Regularity of Weak Solutions for Singular Elliptic Problems Driven by m-Laplace Operator

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November 11, 2015

Abstract

We obtain optimal regularity in the Sobolev space $W_0^{1,\tau}(\Omega)$ for the unique solution of

$$-\Delta_m u = K(x)u^{-p} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Here $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, $m > 1$, $p \geq 0$ and $K \in C(\Omega)$ is a positive function that behaves like $\text{dist}(x, \partial\Omega)^{-q}$ for some $q \geq 0$ with $p + q < 2 - \frac{1-p}{m}$.

We obtain that the unique weak solution to the above problem belongs to $W_0^{1,\tau}(\Omega)$ for

$$m \leq \tau < \frac{m+p-1}{p+q-1} \quad \text{if } p+q > 1,$$

and

$$m \leq \tau < \infty \quad \text{if } p+q = 1.$$

The above range of τ is optimal.

1 Introduction and the main result

In this note we are interested in the study of regularity of the weak solution to

$$\begin{cases} -\Delta_m u = K(x)u^{-p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, $m > 1$, $p \geq 0$ and $K \in C(\Omega)$ is a positive function that behaves like $\delta(x)^{-q}$ for some $q \geq 0$, with $p + q < 2 - \frac{1-p}{m}$ where $\delta(x) := \text{dist}(x, \partial\Omega)$. The existence of a weak solution $u \in W_0^{1,m}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ (if $p + q < 1$) or $u \in W_0^{1,m}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$ (if $p + q \geq 1$) has been obtained in [1, 5, 6] for some α, $\beta \in (0, 1)$. We are here interested in the optimal $W_0^{1,\tau}(\Omega)$ regularity of the unique solution.

In this note, for any two functions f and g defined on Ω we shall write $f \sim g$ to denote that

$$c_1 \leq \frac{f}{g} \leq c_2 \quad \text{in } \Omega,$$

for some positive constants c_1 and c_2.

Our main result in this note is as follows:

Theorem 1.1. *Let $p \geq 0$ and $K : \Omega \to (0, \infty)$ be a continuous function such that $K(x) \sim \delta(x)^{-q}$, $p + q < 2 - \frac{1-p}{m}$. Then, the problem*

$$\begin{cases} -\Delta_m u = K(x)u^{-p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has a unique solution $u \in W_0^{1,m}(\Omega)$ and:

(i) *If $p + q < 1$ then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and $u(x) \sim \delta(x)$.*

(ii) *If $p + q = 1$ then $u \in W_0^{1,\tau}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$ and for all $m \leq \tau < \infty$. Also $u \sim \delta(x) \log^{\frac{1}{m+p-1}} \left(\frac{1}{\delta(x)}\right)$.*

(iii) *If $p + q > 1$ then $u \in W_0^{1,\tau}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$ and for all $m \leq \tau < \frac{m+p-1}{p+q-1}$. Also $u \sim \delta(x)^{\frac{m-q}{m+p-1}}$.*

2 Auxiliary results

The key result in proving Theorem 1.1 is the following:

Proposition 2.1. *Let $u \in W_0^{1,m}(\Omega) \cap C(\overline{\Omega})$, $m > 1$ satisfy*

$$\begin{cases} -\Delta_m u = \theta(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\theta : \Omega \to (0, \infty)$ is a continuous function.

(i) *If $\theta(x) \sim \delta(x)^{-a}$ for $a \in \left(1, 2 - \frac{1}{m}\right)$ then $u \in W_0^{1,p}(\Omega)$ for all $p \in \left[m, \frac{m-1}{a-1}\right)$.*

(ii) *If $\theta(x) \sim \delta(x)^{-1} log^{-a}\left(\frac{1}{\delta(x)}\right)$ for $a \in (0, 1)$ then $u \in W_0^{1,p}(\Omega)$ for all $p \in [m, \infty)$.*

We shall use the following results:

Lemma 2.2. (see [7, Theorem 2]). *Assume $p \geq m > 1$, $u \in W_0^{1,m}(\Omega)$ and $\Phi \in L^{\frac{p}{m-1}}(\Omega; \mathbb{R}^N)$ satisfy*

$$\Delta_m u = div(\Phi) \quad \text{in } \Omega.$$

Then $\nabla u \in L^p(\Omega; \mathbb{R}^N)$ and there exist $c = c(m, p, N)$ such that

$$||\nabla u||_{L^p(\Omega)}^{m-1} \leq c||\Phi||_{L^{\frac{p}{m-1}}(\Omega)}.$$

Lemma 2.3. (see [2, Lemma 2] or [1, Lemma 4.4]). *There exist $c > 0$ such that if $B_{2r}(x_0) \subset \Omega$, $0 < r \leq 1$ and $v \in W_0^{2,p}(\Omega)$, for some $p > N$, then*

$$||\nabla v||_{L^\infty(B_r(x_0))} \leq c\Big[r||\Delta v||_{L^\infty(B_{2r}(x_0))} + \frac{1}{r}||v||_{L^\infty(B_{2r}(x_0))}\Big]. \tag{2.2}$$

Lemma 2.4. (see [9]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded and smooth domain. Then*

$$\int_\Omega \delta(x)^{-a} dx < \infty \quad \text{if and only if } a < 1.$$

Proof of Proposition 2.1. Let $w \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{cases} -\Delta w = \theta(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Denote by $\phi > 0$ the first eigenfunction of $-\Delta$ in Ω.

(i) Assume $\theta(x) \sim \delta(x)^{-a}$ for some $a \in \left(1, 2 - \frac{1}{m}\right)$. Then

$$\underline{w}(x) = \frac{1}{c}\phi(x)^{2-a} \quad \text{and} \quad \overline{w}(x) = c\phi(x)^{2-a},$$

are respectively sub and supersolutions of (2.3) provided $c > 1$ is large enough, so

$$w(x) \sim \delta(x)^{2-a}. \tag{2.4}$$

We claim that

$$|\nabla w(x)| \le c\delta(x)^{1-a} \quad \text{in } \Omega, \tag{2.5}$$

for some $c > 0$. In order to prove this, let $x \in \Omega$ be a fixed point and $r = \frac{\delta(x)}{3}$. Then

$$B_{2r}(x) \subset \Omega_0 = \{z \in \Omega : \frac{\delta(x)}{3} \le \delta(z) \le \frac{5}{3}\delta(x)\} \subset \Omega$$

and by Lemma 2.3 we have:

$$|\nabla w(x)| \le c\Big[r||\Delta w||_{L^\infty(\Omega_0)} + \frac{1}{r}||w||_{L^\infty(\Omega_0)}\Big] \le c\delta(x)^{1-a} \tag{2.6}$$

which proves (2.5). Using the estimate in (2.5) we deduce that $|\nabla w| \in L^{\frac{p}{m-1}}(\Omega)$ whenever $\delta(x)^{1-a} \in L^{\frac{p}{m-1}}(\Omega)$ and by Lemma 2.4 this is equivalent to $p < \frac{m-1}{a-1}$. Using Lemma 2.2 with $\Phi = \nabla w$, we conclude the proof.

(ii) Assume now $\theta(x) \sim \delta(x)^{-1} \log^{-a}\left(\frac{1}{\delta(x)}\right)$ for some $a \in (0, 1)$. Then

$$\underline{w}(x) = \frac{1}{c}\phi(x) \log^{1-a}\left(\frac{A}{\phi(x)}\right) \quad \text{and} \quad \overline{w}(x) = c\phi(x) \log^{1-a}\left(\frac{A}{\phi(x)}\right),$$

are respectively sub and supersolutions of (2.3), where $A > 1$ is large. So,

$$w(x) \sim \delta(x) \log^{1-a}\left(\frac{A}{\delta(x)}\right). \tag{2.7}$$

Using (2.7) and the similar approach as in part(i), we deduce that

$$|\nabla w(x)| \le c \log^{1-a}\left(\frac{A}{\delta(x)}\right) \quad \text{in } \Omega,$$

where $A > 1 + diam(\Omega)$. In particular $|\nabla w| \in L^p(\Omega)$ for all $p > 1$ which, by Lemma 2.2 with $\Phi = \nabla w$, yields $u \in W_0^{1,p}(\Omega)$ for all $p \in [m, \infty)$. This finishes the proof of our result.

□

Remark 2.5. The regularity in Proposition 2.1 is optimal. In order to see this, let (φ, λ) denote the first eigenfunction and eigenvalue of $-\Delta_m$ in Ω, that is

$$\begin{cases} -\Delta_m \varphi = \lambda |\nabla \varphi|^{m-2}\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

It is known that $\lambda > 0$, $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and φ has constant sign in Ω. Thus, by normalizing φ we may assume $\varphi > 0$ in Ω and $||\varphi||_\infty = 1$. To show that $W_0^{1,p}(\Omega)$ regularity in Proposition 2.1(i) is optimal, let $\theta(x) = -\Delta_m(\varphi^{2-a})$. Some straightforward calculations yield $\theta(x) \sim \varphi(x)^{-a} \sim \delta(x)^{-a}$. Thus, $w = \varphi^{2-a}$ is a solution of (2.1) with $\theta(x)$ given by $-\Delta_m(\varphi^{2-a})$. Clearly $w \in W_0^{1,p}(\Omega)$ for all $m \le p < \frac{m-1}{a-1}$, but by Lemma 2.4 one has $w \notin W_0^{1,\frac{m-1}{a-1}}(\Omega)$.

Similarly, to show that regularity $w \in W_0^{1,p}(\Omega)$, $m \le p < \infty$ is optimal we take $\theta(x) = -\Delta_m\Big(\varphi \log^{1-a}\left(\frac{A}{\varphi}\right)\Big)$ where $A > 1$ is a large constant.

3 Proof of Theorem 1.1

The existence of a solution $u \in W_0^{1,m}(\Omega)$ follows from [6, Theorem 3.2].

(i) If $p + q < 1$, then by [6, Theorem 2.1], we have $u \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, 1)$.

(ii) If $p + q = 1$, then by [6, Theorem 2.1], we have $u \in W_0^{1,m}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$. Also the behaviour $u \sim \delta(x) \log^{\frac{1}{m+p-1}}\left(\frac{1}{\delta(x)}\right)$ follows in the same way as in [5, Lemma 3.3] by noting that

$$\underline{u}(x) = \frac{1}{c}\varphi(x)\log^{1-a}\left(\frac{A}{\varphi(x)}\right) \quad \text{and} \quad \overline{u}(x) = c\varphi(x)\log^{1-a}\left(\frac{A}{\varphi(x)}\right),$$

are respectively sub and supersolutions for some large $c > 1$. Using the asymptotic behaviour of u we deduce that

$$\theta(x) = K(x)u^{-p}(x) \sim \delta(x)^{-1}\log^{-\frac{p}{m+p-1}}\left(\frac{1}{\delta(x)}\right).$$

By Proposition 2.1(ii) it follows that $u \in W_0^{1,\tau}(\Omega)$ for all $\tau \in [m, \infty)$.

(iii) If $p + q > 1$, then by [6, Theorem 2.1], we have $u \in W_0^{1,m}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$. Using the fact that

$$\underline{u}(x) = \frac{1}{c}\varphi(x)^{\frac{m-q}{m+p-1}} \quad \text{and} \quad \overline{u}(x) = c\varphi(x)^{\frac{m-q}{m+p-1}},$$

are respectively sub and supersolutions of (1.1) for some large $c > 1$, we easily deduce that

$$u \sim \delta(x)^{\frac{m-q}{m+p-1}}.$$

Then

$$\theta(x) = K(x)u^{-p}(x) \sim \delta(x)^{-\frac{mp+(m-1)q}{m+p-1}},$$

and note that $a = \frac{mp+(m-1)q}{m+p-1} \in \left(1, 2 - \frac{1}{m}\right)$. By Proposition 2.1(ii) it follows that $u \in W_0^{1,\tau}(\Omega)$ for all $\tau \in \left[m, \frac{m+p-1}{p+q-1}\right)$.

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