2.4.(一).2

$y'' + 3y' + 2y = e^x$

特徵方程式: $r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$
$r = -1 \;\; or \;\; r = -2$
$\therefore y_h = c_1 e^{-x} + c_2 e^{-2x}$

$Let \;\; [\times 2] y_p = Ae^x$

$[\times 3] y_p' = Ae^x$

$[\times 1] y_p'' = Ae^x$

$e^x = (2A + 3A + A)e^x$
$6A = 1$
$A = \frac{1}{6}$

$\therefore G.S. \quad y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{6} e^x$

2.4.(一).5

$y'' - 2y' - 3y = 10\,sin\,x$

特徵方程式: $r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3 \;\; or \;\; r = -1$

$\therefore y_h = c_1 e^{-x} + c_2 e^{3x}$

$Let \;\; [\times(-3)]y_p = A\,cos\,x + B\,sin\,x$

$[\times(-2)]y_p' = -A\,sin\,x + B\,cos\,x$

$[\times 1]y_p'' = -A\,cos\,x - B\,sin\,x$

$10\,sin\,x = (-3A - 2B - A)\,cos\,x + (-3B + 2A - B)\,sin\,x$

$\begin{cases} -4A - 2B = 0 \\ 2A - 4B - 10 \end{cases}$

$\begin{cases} A = 1 \\ B = -2 \end{cases}$

$\therefore G.S. \quad y = c_1 e^{-x} + c_2 e^{3x} + cos\,x - 2\,sin\,x$

2.4.(一).7

$$y'' + 4y' + 3y = cos(3x)$$

特徵方程式: $r^2 + 4r + 3 = 0$

$$(r+1)(r+3) = 0$$
$$r = -1 \ \ or \ \ r = -3$$
$$\therefore y_h = c_1 e^{-x} + c_2 e^{-3x}$$

$$Let \ \ [\times 3] y_p = A\,cos(3x) + B\,sin(3x)$$

$$[\times 4] y_p' = -3A\,sin(3x) + 3B\,cos(3x)$$

$$[\times 1] y_p'' = -9A\,cos(3x) - 9B\,sin(3x)$$

$$cos(3x) = (3A + 12B - 9A)\,cos(3x) + (3B - 12A - 9B)\,sin(3x)$$

$$\begin{cases} -6A + 12B = 1 \\ -12A - 6B = 0 \end{cases}$$

$$\begin{cases} A = -\dfrac{1}{30} \\ B = \dfrac{1}{15} \end{cases}$$

$$\therefore G.S. \quad y = c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{30} cos(3x) + \frac{1}{15} sin(3x)$$

2.4.(一).13

$y'' + y = x + x\sin x$

特徵方程式: $r^2 + 1 = 0$

$r = \pm i$

$\therefore y_h = c_1 \cos x + c_2 \sin x$

$Let \ \ [\times 1] y_{p1} = Ax$

$[\times 0] y'_{p1} = A$

$[\times 1] y''_{p1} = 0$

$x = Ax$

$A = 1$

$Let \ \ [\times 1] y_{p2} = x[(Ax + B)\cos x + (Cx + D)\sin x]$

$= (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x$

$[\times 0] y'_p = (2Ax + B)\cos x - (Ax^2 + Bx)\sin x + (2Cx + D)\sin x + (Cx^2 + D)\cos x$

$= [Cx^2 + (2A + D)x + B]\cos x + [-Ax^2 + (-B + 2C)x + D]\sin x$

$[\times 1] y''_p = (2Cx + 2A + D)\cos x - [Cx^2 + (2A + D)x + B]\sin x + (-2Ax - B + 2C)\sin x + [-Ax^2 + (-B + 2C)x + D]\cos x$

$= [-Ax^2 + (-B + 4C)x + 2A + 2D]\cos x + [-Cx^2 + (-4A - D)x - 2B + 2C]\sin x$

$x\sin x = [(A - A)x^2 + (B - B + 4C)x + 2A + 2D]\cos x + [(C - C)x^2 + (D - 4A - D)x - 2B + 2C]\sin x$

$$\begin{cases} 4C = 0 \\ 2A + 2D = 0 \\ -4A = 1 \\ -2B + 2C = 0 \end{cases}$$

$$\begin{cases} A = -\frac{1}{4} \\ B = 0 \\ C = 0 \\ D = \frac{1}{4} \end{cases}$$

$\therefore G.S. \quad y = c_1 \cos x + c_2 \sin x + x - \frac{1}{4}x^2 \cos x + \frac{1}{4}x \sin x$

2.4.(二).4

$y'' + 2y' + y = e^{-x} \ln x$

特徵方程式: $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1$(重根)

$\therefore y_h = (c_1 x + c_2)e^{-x}$

Let $y_p = u_1 y_1 + u_2 y_2$

$$\begin{cases} u_1'(xe^{-x}) + u_2'(e^{-x}) = 0 \\ u_1'(e^{-x} - xe^{-x}) + u_2'(-e^{-x}) = e^{-x} \ln x \end{cases}$$

$$\begin{cases} xu_1' + u_2' = 0 \\ (1-x)u_1' - u_2' = \ln x \end{cases}$$

$$u_1' = \frac{\begin{vmatrix} 0 & 1 \\ \ln x & -1 \end{vmatrix}}{\begin{vmatrix} x & 1 \\ 1-x & -1 \end{vmatrix}} = \frac{-\ln x}{-x-(1-x)} = \ln x$$

$$u_1 = \int \ln x \, dx = x \ln x - \int 1 \, dx = x \ln x - x = x(\ln x - 1)$$

$$u_2' = \frac{\begin{vmatrix} x & 0 \\ 1-x & \ln x \end{vmatrix}}{\begin{vmatrix} x & 1 \\ 1-x & -1 \end{vmatrix}} = \frac{x \ln x}{-x-(1-x)} = -x \ln x$$

$$u_2 = \int -x \ln x \, dx = -\left(\frac{1}{2}x^2 \ln x - \int \frac{1}{2}x \, dx\right) = -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 = -\frac{1}{4}x^2(2\ln x - 1)$$

$$\therefore y_p = xe^{-x}[x(\ln x - 1)] + e^{-x}\left[-\frac{1}{4}x^2(2\ln x - 1)\right] = \frac{1}{4}x^2 e^{-x}(2\ln x - 3)$$

$$\therefore G.S. \quad y = (c_1 x + c_2)e^{-x} + \frac{1}{4}x^2 e^{-x}(2\ln x - 3)$$