

2.2.6

$$x^2 y'' + 3xy' + y = 0, y_1 = x^{-1}$$

$$\text{Let } y_2 = ux^{-1}$$

$$y_2' = u'x^{-1} - ux^{-2}$$

$$y_2'' = u''x^{-1} - 2u'x^{-2} + 2ux^{-3}$$

$$x^2 y_2'' + 3xy_2' + y_2 = 0$$

$$x^2(u''x^{-1} - 2u'x^{-2} + 2ux^{-3}) + 3x(u'x^{-1} - ux^{-2}) + ux^{-1} = 0$$

$$xu'' + u' = 0$$

$$u'' + \frac{1}{x}u' = 0$$

$$\text{Let } v = u', v' = u''$$

$$v' + \frac{1}{x}v = 0$$

$$\frac{dv}{dx} = -\frac{1}{x}v$$

$$\frac{1}{v}dv = -\frac{1}{x}dx$$

$$\int \frac{1}{v}dv = \int -\frac{1}{x}dx$$

$$\ln v = -\ln x$$

$$v = \frac{1}{x}$$

$$u = \int \frac{1}{x}dx = \ln x$$

$$\therefore y_2 = x^{-1} \ln x$$

$$\therefore G.S. \quad y = c_1 x^{-1} + c_2 x^{-1} \ln x$$