Lower and upper bounds of the circumference of a circle

If $S^{(i)}$ and $S^{(u)}$ denote the perimeters of the inscribed and circumscribed polygons, respectively, as shown below, prove that

$$S^{(i)} \leq S \leq S^{(u)}$$

where $S$ is the circumference of the circle.

<Sol>

The perimeters of the inscribed, circumscribed $n$-sided polygons and the circumference of the circle, respectively, are given as

$$S^{(i)} = 2nR \sin \left( \frac{\pi}{n} \right), \quad S^{(u)} = 2nR \tan \left( \frac{\pi}{n} \right), \quad S = 2\pi R,$$

where $n$ is the number of polygon edges and $R$ is the radius of the circle. Since, in general,

$$S^{(i)} = 2nR \sin \left( \frac{\pi}{n} \right) < S = 2\pi R < S^{(u)} = 2nR \tan \left( \frac{\pi}{n} \right),$$

and as $n$ approaches infinity,

$$\lim_{n \to \infty} S^{(i)} = \lim_{n \to \infty} 2nR \sin \left( \frac{\pi}{n} \right) = 2\pi R = S, \quad \lim_{n \to \infty} S^{(u)} = \lim_{n \to \infty} 2nR \tan \left( \frac{\pi}{n} \right) = 2\pi R = S.$$

Therefore, it prove that

$$S^{(i)} \leq S \leq S^{(u)}.$$

The following figure shows how $S^{(i)}/S$ and $S^{(u)}/S$ converge to one as $n$ is increased.