right)^2 - 1}} = j\frac{\eta}{\sqrt{\left(\frac{f_{c,,m}}{f}\right)^2 - 1}} \qquad f < f_{c,m} \quad \Rightarrow \quad \text{no power flow}$$

for loss-free case, time-average power density transmitted

$$\left\langle \underline{S_{av}} \right\rangle = \frac{1}{2} Re\left\{ \underline{\hat{E}} \times \underline{\hat{H}^*} \right\}$$

$$= \frac{1}{2} Re\left\{ \hat{E}_y \underline{a_y} \times \left(\hat{H}_x^* \underline{a_x} + \hat{H}_z^* \underline{a_z} \right) \right\}$$

$$= \frac{1}{2} Re\left\{ \left(\hat{E}_y \cdot \hat{H}_z^* \right) \underline{a_x} - \left(\hat{E}_y \cdot \hat{H}_x^* \right) \underline{a_z} \right\} \qquad \text{(Clayton)}$$

$$= \frac{1}{2} Re\left\{ j\left(-\frac{\omega\mu a}{m\pi} B'^2 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right)\right)\underline{a_x} + \left(j\omega\mu\hat{\gamma}^*\left(\frac{B'a}{m\pi}\right)^2 \sin^2\left(\frac{m\pi}{a}x\right)\cdot\right)\underline{a_z} \right\}$$

$$= \frac{1}{2}\beta_m\omega\mu\left(\frac{B'a}{m\pi}\right)^2 \sin^2\left(\frac{m\pi}{a}x\right)\underline{a_z} \qquad \left[W/m^2\right] \qquad \hat{\gamma}^* = -j\beta_m$$

$$\underline{\hat{S}} = \underline{\hat{E}} \times \underline{\hat{H}}^* = \frac{1}{Z_{TE,m}} \left|\hat{E}_y\right|^2 \underline{a_z} = \frac{\omega\mu\left(\frac{aB'}{m\pi}\right)^2 \sin^2\left(\frac{m\pi}{a}x\right)}{Z_{TE,m}} \underline{a_z} = \frac{\omega\mu\left(\frac{aB'}{m\pi}\right)^2 \sin^2\left(\frac{m\pi}{a}x\right)}{\frac{\sqrt{\mu/\varepsilon}}{\sqrt{1-\left(\frac{f_{c,,m}}{f}\right)^2}}} \underline{a_z}$$

$$= \omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_{c,,m}}{f}\right)^2}\left(\frac{aB'}{m\pi}\right)^2 \sin^2\left(\frac{m\pi}{a}x\right)\underline{a_z} = \beta_m\omega\mu\left(\frac{B'a}{m\pi}\right)^2 \sin^2\left(\frac{m\pi}{a}x\right)\underline{a_z}$$

power transmitted through a cross-sectional area of width *b* and a height *a* meters

$$P_{av} = \int_0^b\int_0^a \frac{1}{2}\beta_m\omega\mu\left(\frac{B'a}{m\pi}\right)^2 sin^2\left(\frac{m\pi}{a}x\right)dxdy = \frac{1}{2}\beta_m\omega\mu\left(\frac{B'a}{m\pi}\right)^2 b\int_0^a sin^2\left(\frac{m\pi}{a}x\right)dx$$

$$= \frac{b}{2}\beta_m\omega\mu\left(\frac{B'a}{m\pi}\right)^2\int_0^a \frac{1-cos\,2\left(\frac{m\pi}{a}x\right)}{2}dx = \frac{b}{2}\beta_m\omega\mu\left(\frac{B'a}{m\pi}\right)^2\left(\frac{a}{2}\right)$$

$$P_{av} = \frac{ab}{4}\beta_m\omega\mu\left(\frac{B'a}{m\pi}\right)^2$$

Complex Poynting vector

$$\underline{S} = \underline{E}\times\underline{H} = \frac{1}{2}\left[\underline{\hat{E}}(\underline{r})e^{j\omega t} + \underline{\hat{E}}^*(\underline{r})e^{-j\omega t}\right]\times\frac{1}{2}\left[\underline{\hat{H}}(\underline{r})e^{j\omega t} + \underline{\hat{H}}^*(\underline{r})e^{-j\omega t}\right]$$

$$= \frac{1}{2}\left[\frac{\underline{\hat{E}}(\underline{r})\times\underline{\hat{H}}^*(\underline{r}) + \underline{\hat{E}}^*(\underline{r})\times\underline{\hat{H}}(\underline{r})}{2}\right] + \frac{1}{2}\left[\frac{\underline{\hat{E}}(\underline{r})\times\underline{\hat{H}}(\underline{r})e^{j2\omega t} + \underline{\hat{E}}^*(\underline{r})\times\underline{\hat{H}}^*(\underline{r})e^{-j2\omega t}}{2}\right]$$

$$\underline{S} = \frac{1}{2}Re\left\{\underline{\hat{E}}(\underline{r})\times\underline{\hat{H}}^*(\underline{r})\right\} + \frac{1}{2}Re\left\{\underline{\hat{E}}(\underline{r})\times\underline{\hat{H}}(\underline{r})e^{j2\omega t}\right\}$$

time-average Poynting vector $\left\langle\underline{S_{av}}\right\rangle = \frac{1}{T}\int_0^T \underline{S}dt = \frac{1}{2}Re\left\{\underline{\hat{E}}(\underline{r})\times\underline{\hat{H}}^*(\underline{r})\right\}$

- **power loss**

attenuation constant $\alpha = \alpha_c + \alpha_d$

α_c : attenuation constant due to ohmic power loss in the imperfectly conducting walls

α_d : attenuation constant due to losses in the dielectric.

surface current on the lower plate

$$\underline{\hat{J}_s} = \underline{a_n}\times\underline{\hat{H}}\Big|_{x=0} = \underline{a_x}\times\left(\hat{H}_x\underline{a_x} + \hat{H}_z\underline{a_z}\right)\Big|_{x=0} = -\hat{H}_z\Big|_{x=0}\underline{a_y} = -B'\cos\left(\frac{m\pi}{a}x\right)\Big\|_{x=0}\underline{a_y}$$

$$= -B'\underline{a_y}$$

$$\left|\hat{J}_{sy}\right| = \left|\hat{H}_z\right| = B'$$

surface impedance for the lower plate

$$\hat{Z}_s = \frac{\hat{E}_t}{\hat{J}_s} = \frac{\hat{E}_y}{\hat{H}_z} = \hat{\eta}_c$$ $\hat{\eta}_c$: complex intrinsic impedance of the plate conductor (Cheng)

$$\hat{Z}_s = \frac{\hat{E}_y}{\hat{H}_z}\Big|_{x=0} = 0$$ for perfectly conducting plate

otherwise

$$\hat{Z}_s = \hat{\eta}_c = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1+j)(\sigma\delta)^{-1} = R_s + jX_s$$

skin depth $\delta = \dfrac{1}{\alpha} = \dfrac{1}{\sqrt{\pi f \mu \sigma}}$

power loss in the conducting plate

$$\left\langle \underline{S_{av}} \right\rangle = \frac{1}{2} Re\left\{ \underline{\hat{E}} \times \underline{\hat{H}^*} \right\}$$

$$= \frac{1}{2} Re\left\{ \hat{E}_y \underline{a_y} \times \left(\hat{H}_x^* \underline{a_x} + \hat{H}_z^* \underline{a_z} \right) \right\}$$

$$= \frac{1}{2} Re\left\{ \left(\hat{E}_y \cdot \hat{H}_z^* \right) \underline{a_x} - \left(\hat{E}_y \cdot \hat{H}_x^* \right) \underline{a_z} \right\}$$

$$= \frac{1}{2} Re\left\{ j\left(-\frac{\omega\mu a}{m\pi} B'^2 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{m\pi}{a}x\right) \right) \underline{a_x} + \left(j\omega\mu\hat{\gamma}^* \left(\frac{B'a}{m\pi}\right)^2 \sin^2\left(\frac{m\pi}{a}x\right) \cdot \right) \underline{a_z} \right\}$$

$$= \frac{1}{2} \beta_m \omega\mu \left(\frac{B'a}{m\pi}\right)^2 \sin^2\left(\frac{m\pi}{a}x\right) \underline{a_z} \qquad \left[W / m^2\right] \qquad \hat{\gamma}^* = -j\beta_m$$

$$\Rightarrow \quad \hat{E}_z = BD \sin k_x x \sin k_y y e^{-\gamma z} = \hat{E}_0 \sin k_x x \sin k_y y e^{-\gamma z} \qquad BD = \hat{E}_0$$

Apply (2) $\rightarrow \quad \hat{E}_z\big|_{x=a} = \hat{E}_0 \sin k_x a \sin k_y y e^{-\gamma z} = 0 \quad \rightarrow \quad k_x = \dfrac{m\pi}{a} \qquad m = 1,2,\cdots,\infty$

$$\Rightarrow \quad \hat{E}_z = \hat{E}_0 \sin k_x a \sin k_y y e^{-\gamma z}$$

Apply (4) $\rightarrow \quad \hat{E}_z\big|_{y=b} = \hat{E}_0 \sin k_x a \sin k_y b e^{-\gamma z} = 0 \quad \rightarrow \quad k_y = \dfrac{n\pi}{a} \qquad n = 1,2,\cdots,\infty$

$$\Rightarrow \quad \hat{E}_z = \hat{E}_0 \sin k_x a \sin k_y y e^{-\gamma z}$$

$$\hat{E}_z = \hat{E}_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-\gamma_{mn}z}$$

$$h_{mn}^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\frac{\partial^2 \hat{E}_z}{\partial x^2} + h^2 \hat{E}_z = 0 \quad \Rightarrow \quad \hat{E}_z(x) = A\sin(hx) + B\cos(hx)$$

Apply B.C.

$$\begin{cases} \hat{E}_z\big|_{x=0} = 0 & (1) \\ \hat{E}_z\big|_{x=a} = 0 & (2) \end{cases}$$