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Existence results of positive solutions for nonlinear cooperative elliptic systems involving fractional Laplacian

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Abstract

In this article, we prove existence results of positive solutions for the following nonlinear elliptic problem with gradient terms:

$$\begin{cases} (-\Delta)^\alpha u = f(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ (-\Delta)^\alpha v = g(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $(-\Delta)^\alpha$ denotes the fractional Laplacian and Ω is a smooth bounded domain in $\mathbb{R}^N$. It shown that under some assumptions on f and g, the problem has at least one positive solution (u,v). Our proof is based on the classical scaling method of Gidas and Spruck and topological degree theory.

Keywords: Fractional Laplacian; nonlinear elliptic system; gradient term; existence result.

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1 Introduction

The problem of the existence of solutions for fractional elliptic systems of the form

$$\begin{cases} (-\Delta)^\alpha u = f(x,u,v) & \text{in } \Omega, \\ (-\Delta)^\alpha v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N > 2\alpha$ and $\alpha \in (0,1)$, has been the object of intensive research during the last years. Variational methods have been frequently used, since there is by now a remarkable collection of abstract results on the existence of critical points. But when come to the problem under consideration is not of variational type, for example gradient terms are present, existence results of solutions are very little. In the case of scalar equation

$$\begin{cases} (-\Delta)^\alpha u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and

$$\begin{cases} (-\Delta)^\alpha u = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

have been studied in [1] provided some growth condition of f with respect to u and ∇u is imposed. In our paper, we consider the system case and some results in [8] and [17] are extended to the fractional Laplacian case.

The fractional Laplacian $(-\Delta)^\alpha$ is defined as

$$(-\Delta)^\alpha u(x) = C_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}} dy \quad \text{for } x \in \mathbb{R}^N. \tag{1.2}$$

Here $P.V.$ denotes the principal value of the integral, that for notational simplicity we omit in what follows and we without loss of generality take $C_{N,\alpha} = 1$. In additional, for some $\sigma > 0$, suppose that $u \in \mathcal{L}_\alpha(\mathbb{R}^N) \cap C^{2\alpha+\sigma}(\mathbb{R}^N)$ if $0 < \alpha < 1/2$ or $u \in \mathcal{L}_\alpha(\mathbb{R}^N) \cap C^{1,2\alpha+\sigma-1}(\mathbb{R}^N)$ if $\alpha \geq 1/2$ where

$$\mathcal{L}_\alpha(\mathbb{R}^N) := \left\{ u \;\middle|\; u : \mathbb{R}^N \to \mathbb{R} \text{ such that } \int_{\mathbb{R}^N} \frac{|u(y)|}{1+|y|^{N+2\alpha}} dy < \infty \right\},$$

then the definition (1.2) is well defined (see Proposition 2.4 in [16]).

If we separate the leading part in (1.1), it becomes a system of the form

$$\begin{cases} (-\Delta)^\alpha u = a(x)u^{\beta_{11}} + b(x)v^{\beta_{12}} + h_1(x,u,v) & \text{in } \Omega, \\ (-\Delta)^\alpha v = c(x)u^{\beta_{21}} + c(x)v^{\beta_{22}} + h_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

The above system is not variational, and consequently we will use topological methods to prove the existence of positive solutions. The main difficulty when using topological methods lies in the need of obtaining a priori bounds. Here the a priori bounds for the solutions of (1.3) are obtained via the so-called blow-up method (cf. Gidas and Spruck [9]).

Assuming the following conditions:

(A1) The coefficients $a, b, c, d : \bar{\Omega} \to [0, \infty)$ are continous functions.

(A2) $\beta_{ij} \geq 0$ for $i, j = 1, 2$.

(A3) $h_1, h_2 \in C(\Omega, \mathbb{R}, \mathbb{R})$, and there exists positive constants c_1 and c_2 such that

$$\begin{aligned} |h_1(x,s,t)| &\leq c_1(1+|s|^{\gamma_{11}}+|t|^{\gamma_{12}}), \\ |h_2(x,s,t)| &\leq c_2(1+|s|^{\gamma_{21}}+|t|^{\gamma_{22}}), \end{aligned}$$

where $1 < \gamma_{ij} < \beta_{ij}$ for $i, j = 1, 2$.

We say that system (1.3) is weakly coupled if $\beta_{11} > 1$, $\beta_{22} > 1$ and

$$\beta_{12} < \frac{\beta_{22}-1}{\beta_{11}-1}\beta_{11}, \quad \beta_{21} < \frac{\beta_{11}-1}{\beta_{22}-1}\beta_{22} \tag{1.4}$$

and it is strongly coupled if $\beta_{12}\beta_{21} > 0$ and

$$\beta_{11} < \frac{\beta_{21}+1}{\beta_{12}+1}\beta_{12}, \quad \beta_{22} < \frac{\beta_{12}+1}{\beta_{21}+1}\beta_{21}. \tag{1.5}$$

Our first main result is

Theorem 1.1 *Suppose (A1)-(A3) hold.*

(i) If system (1.3) is weakly coupled and $a(x), d(x) \geq c_0 > 0$ for $x \in \bar{\Omega}$, assume also that

$$1 < \beta_{11}, \beta_{22} < \frac{N+2\alpha}{N-2\alpha}, \tag{1.6}$$

then system (1.3) has at least one positive viscosity solution.

(ii) If system (1.3) is strongly coupled and $b(x), c(x) \geq c_0 > 0$ for $x \in \bar{\Omega}$, assume also that

$$\min\left\{\frac{2\alpha(\beta_{12}+1)}{\beta_{12}\beta_{21}-1}, \frac{2\alpha(\beta_{21}+1)}{\beta_{12}\beta_{21}-1}\right\} > \frac{N-2\alpha}{2}, \tag{1.7}$$

then system (1.3) has at least one positive viscosity solution.

In this article, we will also consider problems with gradient terms. More precisely, we will study the following system

$$\begin{cases} (-\Delta)^\alpha u = a(x)u^{\beta_{11}} + b(x)v^{\beta_{12}} + h_1(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ (-\Delta)^\alpha v = c(x)u^{\beta_{21}} + c(x)v^{\beta_{22}} + h_2(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.8}$$

We should note that a natural restriction in order that the gradient is meaningful for nonlocal problem is $\alpha > 1/2$ (cf. [1]). To obtain the a priori bounded of system (1.8), a serious difficulty comes when one proceeds to estimate the gradients of sequences of solutions that appear in the blow-up method. To handle it, we have to use some norm with weights depending on the distance of the boundary of domains involved, see [10, 8, 1] and references therein. We define the weakly coupled and strongly coupled of system (1.8) is the same as (1.4) and (1.5) respectively and also assume the following:

(A4) $h_1, h_2 \in C(\Omega, \mathbb{R}, \mathbb{R}, \mathbb{R}^N, \mathbb{R}^N)$ are nonnegative, and there exists positive constants c_1 and c_2 such that

$$\begin{aligned} h_1(x,s,t,\xi,\eta) &\leq c_1(1+|s|^{\gamma_{11}}+|t|^{\gamma_{12}}+|\xi|^{\theta_{11}}+|\eta|^{\theta_{12}}),\\ h_2(x,s,t,\xi,\eta) &\leq c_2(1+|s|^{\gamma_{21}}+|t|^{\gamma_{22}}+|\xi|^{\theta_{21}}+|\eta|^{\theta_{22}}),\end{aligned}$$

where $1 < \gamma_{ij} < \beta_{ij}$, $i, j = 1, 2$.

(A5) If system (1.8) is weakly coupled and θ_{ij} $(i, j = 1, 2)$ satisfying

$$\begin{aligned} 1 < \theta_{11} &< \frac{2\alpha\beta_{11}}{\beta_{11}+2\alpha-1}, \quad 1 < \theta_{22} < \frac{2\alpha\beta_{22}}{\beta_{22}+2\alpha-1},\\ 1 < \theta_{12} &< \min\left\{2\alpha, \frac{2\alpha\beta_{11}(\beta_{22}-1)}{(\beta_{11}-1)(\beta_{22}+2\alpha-1)}\right\},\\ 1 < \theta_{21} &< \min\left\{2\alpha, \frac{2\alpha\beta_{22}(\beta_{11}-1)}{(\beta_{11}+2\alpha-1)(\beta_{22}-1)}\right\}.\end{aligned}$$

(A6) If system (1.8) is strongly coupled and θ_{ij} $(i, j = 1, 2)$ satisfying

$$\begin{aligned} 1 < \theta_{11} &< \frac{2\alpha\beta_{12}(\beta_{21}+1)}{2\alpha\beta_{12}+\beta_{12}\beta_{21}+2\alpha-1},\\ 1 < \theta_{22} &< \frac{2\alpha\beta_{21}(\beta_{12}+1)}{2\alpha\beta_{21}+\beta_{12}\beta_{21}+2\alpha-1},\\ 1 < \theta_{12} &< \min\left\{2\alpha, \frac{2\alpha\beta_{12}(\beta_{21}+1)}{2\alpha\beta_{21}+\beta_{12}\beta_{21}+2\alpha-1}\right\},\\ 1 < \theta_{21} &< \min\left\{2\alpha, \frac{2\alpha\beta_{21}(\beta_{12}+1)}{2\alpha\beta_{12}+\beta_{12}\beta_{21}+2\alpha-1}\right\}.\end{aligned}$$

Remark 1.1 *We note that if $\alpha > 1/2$ and $\beta_{ii} > 1$, then*

$$1 < \frac{2\alpha\beta_{ii}}{\beta_{ii}+2\alpha-1} < 2\alpha$$

for $i = 1, 2$. This implies that $\theta_{ii} < 2\alpha$ $(i = 1, 2)$ in (A5). On the other hand, it permits us to assume that $\theta_{ii} > 1$ $(i = 1, 2)$. Similarly, we have $\theta_{ii} < 2\alpha$ $(i = 1, 2)$ in (A6) if $\alpha > 1/2$ and we can also assume $\theta_{ii} > 1$ $(i = 1, 2)$ in (A6) if $\beta_{ij}\beta_{ji} > 1$ $(i, j = 1, 2,\ i \neq j)$.

We will prove that

Theorem 1.2 *Let $1/2 < \alpha < 1$. Suppose $(A1), (A2)$ and $(A4)$ hold.*

(i) If system (1.8) is weakly coupled, $a(x), d(x) \geq c_0 > 0$ for $x \in \bar{\Omega}$ and $(A5)$ holds, then system (1.8) has at least one positive viscosity solution if

$$1 < \beta_{11}, \beta_{22} \leq \frac{N}{N-2\alpha}. \tag{1.9}$$

(ii) If system (1.8) is strongly coupled, $b(x), c(x) \geq c_0 > 0$ for $x \in \bar{\Omega}$ and $(A6)$ holds, then system (1.8) has at least one positive viscosity solution if

$$\max\left\{\frac{2\alpha(\beta_{12}+1)}{\beta_{12}\beta_{21}-1}, \frac{2\alpha(\beta_{21}+1)}{\beta_{12}\beta_{21}-1}\right\} \geq N-2\alpha. \tag{1.10}$$

We notice that we obtain the existence result of system (1.8) in a small range compared with (1.6) and (1.7) in Theorem 1.1. As mentioned before, we need to consider weighted norms which present some problems since the scaling needed near the boundary is not the same as in the interior. Therefore, we need to split our study into two parts : first, we obtain rough universal bounds for all solutions of (1.8), by using the well-known doubling lemma in [12] and our problems are nonlocal which forces us to strengthen the subcritical hypothesis to (1.8) and to require instead (1.9) and (1.10) (see Lemma 3.1 and Remark 3.1). After that, we reduce the obtention of the priori bounds to an analysis near boundary.

Remark 1.2 *In fact, Theorems 1.1 and 1.2 are also true if we replace the fractional Laplacian to the following more general operator*

$$(-\Delta)^{\alpha}_{K} u(x) = \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2\alpha}} K(y)dy,$$

where $\alpha \in (0,1)$ and K is a measurable function defined in $\mathbb{R}^N$ satisfies $\lambda \leq K(x) \leq \Lambda$ in $\mathbb{R}^N$ for some constants $\lambda \leq \Lambda$ and

$$\lim_{x\to 0} K(x) = 1,$$

which is considered in [1]. In particular, we notice that if set $K \equiv 1$, $(-\Delta)^{\alpha}_{K}$ reduces to the fractional Laplacian.

The paper is organized as follows. In Section 2 we recall some well known regularity results, convergence theorem, weighted norms and the Liouville type theorems in $\mathbb{R}^N$ for nonlocal systems. In Section 3, we obtain a priori bounds of systems (1.3) and (1.8) by the blow-up method. The existence results, Theorems 1.1 and 1.2, are shown by topological degree theory in Sections 4.

2 Preliminaries

The purpose of this section is to introduce some preliminaries. We start this section by recalling the following maximum principle.

Proposition 2.1 *([1], Lemma 7) Let Let Ω be an open and bounded domain of $\mathbb{R}^N$, and assume that $u \in C(\mathbb{R}^N)$ be a viscosity solution of*

$$(-\Delta)^\alpha u \geq 0 \ in \ \Omega$$

with $u \geq 0$ in $\mathbb{R}^N$. Then $u > 0$ or $u \equiv 0$ in Ω.

Next we give a C^β estimate, which is a direct conclusion of Theorem 26 in [4].

Theorem 2.1 *Let Ω be a regular domain. If $u \in C(\bar{\Omega})$ satisfies the inequalities*

$$\Delta^\alpha u \geq -C_0 \quad and \quad \Delta^\alpha u \leq C_0 \quad in \ \Omega,$$

then for any $\Omega' \Subset \Omega$ there exist constant $\beta > 0$ such that $u \in C^\beta(\Omega')$ and

$$\|u\|_{C^\beta(\Omega')} \leq C\{\sup_\Omega |u| + \|u\|_{L^\infty(\Omega)} + C_0\}$$

for some constant $C > 0$ which depends on N.

We also need the following regularity.

Theorem 2.2 *([14], Theorem 2.5) Let g bounded in $\mathbb{R}^N \setminus \Omega$ and $f \in C^\beta_{loc}(\Omega)$. Suppose u is a viscosity solution of*

$$(-\Delta)^\alpha u = f \quad in \ \Omega, \qquad u = g \quad in \ \mathbb{R}^N \setminus \Omega.$$

Then there exists a $\gamma > 0$ such that $u \in C^{2\alpha+\gamma}_{loc}(\Omega)$.

When $\alpha \in (1/2, 1)$ the Hölder estimate for the solution can be improved to obtain an estimate for the first derivertives.

Theorem 2.3 *([11], Theorem 1.2) Assume that $\alpha \in (1/2, 1)$. Suppose u is a viscosity solution of*

$$(-\Delta)^\alpha u = f \quad in \ \Omega,$$

where $f \in L^\infty_{loc}(\Omega)$ Then there exists a $\beta = \beta(N, \alpha) \in (0, 1)$ such that $u \in C^{1,\beta}_{loc}(\Omega)$. Moreover, for every ball $B_R \subset\subset \Omega$ there exists a positive constant $C = C(N.\alpha, R)$ such that

$$\|u\|_{C^{1,\beta}(\overline{B_{R/2}})} \leq C\left(\|f\|_{L^\infty(B_R)} + \|u\|_{L^\infty(\mathbb{R}^N)}\right).$$

We are going to use the following convergence result for fractional Laplacian (see Lemma 5 in [4] for integro differential equation).

Theorem 2.4 *Let $\{u_k\}$, $k \in \mathbb{N}$ be a sequence of functions that are bounded in $\mathbb{R}^N$ and continuous in Ω, f_k and f are continuous in Ω such that*
(1) $\Delta^\alpha u_k \le f_k$ in Ω in viscosity sense.
(2) $u_k \to u$ locally uniformly in Ω.
(3)$u_k \to u$ a.e. in $\mathbb{R}^N$.
(4) $f_k \to f$ locally uniformly in Ω.
Then $\Delta^\alpha u \le f$ in Ω in viscosity sense.

Since the problems under consideration with a right hand side which is possible singular at $\partial\Omega$, we next introduce some norms which will help us to quantify the singularity of both right hand side and the gradient of the solutions in case $\alpha \in (1/2, 1)$. We denote $d(x) = dist(x, \partial\Omega)$ for $x \in \Omega$. It is well know that d is Lipschitz continuous in Ω with Lipschitz constant 1 and it is a C^2 function in a neighborhood of $\partial\Omega$. We modify it outside this neighborhood to make it a C^2 function, still with Lipschitz constant 1, and we extend it to be zero outside.

For $\tau \in \mathbb{R}$ and $u \in C(\Omega)$, we define (cf. Chapter 6 in [10])

$$\|u\|_0^{(\tau)} = \sup_{\Omega} d(x)^\tau |u(x)|.$$

When $u \in C^1(\Omega)$ we also define

$$\|u\|_1^{(\tau)} = \sup_{\Omega} \left(d(x)^\tau |u(x)| + d(x)^{\tau+1} |\nabla u(x)| \right). \tag{2.1}$$

Then the following estimates are prove in [1] for the Dirichlet problems.

Lemma 2.1 *([1], Lemma 3) Suppose $0 < \alpha < 1$. Let $f \in C(\Omega)$ be such that $\|f\|_0^{(\tau)} < +\infty$ for some $\tau \in (\alpha, 2\alpha)$. Then problem*

$$(-\Delta)^\alpha u = f \quad in\ \Omega, \quad u = 0 \quad in\ \mathbb{R}^N \setminus \Omega, \tag{2.2}$$

admits a unique viscosity solution. Moreover, there exists a positive constant C such that

$$\|u\|_0^{(\tau-2\alpha)} \le C \|f\|_0^{(\tau)}. \tag{2.3}$$

The next estimate concerns the gradient of the solutions of (2.2) in case $\alpha \in (1/2, 1)$.

Lemma 2.2 *([1], Lemma 5) Suppose* $1/2 < \alpha < 1$*. Let* $f \in C(\Omega)$ *be such that* $\|f\|_0^{(\tau)} < +\infty$ *for some* $\tau \in (\alpha, 2\alpha)$*. Then the unique solution of (2.2) verifies*

$$\|\nabla u\|_0^{(\tau-2\alpha+1)} \le C_0\left(\|f\|_0^{(\tau)} + \|u\|_0^{(\tau-2\alpha)}\right),$$

where C_0 *is a positive constant depends on* N *and* α *but not on* Ω*.*

The next lemma is devote to take care of the constant in (2.3) when we consider (2.2) in expanding domains, since in general it depends on Ω. This is a crucial point for the scaling method to work properly in our setting. We take $\xi \in \partial\Omega$, $\lambda > 0$ and let

$$\Omega_\lambda := \{x \in \mathbb{R}^N : \ \xi + \lambda x \in \Omega\}.$$

Observe that $d_\lambda(x) := dist(x, \partial\Omega_\lambda) = \lambda^{-1} d(\xi + \lambda x)$.The following lemma show that the constant in (2.3) for the solution of (2.2) posed in Ω_λ will depend on the domain Ω, but not on the dilation parameter λ.

Lemma 2.3 *([1], Lemma 6) Suppose* $0 < \alpha < 1$*. For every* $\tau \in (\alpha, 2\alpha)$ *and* $\lambda_0 > 0$*, there exist* $C, \delta > 0$ *such that*

$$(-\Delta)^\alpha d_\lambda^{2\alpha-\tau} \ge C d_\lambda^{-\tau} \quad in \ (\Omega_\lambda)_\delta,$$

if $0 < \lambda \le \lambda_0$*. Moreover, if* u *satisfying*

$$(-\Delta)^\alpha u \le C_1 d_\lambda^{-\tau} \quad in \ \Omega_\lambda,$$

for some $C_1 > 0$ *with* $u = 0$ *in* $\mathbb{R}^N \setminus \Omega_\lambda$*, then*

$$u(x) \le C_2(C_1 + \|u\|_{L^\infty(\Omega_\lambda)}) d_\lambda^{2\alpha-\tau} \quad for \ x \in (\Omega_\lambda)_\delta,$$

for some $C_2 > 0$ *depending on* α, δ, τ *and* C_0*.*

We finish this section by listing the well-known Liouville type theorems of the limit systems of (1.1) in the whole space has been considered in our previous article [15]. In [15], we proved that

Theorem 2.5 *([15], Theorem 1.1) Let* $p, q > 0$ *and* $pq > 1$*. Suppose*

$$\frac{2\alpha(p+1)}{pq-1}, \frac{2\alpha(q+1)}{pq-1} \in \left[\frac{N-2\alpha}{2}, \ N-2\alpha\right)$$

and

$$\left(\frac{2\alpha(p+1)}{pq-1}, \frac{2\alpha(q+1)}{pq-1}\right) \ne \left(\frac{N-2\alpha}{2}, \ \frac{N-2\alpha}{2}\right).$$

Then, for some $\sigma > 0$, there exists no positive $\mathcal{L}_\alpha(\mathbb{R}^N) \cap C^{2\alpha+\sigma}(\mathbb{R}^N)$ if $0 < \alpha < 1/2$ or in $\mathcal{L}_\alpha(\mathbb{R}^N) \cap C^{1,2\alpha+\sigma-1}(\mathbb{R}^N)$ if $\alpha \geq 1/2$ type solution to system

$$\begin{cases} (-\Delta)^\alpha u = v^q & \text{in } \mathbb{R}^N, \\ (-\Delta)^\alpha v = u^p & \text{in } \mathbb{R}^N. \end{cases} \tag{2.4}$$

In this theorem, we consider the nonexistence of solutions just in the subregion

$$\max\left\{\frac{2\alpha(p+1)}{pq-1}, \frac{2\alpha(q+1)}{pq-1}\right\} < N - 2\alpha.$$

In order to prove Theorem 1.1 completely, we also need to study the nonexistence results in the following region

$$\max\left\{\frac{2\alpha(p+1)}{pq-1}, \frac{2\alpha(q+1)}{pq-1}\right\} \geq N - 2\alpha, \tag{2.5}$$

Using the fundamental solutions of fractional Laplacian and the comparison principle, we can show there are no positive viscosity supersolutions to system (1.7) if and only if (p, q) verifies (2.5). In fact, we prove that

Theorem 2.6 *Suppose $p, q > 0$ and $pq > 1$. Then there are no positive viscosity supersolutions to system (2.4) if and only if (p, q) verifies (2.5).*

Proof. Here we omit the proof since in very similar as the proof of Theorem 1.3 (I) in [13] by using the fundamental solutions of fractional Laplacian (see Theorem 1.3 in [6] and also Theorem 1.3 in [5]) and the comparison principle (cf. Theorem 2.1 in [6]). □

3 A priori bounds

In this section is devote to get the a priori bounds of systems (1.3) and (1.8) by a blow-up method.

Theorem 3.1 *Under the hypotheses of Theorem 1.1, then each couple of positive viscosity solution (u, v) of (1.3) is bounded in the L^∞-norm by a constant.*

Proof. Assume on the contrary that there exists a sequence (u_n, v_n) of positive solutions of (1.3) such that

$$\max\{\|u_n\|_{L^\infty(\Omega)}, \|v_n\|_{L^\infty(\Omega)}\} \to +\infty$$

as $n \to +\infty$. We may assume

$$\lambda_n = \|u_n\|_{L^\infty(\Omega)}^{-1/\sigma_1},$$

if $\|u_n\|^{\sigma_2}_{L^\infty(\Omega)} \geq \|v_n\|^{\sigma_1}_{L^\infty(\Omega)}$ (up to a sequence), and

$$\lambda_n = \|v_n\|_{L^\infty(\Omega)}^{-1/\sigma_2}$$

otherwise, for some constants $\sigma_1, \sigma_2 > 0$ which are to be determined later. Without loss of generality, we suppose that we are in the first of these two situations.

Note that we have $\lambda_n \to 0$ as $n \to \infty$. Let $x_n \in \Omega$ be a point where u_n assumes its maximum. The functions

$$\tilde{u}_n(x) = \lambda_n^{\sigma_1} u_n(\lambda_n x + x_n) \quad \text{and} \quad \tilde{v}_n(x) = \lambda_n^{\sigma_2} v_n(\lambda_n x + x_n)$$

are such that $\tilde{u}_n(0) = 1$ and $0 \leq \tilde{u}_n, \tilde{v}_n \leq 1$ in Ω. One also verifies that the functions $\tilde{u}_n$ and $\tilde{v}_n$ satisying

$$\begin{cases} (-\Delta)^\alpha \tilde{u}_n = \lambda_n^{\sigma_1+2\alpha-\sigma_1\beta_{11}} a(z)\tilde{u}_n^{\beta_{11}} + \lambda_n^{\sigma_1+2\alpha-\sigma_2\beta_{12}} b(z)\tilde{v}_n^{\beta_{12}} & \\ \qquad\qquad + \lambda_n^{\sigma_1+2\alpha} h_1(z, \lambda_n^{-\sigma_1}\tilde{u}_n, \lambda_n^{-\sigma_2}\tilde{v}_n) & \text{in } \Omega_n, \\ (-\Delta)^\alpha \tilde{v}_n = \lambda_n^{\sigma_2+2\alpha-\sigma_1\beta_{21}} c(z)\tilde{u}_n^{\beta_{21}} + \lambda_n^{\sigma_2+2\alpha-\sigma_2\beta_{22}} b(z)\tilde{v}_n^{\beta_{22}} & \\ \qquad\qquad + \lambda_n^{\sigma_2+2\alpha} h_2(z, \lambda_n^{-\sigma_1}\tilde{u}_n, \lambda_n^{-\sigma_2}\tilde{v}_n) & \text{in } \Omega_n, \end{cases} \tag{3.1}$$

where $z = \lambda_n x + x_n$, and

$$\Omega_n = \{x \in \mathbb{R}^N : \ x_n + \lambda_n x \in \Omega\}.$$

By assumption (A3) we have

$$|\lambda_n^{\sigma_1+2\alpha} h_1(z, \lambda_n^{-\sigma_1}\tilde{u}_n, \lambda_n^{-\sigma_2}\tilde{v}_n)| \leq C(\lambda_n^{\sigma_1+2\alpha} + \lambda_n^{\sigma_1+2\alpha-\sigma_1\gamma_{11}}|\tilde{u}_n|^{\gamma_{11}} + \lambda_n^{\sigma_1+2\alpha-\sigma_2\gamma_{12}}|\tilde{v}_n|^{\gamma_{12}})$$

and a similar estimate for $\lambda_n^{\sigma_2+2\alpha} h_2$.

Case I: weakly coupled. Choosing

$$\sigma_1 = \frac{2\alpha}{\beta_{11}-1} \quad \text{and} \quad \sigma_2 = \frac{2\alpha}{\beta_{22}-1},$$

since (1.3) is a weakly coupled system, we obtain

$$\begin{aligned} \sigma_1 + 2\alpha - \sigma_1\beta_{11} &= 0, \quad \sigma_1 + 2\alpha - \sigma_2\beta_{12} > 0, \\ \sigma_2 + 2\alpha - \sigma_1\beta_{21} &> 0, \quad \sigma_2 + 2\alpha - \sigma_2\beta_{22} = 0. \end{aligned} \tag{3.2}$$

By assumption $\gamma_{ij} < \beta_{ij}$ and (3.2), we also obtain inequalities

$$\begin{aligned}\sigma_1 + 2\alpha - \sigma_1\gamma_{11} &> 0, \quad \sigma_1 + 2\alpha - \sigma_2\gamma_{12} > 0,\\ \sigma_2 + 2\alpha - \sigma_1\gamma_{21} &> 0, \quad \sigma_2 + 2\alpha - \sigma_2\gamma_{22} > 0.\end{aligned}$$

For $x \in \Omega$, we denote $d(x) = \mathrm{dist}(x, \partial\Omega)$. By passing to subsequences, two situations may arise: either $d(x_n)/\lambda_n \to +\infty$ or $d(x_n)/\lambda_n \to d \geq 0$.

We first suppose the case $d(x_n)/\lambda_n \to +\infty$ holds. Then $\Omega_n \to \mathbb{R}^N$ as $n \to +\infty$. Since $\tilde{u}_n$ and $\tilde{v}_n$ are uniformly bounded, by Theorem 2.1 with an application of Ascoli-Arzelá theorem and a diagonal argument, we have $\tilde{u}_n \to u$ and $\tilde{v}_n \to v$ locally uniformly in $\mathbb{R}^N$. Then passing to the limit (use Theorem 2.4), we see that (u, v) solves

$$\begin{cases} (-\Delta)^\alpha u = a(x_0)u^{\beta_{11}} & \text{in } \mathbb{R}^N,\\ (-\Delta)^\alpha v = d(x_0)v^{\beta_{22}} & \text{in } \mathbb{R}^N \end{cases}$$

in the viscosity sense. However, by Theorem 1.2 in [14] (see also [2]), we know this problem has no positive viscosity solutions if

$$1 < \beta_{11}, \beta_{22} < \frac{N+2\alpha}{N-2\alpha}.$$

If the case $d(x_n)/\lambda_n \to d \geq 0$ holds, then we may assume $x_n \to x_0 \in \partial\Omega$. Without loss of generality, we may assume the $\nu(x_0) = -e_N$. In this case we consider functions

$$\bar{u}_n(x) = \lambda_n^{\sigma_1} u_n(\lambda_n x + \xi_n) \quad \text{and} \quad \bar{v}_n(x) = \lambda_n^{\sigma_2} v_n(\lambda_n x + \xi_n) \quad \text{in } D_n,$$

where $\xi_n \in \partial\Omega$ is the projection of x_n on $\partial\Omega$ and

$$D_n = \{x \in \mathbb{R}^N : \xi_n + \lambda_n x \in \Omega\}.$$

Observe that

$$0 \in \partial D_n, \tag{3.3}$$

and

$$D_n \to \mathbb{R}^N_+ = \{x \in \mathbb{R}^n : \ x_N > 0\} \quad \text{as } n \to +\infty.$$

It also follows that $(\bar{u}_n, \bar{v}_n)$ satisfying (3.1) in D_n with a slightly different functions h_1 and h_2, but with same bounds.

Furthermore, let

$$\tilde{x}_n = \frac{x_n - \xi_n}{\lambda_n},$$

we have $|\tilde{x}_n| = d(x_n)/\lambda_n$ and $\bar{u}_n(\tilde{x}_n) = 1$. We claim that

$$d := \lim_{n\to+\infty} \frac{d(x_n)}{\lambda_n} > 0.$$

This is in particular guarantees that by passing to a subsequence $x_n \to x_0$, where $|x_0| = d > 0$. Therefore, x_0 is in the interior of the half space $\mathbb{R}^N_+$. Now we are in the position to prove the claim. Observe that by (3.1), we have

$$(-\Delta)^\alpha \bar{u}_n \le C \le \tilde{C} d_n^{-\theta} \quad \text{in } D_n,$$

where $\theta \in (\alpha, 2\alpha)$ and $d_n = \mathrm{dist}(x, \partial D_n)$. By Lemma 2.3, for fixed θ, there exists a constant $C_0 > 0$ and $\delta > 0$ such that $\bar{u}_n(x) \le C_0 d_n(x)^{2\alpha-\theta}$ if $d_n(x) < \delta$. In particular, by (3.3), we know $|\tilde{x}_n| \ge d_n(\tilde{x}_n)$. Therefore, if $d_n(\tilde{x}_n) < \delta$, then $1 = \bar{u}_n(\tilde{x}_n) \le C_0 d_n(\tilde{x}_n)^{2\alpha-\theta} \le C_0|\tilde{x}_n|^{2\alpha-\theta}$. This implies $|\tilde{x}_n|$ is bounded from blow and thus $d > 0$.

Now we can employ regularity Theorem 2.4 as before to obtain that $\bar{u}_n \to u$ and $\bar{v}_n \to v$ on compact sets of $\mathbb{R}^N_+$, where (u, v) verifies that $0 \le u, v \le 1$ in $\mathbb{R}^N_+$, $u(x_0) = 1$ and solves

$$\begin{cases} (-\Delta)^\alpha u = a(x_0) u^{\beta_{11}} & \text{in } \mathbb{R}^N_+, \\ (-\Delta)^\alpha v = d(x_0) v^{\beta_{22}} & \text{in } \mathbb{R}^N_+, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+, \end{cases}$$

in viscosity sense. However, by Theorems 1.1 in [14], we know the above problem have no positive viscosity solutions if

$$1 < \beta_{11}, \beta_{22} < \frac{(N-1)+2\alpha}{(N-1)-2\alpha}.$$

This contradicts with our assumption since $\frac{(N-1)+2\alpha}{(N-1)-2\alpha} > \frac{N+2\alpha}{N-2\alpha}$ if $N > 1 + 2\alpha$.

Case II: strongly coupled. Choosing

$$\sigma_1 = \frac{2\alpha(\beta_{12}+1)}{\beta_{12}\beta_{21}-1} \quad \text{and} \quad \sigma_2 = \frac{2\alpha(\beta_{21}+1)}{\beta_{12}\beta_{21}-1}.$$

Since the system (1.3) is strongly coupled, we obtain

$$\begin{aligned} \sigma_1 + 2\alpha - \sigma_1\beta_{11} &> 0, \quad \sigma_1 + 2\alpha - \sigma_2\beta_{12} = 0, \\ \sigma_2 + 2\alpha - \sigma_1\beta_{21} &= 0, \quad \sigma_2 + 2\alpha - \sigma_2\beta_{22} > 0. \end{aligned} \tag{3.4}$$

and assumption $\gamma_{ij} < \beta_{ij}$ implies

$$\begin{aligned} \sigma_1 + 2\alpha - \sigma_1\gamma_{11} &> 0, \quad \sigma_1 + 2\alpha - \sigma_2\gamma_{12} > 0, \\ \sigma_2 + 2\alpha - \sigma_1\gamma_{21} &> 0, \quad \sigma_2 + 2\alpha - \sigma_2\gamma_{22} > 0. \end{aligned}$$

By a similar argument as before, we know the limit system of (3.1) is

$$\begin{cases} (-\Delta)^\alpha u = b(x_0)v^{\beta_{12}} \\ (-\Delta)^\alpha v = c(x_0)v^{\beta_{21}} \end{cases}$$

in $\mathbb{R}^N$ or $\mathbb{R}^N_+$ with $u = v = 0$ in $\mathbb{R}^N \setminus \mathbb{R}^N_+$. Therefore, by Liouville type results in $\mathbb{R}^N$ (see Theorems 2.5 and 2.6) and $\mathbb{R}^N_+$ (see Theorem 4.2 in [14]) and regularity results (cf. [16]), we come to a contradiction as before if (1.7) holds. We complete the prove. □

Next, we prove a priori bound for system (1.8). Firstly, we obtain rough bounds for all solutions of the system (1.8) which are universal, in the spirit of [12].

Lemma 3.1 *Under assumptions in Theorem 1.2, assume that positive function $u, v \in C^1(\Omega) \cap L^\infty(\mathbb{R}^N)$ satisfying*

$$\begin{cases} (-\Delta)^\alpha u = a(x)u^{\beta_{11}} + b(x)v^{\beta_{12}} + h_1(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \\ (-\Delta)^\alpha v = c(x)u^{\beta_{21}} + c(x)v^{\beta_{22}} + h_2(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \end{cases} \tag{3.5}$$

in the viscosity sense, then there exists a positive constant C such that

$$u(x) \le C(1 + \operatorname{dist}(x, \partial\Omega)^{-\sigma_1}), \quad |\nabla u(x)| \le C(1 + \operatorname{dist}(x, \partial\Omega)^{-(\sigma_1+1)}),$$

$$v(x) \le C(1 + \operatorname{dist}(x, \partial\Omega)^{-\sigma_2}), \quad |\nabla v(x)| \le C(1 + \operatorname{dist}(x, \partial\Omega)^{-(\sigma_2+1)}),$$

for $x \in \Omega$, where

$$\sigma_1 = \frac{2\alpha}{\beta_{11} - 1} \quad \text{and} \quad \sigma_2 = \frac{2\alpha}{\beta_{22} - 1},$$

verifies (1.9) in weakly coupled case (see (1.4)), and

$$\sigma_1 = \frac{2\alpha(\beta_{12} + 1)}{\beta_{12}\beta_{21} - 1} \quad \text{and} \quad \sigma_2 = \frac{2\alpha(\beta_{21} + 1)}{\beta_{12}\beta_{21} - 1}.$$

verifies (1.10) in strongly coupled case (see (1.5)).

Proof. Assume that the Lemma fails. Then, there exist sequences of positive function $u_n, v_n \in C^1(\Omega) \cap L^\infty(\mathbb{R}^N)$ and $y_n \in \Omega$ satisfying

$$\begin{cases} (-\Delta)^\alpha u_n = a(x)u_n^{\beta_{11}} + b(x)v_n^{\beta_{12}} + h_1(x, u_n, v_n, \nabla u_n, \nabla v_n) & \text{in } \Omega, \\ (-\Delta)^\alpha v_n = c(x)u_n^{\beta_{21}} + c(x)v_n^{\beta_{22}} + h_2(x, u_n, v_n, \nabla u_n, \nabla v_n) & \text{in } \Omega \end{cases}$$

and

$$M_n := u_n^{\frac{1}{\beta_1}} + |\nabla u_n|^{\frac{1}{\beta_1+1}} + v_n^{\frac{1}{\beta_2}} + |\nabla v_n|^{\frac{1}{\beta_2+1}}$$

satisfies

$$M_n(y_n) > 2n(1 + \mathrm{dist}(y_n, \partial\Omega)^{-1}). \tag{3.6}$$

By Lemma 5.1 in [12], there exists a sequence of points $x_n \in \Omega$ with the property that $M_n(x_n) \geq M_n(y_n)$, $M_n(x_n) > 2n\mathrm{dist}(x_n, \partial\Omega)^{-1}$ and

$$M_n(z) \leq 2M_n(x_n) \quad \text{in } B(x_n, nM_n(x_n)^{-1}). \tag{3.7}$$

Observe that (3.6) implies $M_n(x_n) \to +\infty$. Let $\lambda_n = M_n(x_n)^{-1} \to 0$ and define

$$\tilde{u}_n(x) = \lambda_n^{\sigma_1} u_n(x_n + \lambda_n x) \quad \text{and} \quad \tilde{v}_n(x) = \lambda_n^{\sigma_2} v_n(x_n + \lambda_n x)$$

in $B_n := \{x \in \mathbb{R}^N : \ |x| < n\}$. Then function $(\tilde{u}_n, \tilde{v}_n)$ satisfies

$$\begin{cases} (-\Delta)^\alpha \tilde{u}_n = \lambda_n^{\sigma_1+2\alpha-\sigma_1\beta_{11}} a(z)\tilde{u}_n^{\beta_{11}} + \lambda_n^{\sigma_1+2\alpha-\sigma_2\beta_{12}} b(z)\tilde{v}_n^{\beta_{12}} \\ \qquad +\lambda_n^{\sigma_1+2\alpha} h_1(z, \lambda_n^{-\sigma_1}\tilde{u}_n, \lambda_n^{-\sigma_2}\tilde{v}_n, \lambda_n^{-(\sigma_1+1)}\nabla\tilde{u}_n, \lambda_n^{-(\sigma_2+1)}\nabla\tilde{v}_n) \quad \text{in } B_n, \\ (-\Delta)^\alpha \tilde{v}_n = \lambda_n^{\sigma_2+2\alpha-\sigma_1\beta_{21}} c(z)\tilde{u}_n^{\beta_{21}} + \lambda_n^{\sigma_2+2\alpha-\sigma_2\beta_{22}} b(z)\tilde{v}_n^{\beta_{22}} \\ \qquad +\lambda_n^{\sigma_2+2\alpha} h_2(z, \lambda_n^{-\sigma_1}\tilde{u}_n, \lambda_n^{-\sigma_2}\tilde{v}_n, \lambda_n^{-(\sigma_1+1)}\nabla\tilde{u}_n, \lambda_n^{-(\sigma_2+1)}\nabla\tilde{v}_n) \quad \text{in } B_n, \end{cases} \tag{3.8}$$

where $z = \lambda_n x + x_n$. By assumption (A4) we have

$$\begin{aligned} &|\lambda_n^{\sigma_1+2\alpha} h_1(z, \lambda_n^{-\sigma_1}\tilde{u}_n, \lambda_n^{-\sigma_2}\tilde{v}_n, \lambda_n^{-(\sigma_1+1)}\nabla\tilde{u}_n, \lambda_n^{-(\sigma_2+1)}\nabla\tilde{v}_n)| \\ &\quad \leq C(\lambda_n^{\sigma_1+2\alpha} + \lambda_n^{\sigma_1+2\alpha-\sigma_1\gamma_{11}}|\tilde{u}_n|^{\gamma_{11}} + \lambda_n^{\sigma_1+2\alpha-\sigma_2\gamma_{12}}|\tilde{v}_n|^{\gamma_{12}} \\ &\qquad +\lambda_n^{\sigma_1+2\alpha-(\sigma_1+1)\theta_{11}}|\nabla\tilde{u}_n|^{\theta_{11}} + \lambda_n^{\sigma_1+2\alpha-(\sigma_2+1)\theta_{12}}|\nabla\tilde{v}_n|^{\theta_{12}}) \end{aligned}$$

and a similar estimate for $\lambda_n^{\sigma_2+2\alpha} h_2$.

Case I: weakly coupled. Choosing

$$\sigma_1 = \frac{2\alpha}{\beta_{11} - 1} \quad \text{and} \quad \sigma_2 = \frac{2\alpha}{\beta_{22} - 1},$$

since (3.5) is a weakly coupled system, we obtain

$$\begin{aligned} \sigma_1 + 2\alpha - \sigma_1\beta_{11} &= 0, \quad \sigma_1 + 2\alpha - \sigma_2\beta_{12} > 0, \\ \sigma_2 + 2\alpha - \sigma_1\beta_{21} &> 0, \quad \sigma_2 + 2\alpha - \sigma_2\beta_{22} = 0. \end{aligned}$$

By assumption $\gamma_{ij} < \beta_{ij}$, we also obtain inequalities

$$\begin{aligned} \sigma_1 + 2\alpha - \sigma_1\gamma_{11} &> 0, \quad \sigma_1 + 2\alpha - \sigma_2\gamma_{12} > 0, \\ \sigma_2 + 2\alpha - \sigma_1\gamma_{21} &> 0, \quad \sigma_2 + 2\alpha - \sigma_2\gamma_{22} > 0. \end{aligned}$$

Using (A5), we have

$$\begin{aligned} \sigma_1 + 2\alpha - (\sigma_1 + 1)\theta_{11} &> 0, \quad \sigma_1 + 2\alpha - (\sigma_2 + 1)\theta_{12} > 0, \\ \sigma_2 + 2\alpha - (\sigma_1 + 1)\theta_{21} &> 0, \quad \sigma_2 + 2\alpha - (\sigma_2 + 1)\theta_{22} > 0. \end{aligned}$$

Moreover, by (3.7),

$$u_n(x)^{\frac{1}{\beta_1}} + |\nabla u_n(x)|^{\frac{1}{\beta_1+1}} + v_n(x)^{\frac{1}{\beta_2}} + |\nabla v_n(x)|^{\frac{1}{\beta_2+1}} \le 2, \quad x \in B_n.$$

We also know that

$$u_n(0)^{\frac{1}{\beta_1}} + |\nabla u_n(0)|^{\frac{1}{\beta_1+1}} + v_n(0)^{\frac{1}{\beta_2}} + |\nabla v_n(0)|^{\frac{1}{\beta_2+1}} = 1.$$

Since $\lambda_n \to 0$, $\tilde{u}_n$, $\tilde{v}_n$, $|\nabla \tilde{u}_n|$ and $|\nabla \tilde{v}_n|$ are uniformly bounded in B_n, by Theorem 2.3 to obtain, with the help of Ascoli-Arzelá's theorem and a diagonal argument, that there exists a subsequence, still denoted $(\tilde{u}_n, \tilde{v}_n)$ such that $\tilde{u}_n \to u$ and $\tilde{v}_n \to v$ in $C^1_{loc}(\mathbb{R}^N)$ as $n \to +\infty$. Since $u(0)^{\frac{1}{\beta_1}} + |\nabla u(0)|^{\frac{1}{\beta_1+1}} + v(0)^{\frac{1}{\beta_2}} + |\nabla v(0)|^{\frac{1}{\beta_2+1}} = 1$, then $(u, v) \neq (0, 0)$.

Next, let $(\bar{u}_n, \bar{v}_n)$ be the functions obtained by extending $(\tilde{u}_n, \tilde{v}_n)$ to be zero outsider B_n. Then we can check that $(\bar{u}_n, \bar{v}_n)$ satisfying

$$\begin{cases} (-\Delta)^\alpha \bar{u}_n \ge a(z)\bar{u}_n^{\beta_{11}} & \text{in } B_n, \\ (-\Delta)^\alpha \bar{v}_n \ge d(z)\bar{v}_n^{\beta_{22}} & \text{in } B_n. \end{cases}$$

Passing the limit by using Theorem 2.4, we have

$$\begin{cases} (-\Delta)^\alpha u \ge a(x_0)u^{\beta_{11}} & \text{in } \mathbb{R}^N, \\ (-\Delta)^\alpha v \ge d(x_0)v^{\beta_{22}} & \text{in } \mathbb{R}^N, \end{cases}$$

which contradicts Theorem 1.3 in [6] since (1.9).

Case II: strongly coupled. Choosing

$$\sigma_1 = \frac{2\alpha(\beta_{12}+1)}{\beta_{12}\beta_{21}-1} \quad \text{and} \quad \sigma_2 = \frac{2\alpha(\beta_{21}+1)}{\beta_{12}\beta_{21}-1}.$$

Since the system (3.5) is strongly coupled, we obtain

$$\begin{aligned} \sigma_1 + 2\alpha - \sigma_1\beta_{11} &> 0, \quad \sigma_1 + 2\alpha - \sigma_2\beta_{12} = 0, \\ \sigma_2 + 2\alpha - \sigma_1\beta_{21} &= 0, \quad \sigma_2 + 2\alpha - \sigma_2\beta_{22} > 0. \end{aligned} \tag{3.9}$$

and assumption $\gamma_{ij} < \beta_{ij}$ implies

$$\begin{aligned} \sigma_1 + 2\alpha - \sigma_1\gamma_{11} &> 0, \quad \sigma_1 + 2\alpha - \sigma_2\gamma_{12} > 0, \\ \sigma_2 + 2\alpha - \sigma_1\gamma_{21} &> 0, \quad \sigma_2 + 2\alpha - \sigma_2\gamma_{22} > 0. \end{aligned}$$

Using (A6), we have

$$\begin{aligned} \sigma_1 + 2\alpha - (\sigma_1+1)\theta_{11} &> 0, \quad \sigma_1 + 2\alpha - (\sigma_2+1)\theta_{12} > 0, \\ \sigma_2 + 2\alpha - (\sigma_1+1)\theta_{21} &> 0, \quad \sigma_2 + 2\alpha - (\sigma_2+1)\theta_{22} > 0. \end{aligned}$$

By a similar argument as before, we know the limit system of (3.5) is

$$\begin{cases} (-\Delta)^\alpha u \geq b(x_0) v^{\beta_{12}} \\ (-\Delta)^\alpha v \geq c(x_0) v^{\beta_{21}} \end{cases}$$

in $\mathbb{R}^N$. Therefore, by Liouville type results Theorem 2.6, we come to a contradiction as before if (1.10) holds. We complete the prove. □

Remark 3.1 *We except Lemma 3.1 to hold in a large range such as (1.6) and (1.7). Unfortunately, this method of proof seems purely local and needs to be properly adapted to del with non-local problems. Notice that there is no information for the functions $\tilde{u}_n$ and $\tilde{v}_n$ in $\mathbb{R}^N \setminus \Omega_n$, which leads difficult to pass limit appropriately in the system satisfies by $(\tilde{u}_n, \tilde{v}_n)$.*

Now, we are in position to obtain a priori bounds of system (1.8). We have already remarked that due to the expected singularity of the gradient of solutions near boundary we need to work in spaces with weights which take care of the singularity. Hence, we fix s satisfying

$$0 < s < 1 - \frac{\alpha}{\theta_{ij}} < 1, \tag{3.10}$$

where $\theta_{ij}(i, j = 1, 2)$ satisfying (A5) or (A6). Let

$$E = \{(u, v) \in C(\Omega) \times C(\Omega) : \ \|u\|_1^{(-s)} < +\infty, \quad \|v\|_1^{(-s)} < +\infty\}, \tag{3.11}$$

where norm $\|\cdot\|_1^{(\tau)}$ is given as in (2.1) with $\tau = -s$.

Then we can prove

Theorem 3.2 *Under the hypotheses of Theorem 1.2, then there exists a constant $C > 0$ such that for each couple of positive viscosity solutions (u, v) of (1.8) in E with s satisfying (3.10), we have*

$$\|u\|_1^{(-s)}, \|v\|_1^{(-s)} < C.$$

Proof. Assume that the conclusion of the theorem fails. Then there exists a sequence of positive solutions $(u_n, v_n) \in E$ such that

$$\max\{\|u_n\|_1^{(-s)}, \|v_n\|_1^{(-s)}\} \to +\infty$$

as $n \to +\infty$. We may assume

$$\lambda_n = \left(\|u_n\|_1^{(-s)}\right)^{-1/(\sigma_1+s)},$$

if $\left(\|u_n\|_1^{(-s)}\right)^{\sigma_2+s} \geq \left(\|v_n\|_1^{(-s)}\right)^{\sigma_1+s}$ (up to a sequence), and

$$\lambda_n = \left(\|v_n\|_1^{(-s)}\right)^{-1/(\sigma_2+s)}$$

otherwise, for some constants $\sigma_1, \sigma_2 > 0$ which are to be determined later. Without loss of generality, we suppose that we are in the first of these two situations.

Define

$$M_n(x) = d(x)^{-s}u_n(x) + d(x)^{1-s}|\nabla u_n(x)|.$$

Next, we choosing point $x_n \in \Omega$ such that $M_n(x_n) \geq \sup_\Omega M_n - \frac{1}{n}$ (the supremum may not be achieved). By our assumptions, we know $M_n(x_n) \to +\infty$. Let ξ_n be a projection of x_n on $\partial\Omega$ and let

$$\tilde{u}_n(x) = \lambda_n^{\sigma_1} u_n(\xi_n + \lambda_n x) \quad \text{and} \quad \tilde{v}_n(x) = \lambda_n^{\sigma_2} v_n(\xi_n + \lambda_n x)$$

in $D_n = \{x \in \mathbb{R}^N : \ \xi_n + \lambda_n x \in \Omega\}$. Then function $(\tilde{u}_n, \tilde{v}_n)$ satisfies

$$\begin{cases} (-\Delta)^\alpha \tilde{u}_n = \lambda_n^{\sigma_1+2\alpha-\sigma_1\beta_{11}} a(z)\tilde{u}_n^{\beta_{11}} + \lambda_n^{\sigma_1+2\alpha-\sigma_2\beta_{12}} b(z)\tilde{v}_n^{\beta_{12}} \\ \qquad +\lambda_n^{\sigma_1+2\alpha} h_1(z, \lambda_n^{-\sigma_1}\tilde{u}_n, \lambda_n^{-\sigma_2}\tilde{v}_n, \lambda_n^{-(\sigma_1+1)}\nabla\tilde{u}_n, \lambda_n^{-(\sigma_2+1)}\nabla\tilde{v}_n) \quad \text{in } D_n \\ (-\Delta)^\alpha \tilde{v}_n = \lambda_n^{\sigma_2+2\alpha-\sigma_1\beta_{21}} c(z)\tilde{u}_n^{\beta_{21}} + \lambda_n^{\sigma_2+2\alpha-\sigma_2\beta_{22}} b(z)\tilde{v}_n^{\beta_{22}} \\ \qquad +\lambda_n^{\sigma_2+2\alpha} h_2(z, \lambda_n^{-\sigma_1}\tilde{u}_n, \lambda_n^{-\sigma_2}\tilde{v}_n, \lambda_n^{-(\sigma_1+1)}\nabla\tilde{u}_n, \lambda_n^{-(\sigma_2+1)}\nabla\tilde{v}_n) \quad \text{in } D_n, \end{cases} \tag{3.12}$$

where $z = \lambda_n x + \xi_n$. By assumption (A4) we have

$$\begin{aligned} &|\lambda_n^{\sigma_1+2\alpha} h_1(z, \lambda_n^{-\sigma_1}\tilde{u}_n, \lambda_n^{-\sigma_2}\tilde{v}_n, \lambda_n^{-(\sigma_1+1)}\nabla\tilde{u}_n, \lambda_n^{-(\sigma_2+1)}\nabla\tilde{v}_n)| \\ &\quad \leq C(\lambda_n^{\sigma_1+2\alpha} + \lambda_n^{\sigma_1+2\alpha-\sigma_1\gamma_{11}}|\tilde{u}_n|^{\gamma_{11}} + \lambda_n^{\sigma_1+2\alpha-\sigma_2\gamma_{12}}|\tilde{v}_n|^{\gamma_{12}} \\ &\qquad +\lambda_n^{\sigma_1+2\alpha-(\sigma_1+1)\theta_{11}}|\nabla\tilde{u}_n|^{\theta_{11}} + \lambda_n^{\sigma_1+2\alpha-(\sigma_2+1)\theta_{12}}|\nabla\tilde{v}_n|^{\theta_{12}}) \end{aligned}$$

and a similar estimate for $\lambda_n^{\sigma_2+2\alpha}h_2$. Moreover, $(\tilde{u}_n, \tilde{v}_n)$ satisfying

$$\lambda_n^s d(\xi_n + \lambda_n x)^{-s}\tilde{u}_n(x) + \lambda_n^{s-1} d(\xi_n + \lambda_n x)^{1-s}|\nabla\tilde{u}_n(x)| = \frac{M_n(\xi_n + \lambda_n x)}{M_n(x_n)}$$

and

$$\lambda_n^s d(\xi_n + \lambda_n x)^{-s}\tilde{v}_n(x) + \lambda_n^{s-1} d(\xi_n + \lambda_n x)^{1-s}|\nabla\tilde{v}_n(x)| \leq \lambda_n^{\sigma_2+t}\|v_n\|_1^{-s}.$$

Then, using the fact $\lambda_n^{-1}d(\xi_n + \lambda_n x) = dist(x, \partial D_n) =: d_n(x)$ and the choice of the points x_n, we obtain for large n

$$d_n(x)^{-s}\tilde{u}_n(x) + d_n(x)^{1-s}|\nabla\tilde{u}_n(x)| \leq 2 \quad \text{in } D_n \tag{3.13}$$

and

$$d_n(x)^{-s}\tilde{v}_n(x) + d_n(x)^{1-s}|\nabla \tilde{v}_n(x)| \le 1 \quad \text{in } D_n \tag{3.14}$$

and moreover we know

$$d_n(y_n)^{-s}\tilde{u}_n(y_n) + d_n(y_n)^{1-s}|\nabla \tilde{u}_n(y_n)| = 1, \tag{3.15}$$

where $y_n = \frac{x_n - \xi_n}{\lambda_n}$.

Next, since (u_n, v_n) solves system (1.1), we can use Lemma 3.1 to obtain that

$$M_n(x_n) \le C d(x_n)^{-s}(1 + d(x_n)^{-\sigma_1})$$

for some positive constant C independent of n. This implies that $d(x_n)\lambda_n^{-1} \le C$. This bound entails that, passing to subsequence, $x_n \to x_0 \in \partial\Omega$ and $|y_k| = d(x_n)\lambda_n^{-1} \to d \ge 0$ (in particular the points ξ_n are uniquely determined at least for large n). Assuming that the outward unit normal to $\partial\Omega$ at x_0 is $-e_N$, we also obtain that $D_n \to \mathbb{R}^N_+$ as $n \to +\infty$.

We claim that $d > 0$. To show this, notice that from (3.13) and (3.14) we have

$$(-\Delta)^\alpha \tilde{u}_n \le C d_k^{\min\{(s-1)\theta_{11},(s-1)\theta_{12}\}}$$

in D_n, for some constant independent of n. By our choice of s and θ_{ij} $(i, j = 1, 2)$, we have that

$$s > \frac{\theta_{ij} - 2\alpha}{\theta_{ij}} \tag{3.16}$$

since $1 < \theta_{ij} < 2\alpha$. Together with (3.10), we see that

$$\alpha < \max\{(1-s)\theta_{11}, (1-s)\theta_{12}\} < 2\alpha. \tag{3.17}$$

Thus, by Lemma 2.3, we can obtain that

$$\tilde{u}_n(x) \le C d_n(x)^{2\alpha+\min\{(s-1)\theta_{11},(s-1)\theta_{12}\}}, \quad \text{when } d_n(x) < \delta. \tag{3.18}$$

Furthermore, since $1 < \theta_{ij} < 2\alpha$ for $i, j = 1, 2$,

$$s > \frac{\theta_{ij} - 2\alpha}{\theta_{ij} - 1}. \tag{3.19}$$

Hence, $-s + 2\alpha + (s-1)\theta_{ij} = s(\theta_{ij} - 1) + 2\alpha - \theta_{ij} > 0$. So, by (3.13), we have

$$\tilde{u}_n(x) \le 2d_n(x)^s \le \delta^{s-2\alpha-\min\{(s-1)\theta_{11},(s-1)\theta_{12}\}} d_n(x)^{2\alpha+\min\{(s-1)\theta_{11},(s-1)\theta_{12}\}}$$

when $d_n(x) \ge \delta$. Hence $\|\tilde{u}_n\|_0^{(-2\alpha-\min\{(s-1)\theta_{11},(s-1)\theta_{12}\})}$ is bounded. Then we can use Lemma 2.2 with $(\tau = -\min\{(s-1)\theta_{11}, (s-1)\theta_{12}\})$ to obtain that

$$|\nabla \tilde{u}_n(x)| \le C d_n(x)^{2\alpha+\min\{(s-1)\theta_{11},(s-1)\theta_{12}\}-1} \quad \text{in } D_n, \tag{3.20}$$

where C is also independent of n.

Taking (3.18) and (3.20) in (3.15), we deduce that

$$1 \le C d_n(y_n)^{2\alpha+\min\{(s-1)\theta_{11},(s-1)\theta_{12}\}-s}.$$

where C is also independent of n. This implies that $d_n(y_n)$ is bounded away from zero. Hence, $|y_n|$ is also since $0 \in \partial D_n$. So that $d > 0$ as claimed.

Finally, we can use Theorem 2.3 together with Ascoli-Arzelá's theorem and a diagonal argument to obtain that $\tilde{u}_n \to u$ and $\tilde{v}_n \to v$ in $C^1_{loc}(\mathbb{R}^N_+)$. Moreover, by (3.15), we have

$$d^{-s}u(y_0) + d^{1-s}|\nabla u(y_0)| = 1$$

for some $y_0 \in \mathbb{R}^N_+$. Hence, $(u, v) \neq (0, 0)$ and $u(x) \le Cx_N^{2\alpha+\min\{(s-1)\theta_{11},(s-1)\theta_{12}}$ and $v(x) \le Cx_N^{2\alpha+\min\{(s-1)\theta_{12},(s-1)\theta_{22}}$ if $0 < x_N < \delta$. Thus $u, v \in C(\mathbb{R}^N)$ and $u = v = 0$ in $\mathbb{R}^N \backslash \mathbb{R}^N_+$. Passing to the limit in (3.12) with the help of Theorem 2.4, we obtain

$$\begin{cases} (-\Delta)^\alpha u = a(x_0)u^{\beta_{11}} & \text{in } \mathbb{R}^N_+, \\ (-\Delta)^\alpha v = d(x_0)v^{\beta_{22}} & \text{in } \mathbb{R}^N_+, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+ \end{cases}$$

in weakly coupled case and

$$\begin{cases} (-\Delta)^\alpha u = b(x_0)v^{\beta_{12}} & \text{in } \mathbb{R}^N_+, \\ (-\Delta)^\alpha v = c(x_0)u^{\beta_{21}} & \text{in } \mathbb{R}^N_+, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+ \end{cases}$$

in strongly coupled case. By Lemma 3.1, we also know $u(x) \le Cx_N^{-\sigma_1}$ and $v(x) \le Cx_N^{-\sigma_2}$ in $\mathbb{R}^N_+$ and thus u and v are bounded. By Theorems 2.1 and 2.2, we know (u, v) is a classical solution. By the strong maximum principle, we have that $u > 0$ and $v > 0$ in $\mathbb{R}^N_+$. This is contradiction with Theorem 1.2 in [14] for scalar case and Theorem 2.6 and Theorem 4.2 in [14] for system. We complete the proof. □

4 Existence results

This section is devoted to prove our existence results, Theorems 1.1 and 1.2. Both proofs are very similar, only that of Theorem 1.2 is slightly more involved. Hence, we only prove Theorem 1.2 here.

We assume $1/2 < \alpha < 1$. Fix s verifying (3.10) and consider the Banach space E (see (3.11)) with norm

$$\|(u,v)\|_E = \max\{\|u\|_1^{(-s)}, \|v\|_1^{(-s)}\},$$

which is an ordered Banach space with the cone of nonnegative functions

$$K = \{(u,v) \in E : \ u \geq 0 \text{ and } v \geq 0 \text{ in } \Omega\}.$$

Observe that for every $(u,v) \in K$ we have

$$\begin{array}{rcl} h_1(x,u,v,\nabla u,\nabla v) & \leq & Cd(x)^{\min\{(s-1)\beta_{11},(s-1)\beta_{12}\}} \\ h_2(x,u,v,\nabla u,\nabla v) & \leq & Cd(x)^{\min\{(s-1)\beta_{12},(s-1)\beta_{22}\}}, \end{array}$$

where positive constant C depending on the norms $\|u\|_1^{(-s)}$ and $\|v\|_1^{(-s)}$. Moreover, as in the proof of Theorem 3.2, we know

$$\alpha < (s-1)\beta_{ij} < 2\alpha, \quad i,j = 1,2.$$

Hence, applying Lemma 2.1 to system

$$\begin{cases} (-\Delta)^\alpha u = a(x)\tilde{u}^{\beta_{11}} + b(x)\tilde{v}^{\beta_{12}} + h_1(x,\tilde{u},\tilde{v},\nabla\tilde{u},\nabla\tilde{v}) & \text{in } \Omega, \\ (-\Delta)^\alpha v = c(x)\tilde{u}^{\beta_{12}} + d(x)\tilde{v}^{\beta_{22}} + h_2(x,\tilde{u},\tilde{v},\nabla\tilde{u},\nabla\tilde{v}) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (4.1)$$

where a,b,c,d satisfying (A1) and h_1,h_2 satisfying (A4), then system (4.1) has a unique nonnegative solution (u,v) with $\|u\|_0^{(-s)} < +\infty$ and $\|v\|_0^{(-s)} < +\infty$.Therefore, $(u,v) \in E$. So we define an operator $\Phi : \ K \to K$ by means of $(u,v) = \Phi(\tilde{u},\tilde{v})$. It is clear that nonnegative solutions of (1.1) in E coincide with the fixed points of the operator.

We first show the basic property of operator Φ.

Lemma 4.1 *The operator* $\Phi : \ K \to K$ *is compact.*

Proof. We just need to do a slight modification of the proof of Lemma 11 in [1] and thus we omit it here. □

The proof of our existence result is an application of degree theory for compact operators in cones. We start by recalling the following well-known result (cf. Theorem 3.6.3 in [11]).

Theorem 4.1 *Let E is an ordered Banach space with positive cone K, and $U \subset K$ is an open bounded set containing 0. Let $r > 0$ such that $B_r(0) \cap K \subset U$. Assume that $\Phi : U \to K$ is compact and satisfies*
(i): for every $\mu \in [0, 1)$, we have $u \neq \mu\Phi(u)$ for every $u \in K$ with $\|u\| = r$;
(ii): there exists $\phi \in K \setminus \{0\}$ such that $u - \Phi(u) = \rho\phi$, for every $u \in \partial U$ and every $\rho \geq 0$.

Then Φ has a fixed point in $U \setminus B_r(0)$.

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We will show Theorem 4.1 is applicable to the operator Φ in $K \subset E$.

We first check first hypothesis (i) in Theorem 4.1. Assume we have $(u, v) = \mu\Phi(u, v)$ for some $\mu \in [0, 1)$ and $(u, v) \in K$. This equivalent

$$\begin{cases} (-\Delta)^\alpha u = \mu(a(x)u^{\beta_{11}} + b(x)v^{\beta_{12}} + h_1(x, u, v, \nabla u, \nabla v)) & \text{in } \Omega, \\ (-\Delta)^\alpha v = \mu(c(x)u^{\beta_{12}} + d(x)v^{\beta_{22}} + h_2(x, u, v, \nabla u, \nabla v)) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By our assumptions (A1) and (A4), we get that the right hand side of the above system can be bounded by

$$\mu(a(x)u^{\beta_{11}} + b(x)v^{\beta_{12}} + h_1(x, u, v, \nabla u, \nabla v)) \leq$$
$$Cd^{\min\{(s-1)\theta_{11},(s-1)\theta_{12}\}} \left(\|w\|_E^{\beta_{11}} + \|w\|_E^{\beta_{12}} + \|w\|_E^{\gamma_{11}} + \|w\|_E^{\gamma_{12}} + \|w\|_E^{\theta_{11}} + \|w\|_E^{\theta_{12}} \right).$$

where $w = (u, v)$ here and what follows and a similar estimate for $\mu(c(x)u^{\beta_{12}} + d(x)v^{\beta_{22}} + h_2(x, u, v, \nabla u, \nabla v))$.

Hence, by Lemmas 2.1 and 2.2 and $\alpha < (1 - s)\theta_{ij} < 2\alpha$ for $i, j = 1, 2$, we have

$$\|w\|_E \leq C\Sigma_{i,j=1,2} \left(\|w\|_E^{\beta_{ij}} + \|w\|_E^{\beta_{ij}} + \|w\|_E^{\gamma_{ij}} + \|w\|_E^{\gamma_{ij}} + \|w\|_E^{\theta_{ij}} + \|w\|_E^{\theta_{ij}} \right),$$

here we have used the fact $s - 2\alpha < (s - 1)\theta_{ij} < 0$ $(i, j = 1, 2)$. Since $\beta_{ij}, \gamma_{ij}, \theta_{ij} > 1$ for $i, j = 1, 2$, this implies that $\|w\|_E \geq r$ for some small positive $r > 0$. Thus, there are no positive solution of $(u, v) = \mu\Phi(u, v)$ if $\|(u, v)\|_E = r$ and $\mu \in (0, 1)$, and (i) follows.

Next, we check (ii). We take $\phi \in K$ to be the unique solution (cf. Theorem 3.1 in [7]) of the problem

$$\begin{cases} (-\Delta)^\alpha \phi = 1 & \text{in } \Omega, \\ \phi = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We claim that there are no solutions in K of the equation $(u, v) - \Phi(u, v) = \rho(\phi, \phi)$ if ρ is large enough. For this purpose, we note that this equation equivalent to

$$\begin{cases} (-\Delta)^\alpha u = a(x)u^{\beta_{11}} + b(x)v^{\beta_{12}} + h_1(x, u, v, \nabla u, \nabla v) + \rho & \text{in } \Omega, \\ (-\Delta)^\alpha v = c(x)u^{\beta_{12}} + d(x)v^{\beta_{22}} + h_2(x, u, v, \nabla u, \nabla v) + \rho & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.2}$$

Fix $\mu_1, \mu_2 > \lambda_1$, where λ_1 is

$$\lambda_1 = \sup\{\lambda \in \mathbb{R} : \text{there exists } u \in C(\mathbb{R}^N),\ u > 0 \text{ in } \Omega \text{ with } u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ such that } (-\Delta)^\alpha u \geq \lambda u \text{ in } \Omega\}.$$

By Lemma 13 in [1], we know $\lambda_1 < +\infty$.

Since $\theta_{ij} > 1$ for $i, j = 1, 2$ and h_1, h_2 are nonnegative, there exists positive constants C_1 and C_2 such that we have either (Case I)

$$\begin{aligned} (-\Delta)^\alpha u &\geq \mu_1 u - C_1 + \rho, \\ (-\Delta)^\alpha v &\geq \mu_2 v - C_2 + \rho, \end{aligned}$$

in Ω, or (Case II)

$$\begin{aligned} (-\Delta)^\alpha u &\geq \mu_1 v - C_1 + \rho, \\ (-\Delta)^\alpha v &\geq \mu_2 u - C_2 + \rho, \end{aligned}$$

in Ω. For Case I, we can take a similar argument as the proof in [1] since $\mu_1, \mu_2 > \lambda_1$ and choose $\rho \geq C_i$ $(i = 1, 2)$. For Case II, we let $\rho \geq \max\{C_1, C_2\}/2$, then

$$(-\Delta)^\alpha (u + v) \geq \min\{\mu_1, \mu_2\}(u + v) \quad \text{in } \Omega.$$

This contradicts the choice of μ_1, μ_2 and the definition of λ_1. Therefore, $\rho \leq C$, and (4.2) does not admit positive solutions in E if ρ is large.

Finally, since $h_1 + \rho$ and $h_2 + \rho$ also verifies (A4) for $\rho \leq C$, we can apply Theorem 4.1 to obtain that the solutions of (4.2) are a priori bounded, that is, there exists $R > r$ such that $\|(u, v)\|_E \leq R$ for every positive solution of (4.2) with $\rho \geq 0$. Thus Theorem 4.1 is applicable with $U = B_R(0) \cap K$ and the existence of a solution in K follows. By the maximum principle, the solution is also positive. We complete the prove. □

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