Terminal Cost Problem (continuing from last week)

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$$\delta J = \int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}} \right)^* - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)^* \right]^{\mathrm{T}} \delta \mathbf{x}(t) dt + \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}} \right)^{\mathrm{T}} \delta \mathbf{u}(t) dt + \mathcal{L}^* \Big|_{t_f} \delta t_f + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)^{*\mathrm{T}} \delta \mathbf{x} \right] \Bigg|_{t_f}$$

$$\Rightarrow \begin{cases} \left(\dfrac{\partial \mathcal{L}}{\partial \mathbf{x}} \right)^* - \dfrac{d}{dt} \left(\dfrac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)^* = 0 \\ \left(\dfrac{\partial \mathcal{L}}{\partial \mathbf{u}} \right) = 0 \end{cases} \quad \text{in the integral terms.}$$

Also, $\mathcal{L}^* \Big|_{t_f} \delta t_f + \left[\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)^{*\mathrm{T}} \delta \mathbf{x}(t) \right] \Bigg|_{t_f} = 0$

In order to convert the expression containing $\delta \mathbf{x}(t)$ in the above equation into an expression containing $\delta \mathbf{x}_f$, we approximate the slope of $\mathbf{x}^*(t) + \delta \mathbf{x}(t)$ at t_f as

$$\dot{\mathbf{x}}^*(t_f) + \delta \dot{\mathbf{x}}(t_f) \approx \frac{\delta \mathbf{x}_f - \delta \mathbf{x}(t_f)}{\delta t_f}$$

$$\Rightarrow \delta \mathbf{x}_f = \delta \mathbf{x}(t_f) + [\dot{\mathbf{x}}^*(t_f) + \delta \dot{\mathbf{x}}(t_f)] \delta t_f$$

$$\delta \mathbf{x}(t_f) = \delta \mathbf{x}_f - \dot{\mathbf{x}}^*(t_f) \delta t_f \qquad \text{(2nd order term neglected!)}$$

$$\Rightarrow \mathcal{L}^* \Big|_{t_f} + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)_{t_f}^{*\mathrm{T}} \left[\delta \mathbf{x}_f - \dot{\mathbf{x}}^*(t_f) \delta t_f \right] = 0$$

$$\left[\mathcal{L}^* \Big|_{t_f} - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)_{t_f}^{*\mathrm{T}} \dot{\mathbf{x}}^*(t_f) \right] \delta t_f + \left[\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right]_{t_f}^{*\mathrm{T}} \delta \mathbf{x}_f = 0$$

Since δt_f and δx_f are arbitrary, the above equation leads to the following boundary conditions.

$$\Rightarrow \begin{cases} \mathcal{L}^* \Big|_{t_f} - \left(\dfrac{\partial \mathcal{L}}{\partial \mathbf{x}} \right)_{t_f}^{*\mathrm{T}} \dot{\mathbf{x}}^*(t_f) = 0 \\ \left(\dfrac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)_{t_f}^{*} = 0 \end{cases} \qquad \text{B.C.}$$

Define the term *Hamiltonian* $\mathcal{H}^*$

$$\mathcal{H}^* = V^* + \boldsymbol{\lambda}^{*\mathrm{T}}\mathbf{f}^* \qquad \text{plant eq.: } \begin{cases} \dot{\mathbf{x}}^* = \mathbf{f} \\ \mathbf{g} = \mathbf{f} - \dot{\mathbf{x}} = \mathbf{0} \end{cases}$$

Cp. (i) $\mathcal{L} = V + \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{g} = V + \boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{f} - \dot{\mathbf{x}})$

(ii) $\mathcal{L}^* = \mathcal{L}^*(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t)$

$$= \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)^{*\mathrm{T}} \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)^* - \boldsymbol{\lambda}^{*\mathrm{T}}(t)\dot{\mathbf{x}}^*(t)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)^* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)^* = 0$$

$$\frac{\partial}{\partial \mathbf{x}}\left[\mathcal{H}^* + \left(\frac{\partial S}{\partial \mathbf{x}}\right)^{*\mathrm{T}} \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)^* - \boldsymbol{\lambda}^{*\mathrm{T}}(t)\dot{\mathbf{x}}^*(t)\right]$$

$$- \frac{d}{dt}\left[\frac{\partial}{\partial \dot{\mathbf{x}}}\left(\mathcal{H}^* + \left(\frac{\partial S}{\partial \mathbf{x}}\right)^{*\mathrm{T}} \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)^* - \boldsymbol{\lambda}^{*\mathrm{T}}(t)\dot{\mathbf{x}}^*(t)\right)\right] = 0$$

$$\Rightarrow \frac{\partial \mathcal{H}^*}{\partial \mathbf{x}} + \left(\frac{\partial^2 S}{\partial \mathbf{x}^2}\right)^{*\mathrm{T}} \dot{\mathbf{x}}^*(t) + \left(\frac{\partial^2 S}{\partial \mathbf{x}\partial t}\right)^* - \frac{d}{dt}\left(\frac{\partial S}{\partial \mathbf{x}}\right)^* + \frac{d}{dt}\boldsymbol{\lambda}^*(t) = 0$$

$$\Rightarrow \frac{\partial \mathcal{H}^*}{\partial \mathbf{x}} + \left(\frac{\partial^2 S}{\partial \mathbf{x}^2}\right)^{*\mathrm{T}} \dot{\mathbf{x}}^*(t) + \left(\frac{\partial^2 S}{\partial \mathbf{x}\partial t}\right)^* - \left[\left(\frac{\partial^2 S}{\partial \mathbf{x}\partial t}\right)^* + \left(\frac{\partial^2 S}{\partial \mathbf{x}^2}\right)^{*\mathrm{T}} \dot{\mathbf{x}}^*(t)\right] + \dot{\boldsymbol{\lambda}}^*(t) = 0$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)^* + \dot{\boldsymbol{\lambda}}^*(t) = 0 \Rightarrow \left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)^* = -\dot{\boldsymbol{\lambda}}^*(t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)^* = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)^* = 0 \Rightarrow \left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)^* = \dot{\mathbf{x}}^*(t)$$

B.C.

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)^* - \boldsymbol{\lambda}^*(t)\right]_{t_f}^{\mathrm{T}} \delta \mathbf{x}_f = 0 \qquad \text{(p.65)}$$

$$\text{Cp:} \left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)^{*\mathrm{T}} \dot{\mathbf{x}}(t)\right]_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)^{*\mathrm{T}}\Bigg|_{t_f} \delta \mathbf{x}_f = 0 \quad \text{(p.63)}$$

1. *Fixed Final Time* $(\delta t_f = 0)$ and *Fixed Final State* $(\delta \mathbf{x}_f = 0)$

2. *Free Final Time* (δt_f is arbitrary) and *Fixed Final State* $(\delta \mathbf{x}_f = 0)$

$$\left(\mathcal{H}^* + \frac{\partial S^*}{\partial t}\right)_{t_f} = 0$$

3. *Fixed Final Time* ($\delta t_f = 0$) and *Free Final State*($\delta \mathbf{x}_f$ is arbitrary)

$$\left(\frac{\partial S}{\partial \mathbf{x}}\right)^*\Bigg|_{t_f} - \boldsymbol{\lambda}^*(t_f) = 0$$

4. *Free Final Time* (δt_f is arbitrary) and *Dependent Free Final State* ($\delta \mathbf{x}_f$ is dependent on δt_f)

   $\delta \mathbf{x}_f = \dot{\mathbf{x}}_f \delta t_f = \dot{\boldsymbol{\theta}}_f \delta t_f$, where $\mathbf{x}$ is approximated by a function $\boldsymbol{\theta}$ at t_f, which

is usually linear. It is assumed $\mathbf{x}_f = \boldsymbol{\theta}_f$ and $\dot{\mathbf{x}}_f \approx \dot{\boldsymbol{\theta}}_f \approx \dot{\boldsymbol{\theta}}(t_f)$.

$$\Rightarrow \left[\mathcal{H}^* + \frac{\partial S^*}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)^* - \boldsymbol{\lambda}^*(t)\right]_{t_f}^{\mathrm{T}} \dot{\boldsymbol{\theta}}_f \delta t_f = 0$$

$$\Rightarrow \left\{\mathcal{H}^* + \frac{\partial S^*}{\partial t} + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)^* - \boldsymbol{\lambda}^*(t)\right]^{\mathrm{T}} \dot{\boldsymbol{\theta}}_f\right\}_{t_f} \delta t_f = 0$$

5. *Free Final Time* and *Free Final State* (δt_f and $\delta \mathbf{x}_f$ are arbitrary)

$$\Rightarrow \begin{cases} \left(\mathcal{H}^* + \dfrac{\partial S^*}{\partial t}\right)_{t_f} = 0 \\ \left(\dfrac{\partial S}{\partial \mathbf{x}}\right)\Bigg|_{t_f}^* - \lambda^*(t_f) = 0 \end{cases}$$

$$\begin{matrix} \delta^2 J > 0 \Rightarrow \min. \\ \delta^2 J < 0 \Rightarrow \max. \end{matrix} \quad \Leftrightarrow \quad \begin{bmatrix} \dfrac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2} & \dfrac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} \\ \dfrac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} & \dfrac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} \end{bmatrix}^* \text{ p.d.}$$

Example 2.12

A 2nd order system is given as follows:

$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) \end{cases}$ and the PI is $J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt$ with B.C. $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and

$\mathbf{x}(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Sol:

$$V = \frac{1}{2}u^2(t), \quad \mathbf{f} = \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}$$

Form the Hamiltonian as $\mathcal{H} = \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)u(t)$

$$\because \left(\frac{\partial \mathcal{H}}{\partial u}\right)^* = 0 \qquad \therefore \; u^*(t) + \lambda_2^*(t) = 0$$

$$\mathcal{H}^* = \frac{1}{2}\left[\left(-\lambda_2^*(t)\right)\right]^2 + \lambda_1^*(t)x_2^*(t) + \lambda_2^*(t)\left[-\lambda_2^*(t)\right] = \lambda_1^*(t)x_2^*(t) - \frac{1}{2}\lambda_2^{*2}(t)$$

State eq. and costate eq.

$$\begin{cases} \dot{x}_1^*(t) = \left(\dfrac{\partial \mathcal{H}}{\partial \lambda_1}\right)^* = x_2^*(t) & (1) \\ \dot{x}_2^*(t) = \left(\dfrac{\partial \mathcal{H}}{\partial \lambda_2}\right)^* = -\lambda_2^*(t) & (2) \\ \dot{\lambda}_1^*(t) = -\left(\dfrac{\partial \mathcal{H}}{\partial x_1}\right)^* = 0 & (3) \\ \dot{\lambda}_2^*(t) = -\left(\dfrac{\partial \mathcal{H}}{\partial x_2}\right)^* = -\lambda_1^*(t) & (4) \end{cases} \Rightarrow \begin{cases} \lambda_1^*(t) = C_1 & [\text{using eq.}(3)] \\ \lambda_2^*(t) = -C_1 t + C_2 & [\text{using eq.}(4)] \\ x_2^*(t) = \dfrac{1}{2}C_1 t^2 - C_2 t + C_3 & [\text{using eq.}(2)] \\ x_1^*(t) = \dfrac{1}{6}C_1 t^3 - \dfrac{1}{2}C_2 t^2 + C_3 t + C_4 & [\text{using eq.}(1)] \end{cases}$$

$u^*(t) = -\lambda_2^*(t) = C_1 t - C_2$

$\because x_1(0) = C_4 = 1$ and $x_2(0) = C_3 = 2$

$$\begin{cases} x_1(2) = \dfrac{4}{3}C_1 - 2C_2 + 4 + 1 = 1 \\ x_2(2) = 2C_1 - 2C_2 + 2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 3 \\ C_2 = 4 \end{cases}$$

$u^*(t) = 3t - 4$

$$J^* = \frac{1}{2}\int_0^2 (3t-4)^2 dt = \frac{1}{18}(3t-4)^3\Big|_0^2 = \frac{1}{18}(8+64) = 4$$

$$\begin{cases} x_1^*(t) = \dfrac{1}{2}t^3 - 2t^2 + 2t + 1 \\ x_2^*(t) = \dfrac{3}{2}t^2 - 4t + 2 \end{cases} \Rightarrow \begin{cases} 3t^2 - 8t + 4 = 0 \\ (3t-2)(t-2) = 0 \end{cases}$$

$t = \frac{2}{3}, 2 \Rightarrow$ relative max. $x_1(\frac{2}{3}) = 1$, and relative min. $x_2(2) = 1$

Example.2.13

A 2nd order system is given as $\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) \end{cases}$, and the PI is $J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt.$

B.C.: $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x_1(2) = 0$, and $x_2(2)$ is free.

Sol:

$$\begin{cases} \dot{x}_1^*(t) = \left(\frac{\partial \mathcal{H}}{\partial \lambda_1}\right)^* = x_2^*(t) & (1) \\ \dot{x}_2^*(t) = \left(\frac{\partial \mathcal{H}}{\partial \lambda_2}\right)^* = -\lambda_2^*(t) & (2) \\ \dot{\lambda}_1^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial x_1}\right)^* = 0 & (3) \\ \dot{\lambda}_2^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial x_2}\right)^* = -\lambda_1^*(t) & (4) \end{cases} \Rightarrow \begin{cases} \lambda_1^*(t) = C_1 & [\text{using eq.}(3)] \\ \lambda_2^*(t) = -C_1 t + C_2 & [\text{using eq.}(4)] \\ x_2^*(t) = \frac{1}{2}C_1 t^2 - C_2 t + C_3 & [\text{using eq.}(2)] \\ x_1^*(t) = \frac{1}{6}C_1 t^3 - \frac{1}{2}C_2 t^2 + C_3 t + C_4 & [\text{using eq.}(1)] \end{cases}$$

$$u^*(t) = -\lambda_2^*(t) = C_1 t - C_2 \qquad x_1(0) = C_4 = 1 \qquad x_2(0) = C_3 = 2$$

$$x_1(2) = \frac{4}{3}C_1 - 2C_2 + 4 + 1 = 0$$

Though $x_2(2)$ is arbitrary, we can use $\left[\left(\frac{\partial S}{\partial x_2}\right)^* - \lambda_2^*(t)\right]_{t_f} = 0$ to provide an additional condition. Hence, $\lambda_2^*(t_f) = \left(\frac{\partial S}{\partial x_2}\right)_{t_f}^*$

$\because$ no terminal cost $\Leftrightarrow S = 0$ $\quad\therefore\ \lambda_2^*(t_f) = \left(\frac{\partial S}{\partial x_2}\right)_{t_f}^* = 0$

$$-2C_1 + C_2 = 0 \Rightarrow \frac{4}{3}C_1 - 4C_1 + 5 = 0 \Rightarrow C_1 = \frac{15}{8},\quad C_2 = \frac{15}{4}$$

$$u^*(t) = \frac{15}{8}t - \frac{15}{4}$$

$$J^* = \frac{1}{2}\int_0^2 \left(\frac{15}{8}t - \frac{15}{4}\right)^2 dt = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{8}{15}\left(\frac{15}{8}t - \frac{15}{4}\right)^3\Bigg|_0^2 = \frac{1}{6}\cdot\frac{8}{15}\left(\frac{15}{8}\right)^3 (t-2)^3\Bigg|_0^2$$

$$= \frac{1}{6}\left(\frac{15}{8}\right)^2 (8) = \frac{75}{16} > 4$$

Cp. $\begin{cases} x_1(2) = 1 \\ x_2(2) = 0 \end{cases}$ and $J^* = 4$