# Critical growth fractional elliptic systems with exponential nonlinearity

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**Abstract**

We study the existence of positive solutions for the system of fractional elliptic equations of the type,

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}u &= \tfrac{p}{p+q}\lambda f(x)|u|^{p-2}u|v|^q + h_1(u,v)e^{u^2+v^2}, \text{ in } (-1,1),\\ (-\Delta)^{\frac{1}{2}}v &= \tfrac{q}{p+q}\lambda f(x)|u|^{p}|v|^{q-2}v + h_2(u,v)e^{u^2+v^2}, \text{ in } (-1,1),\\ u,v &> 0 \text{ in } (-1,1),\\ u &= v = 0 \text{ in } \mathbb{R}\setminus(-1,1). \end{aligned}$$

where $1 < p+q < 2$, $h_1(u,v) = (\alpha+2u^2)|u|^{\alpha-2}u|v|^\beta, h_2(u,v) = (\beta+2v^2)|u|^\alpha|v|^{\beta-2}v$ and $\alpha+\beta > 2$. Here $(-\Delta)^{\frac{1}{2}}$ is the fractional Laplacian operator. We show the existence of multiple solutions for suitable range of $\lambda$ by analyzing the fibering maps and the corresponding Nehari manifold. We also study the existence of positive solutions for a superlinear system with critical growth exponential nonlinearity.

## 1 Introduction

We study the following system for existence and multiplicity of solutions

$$(P_\lambda)\quad \begin{cases} (-\Delta)^{\frac{1}{2}}u &= \frac{p}{p+q}\lambda f(x)|u|^{p-2}u|v|^q + h_1(u,v)e^{u^2+v^2}, \text{ in } (-1,1),\\ (-\Delta)^{\frac{1}{2}}v &= \frac{q}{p+q}\lambda f(x)|v|^{q-2}v|u|^{p} + h_2(u,v)e^{u^2+v^2}, \text{ in } (-1,1),\\ \quad u,v &> 0 \text{ in } (-1,1),\\ \quad u &= v = 0 \text{ in } \mathbb{R}\setminus(-1,1). \end{cases}$$

where $1 < p+q < 2$, $h_1(u,v) = (\alpha+2u^2)|u|^{\alpha-2}u|v|^\beta$ and $h_2(u,v) = (\beta+2v^2)|u|^\alpha|v|^{\beta-2}v$, $\alpha+\beta > 2$, $\lambda > 0$ and $f \in L^r(-1,1)$, for suitable choice of $r > 1$, is sign changing. Here

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$(-\Delta)^{\frac{1}{2}}$ is the $\frac{1}{2}$-Laplacian operator defined as

$$(-\Delta)^{\frac{1}{2}}u = \int_{\mathbb{R}} \frac{(u(x+y)+u(x-y)-2u(x))}{|y|^2}dy \quad \text{for all } \; x \in \mathbb{R}.$$

The fractional Laplacian operator has been a classical topic in Fourier analysis and nonlinear partial differential equations. Fractional operators are involved in financial mathematics, where Levy processes with jumps appear in modeling the asset prices (see [5]). Recently the semilinear equations involving the fractional Laplacian has attracted many researchers. The critical exponent problems for fractional Laplacian have been studied in [28, 29]. Among the works dealing with fractional elliptic equations with critical exponents we cite also [33, 8, 23, 24] and references there-in, with no attempt to provide a complete list.

In the local setting, the semilinear ellitpic systems involving Laplace operator with exponential nonlinearity has been investigated in [14, 22]. The case of polynomial nonlinearities involving linear and quasilinear operators has been studied in [3, 4, 12, 15, 16, 27, 30]. Furthermore, these results for sign changing nonlinearities with polynomial type subcritical and critical growth have been obtained in [6, 7, 10, 26, 31] using Nehari manifold and fibering map analysis. These problems for exponential growth nonlinearties is studied in [14]:

$$\begin{cases} -\Delta u &= g(v), \; -\Delta v = f(u), \text{ in } \Omega \\ \quad u &= v = 0 \text{ on } \partial\Omega. \end{cases}$$

where functions $f$ and $g$ have critical growth in the sense of Trudinger-Moser inequality and have shown the existence of a nontrivial weak solutions in both sub-critical as well as critical growth case. In [22], authors have considered the elliptic system with exponential growth perturbed by a concave growth term and established the global multiplicity results with respect to the parameter.

Recently semilinear equations involving fractional Laplacian with exponential nonlinearities have been studied by many authors. Among them we cite [19, 21, 32, 17] and the references therein. The system of equations with fractional Laplacian operator with polynomial sub-critical and critical Sobolev exponent have been studied in [11, 18]. Our aim in this article is to generalize the result in [17] for fractional elliptic systems. In [18], authors considered the problem

$$(-\Delta)^s u = \lambda|u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \;\; (-\Delta)^s = \lambda|v|^{q-2}v + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v \; in \; \Omega,$$
$$u = v = 0 \; on \; \partial\Omega.$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda, > 0, 1 < q < 2$ and $\alpha > 1, \beta > 1$ satisfy $\alpha+\beta = 2N/(N-2s)$, $s \in (0,1)$ and $N > 2s$. They studied the associated Nehari manifold using the fibering maps and show the existence of non-negative solutions arising out of structure of manifold. Our results in section 3 extends these results to the exponential case.

The variational functional $J_\lambda$ associated to the problem$(P_\lambda)$ is given as

$$J_\lambda(u,v) = \frac{1}{2}\int_{-1}^{1}\left(|(-\Delta)^{\frac{1}{4}}u|^2 + |(-\Delta)^{\frac{1}{4}}v|^2\right)dx - \frac{\lambda}{p+q}\int_{-1}^{1} f(x)|u|^p|v|^q dx - \int_{-1}^{1} G(u,v)dx.$$

**Definition 1.1.** $(u,v) \in H_0^s(-1,1)\times H_0^s(-1,1)$ *is called weak solution of* $(P_\lambda)$ *if*

$$\int_{-1}^{1}\left((-\Delta)^{\frac{1}{4}}u(-\Delta)^{\frac{1}{4}}\phi + (-\Delta)^{\frac{1}{4}}v(-\Delta)^{\frac{1}{4}}\psi\right)dx = \lambda\int_{-1}^{1}\left(|u|^{p-2}|v|^q u\phi + |v|^{q-2}|u|^p v\psi\right)dx$$
$$+\int_{-1}^{1}\left(h_1(u,v)\phi + h_2(u,v)\psi\right)e^{u^2+v^2}dx$$

*for all* $(\phi,\psi) \in H_0^s(-1,1)\times H_0^s(-1,1)$.

In the beautiful work [9], Caffarelli and Silvestre used Dirichlet-Neumann maps to transform the non-local equations involving fractional Laplacian into a local problem. This approach attracted lot of interest by many authors recently to address the existence and multiplicity of solutions using variational methods. In [9], it was shown that for any $v \in H^{\frac{1}{2}}(\mathbb{R})$, the unique function $w(x,y)$ that minimizes the weighted integral

$$\mathcal{E}_{\frac{1}{2}}(w) = \int_0^\infty\int_{\mathbb{R}} |\nabla w(x,y)|^2 dxdy$$

over the set $\left\{w(x,y) : \mathcal{E}_{\frac{1}{2}}(w) < \infty, w(x,0) = v(x)\right\}$ satisfies $\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}}v|^2 = \mathcal{E}_{\frac{1}{2}}(w)$. Moreover $w(x,y)$ solves the boundary value problem

$$-\text{div}(\nabla w) = 0 \text{ in } \mathbb{R}\times\mathbb{R}_+,\ \ w(x,0) = v(x)\quad \frac{\partial w}{\partial\nu} = (-\Delta)^{1/2}v(x)$$

where $\frac{\partial w}{\partial \nu} = \lim\limits_{y\to 0^+}\frac{\partial w}{\partial y}(x,y)$. In case of bounded domains, in [8], it was observed that the harmonic extension problem is

$$(P_E)\left\{\begin{array}{rcl} -\Delta w &=& 0,\ w>0 \ \text{ in } \mathcal{C} = (-1,1)\times(0,\infty),\\ w &=& 0 \text{ on } \{-1,1\}\times(0,\infty),\\ \frac{\partial w}{\partial\nu} &=& (-\Delta)^{1/2}v \text{ on } (-1,1)\times\{0\}.\end{array}\right.$$

This can be solved on the space $H^1_{0,L}(\mathcal{C})$, which is defined as

$$H^1_{0,L}(\mathcal{C}) = \{v\in H^1(\mathcal{C}) : v = 0 \text{ a.e. in } \ \{-1,1\}\times(0,\infty)\}$$

equipped with the norm $\|w\| = \left(\int_{\mathcal{C}} |\nabla w|^2 dxdy\right)^{\frac{1}{2}}$. The space $H^{\frac{1}{2}}(\mathbb{R})$ is the Hilbert space with the norm defined as

$$\|u\|_H^2 = \|u\|_{L^2(\mathbb{R})} + \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx.$$

The space $H_0^{\frac{1}{2}}(\mathbb{R})$ is the completion of $C_0^\infty(\mathbb{R})$ under $[u] = \left(\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx\right)^{\frac{1}{2}}$. In case of bounded intervals, say $I = (-1, 1)$, the function spaces are defined as

$$X = \{u \in H^{\frac{1}{2}}(\mathbb{R}) : \ u = 0 \text{ in } \mathbb{R}\backslash(-1, 1)\}$$

equipped with the norm

$$\|u\|_X = [u]_{H^{1/2}(\mathbb{R})} = \sqrt{2\pi}\|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})}.$$

Using the above idea, in the case of systems we have the following extension problem

$$(P_E)_\lambda \begin{cases} -div(\nabla w_1) = 0, \quad -div(\nabla w_2) = 0 & \text{in } \mathcal{C} = (-1,1)\times(0,\infty), \\ w_1, w_2 > 0 & \text{in } \mathcal{C} = (-1,1)\times(0,\infty), \\ w_1 = w_2 = 0 \text{ on } \{-1,1\}\times(0,\infty), & \\ \frac{\partial w_1}{\partial \nu} = \frac{p}{p+q}\lambda f(x)|u|^{p-2}u|v|^q + h_1(u,v)e^{u^2+v^2} & \text{on } (-1,1)\times\{0\}, \\ \frac{\partial w_2}{\partial \nu} = \frac{q}{p+q}\lambda f(x)|v|^{q-2}v|u|^p + h_2(u,v)e^{u^2+v^2} & \text{on } (-1,1)\times\{0\} \end{cases}$$

where $\frac{\partial w_1}{\partial \nu} = \lim\limits_{y\to 0^+} \dfrac{\partial w_1}{\partial y}(x,y)$ and $\frac{\partial w_2}{\partial \nu} = \lim\limits_{y\to 0^+} \dfrac{\partial w_2}{\partial y}(x,y)$.
The problem $(P_E)_\lambda$ can be solved on the space $\mathcal{W}_{0,L}^1(\mathcal{C})$, which is defined as

$$\mathcal{W}_{0,L}^1(\mathcal{C}) = \{(w_1, w_2) \in H_{0,L}^1(\mathcal{C}) \times H_{0,L}^1(\mathcal{C}) : w_1 = w_2 = 0 \text{ a.e. in } \ \{-1,1\}\times(0,\infty)\}$$

equipped with the norm

$$\|w\| = \left(\int_{\mathcal{C}} |\nabla w_1|^2 dxdy + \int_{\mathcal{C}} |\nabla w_2|^2 dxdy\right)^{\frac{1}{2}}.$$

The Moser-Trudinger trace inequality for $\mathcal{W}_{0,L}^1(\mathcal{C})$ follows from Martinazzi [21] which improves the former results of Ozawa [25] and Kozono, Sato & Wadade [20] and uses Lemma 1.1 in Megrez, Sreenadh & Khalidi [22] which adapts the Trudinger-Moser Inequality for systems:

**Theorem 1.1.** *For any $\alpha \in (0, \pi]$ there exists $C_\alpha > 0$ such that*

$$\sup_{\|w\|\le 1} \int_{-1}^{1} e^{\alpha(w_1(x,0)^2 + w_2(x,0)^2)} dx \le C_\alpha. \tag{1.1}$$

The variational functional $I_\lambda : \mathcal{W}^1_{0,L}(\mathcal{C}) \to \mathbb{R}$ associated to $(P_E)_\lambda$ is defined as,

$$I_\lambda(w) = \frac{1}{2}\int_{\mathcal{C}} \left(|\nabla w_1|^2 + |\nabla w_2|^2\right) dxdy - \frac{\lambda}{p+q}\int_{-1}^{1}(f(x)|w_1(x,0)|^p|w_2(x,0)|^q - G(w(x,0)))dx$$

where $w = (w_1, w_2)$. Any $w = (w_1, w_2) \in \mathcal{W}^1_{0,L}(\mathcal{C})$ is called the weak solution of the problem $(P_E)_\lambda$ if for any $\phi = (\phi_1, \phi_2) \in \mathcal{W}^1_{0,L}(\mathcal{C})$

$$\begin{aligned}\int_{\mathcal{C}} (\nabla w_1 \nabla \phi_1 d + \nabla w_2 \nabla \phi_2)\, dxdy &= \lambda\frac{p}{p+q}\int_{-1}^{1} f(x)|w_1(x,0)|^{p-2}w_1(x,0)|w_2(x,0)|^q\phi_1(x,0)dx\\ &+ \lambda\frac{q}{p+q}\int_{-1}^{1} f(x)|w_1(x,0)|^p|w_2(x,0)|^{q-2}w_2(x,0)\phi_2(x,0)dx\\ &+ \int_{-1}^{1} g_1(w_1(x,0), w_2(x,0))\phi_1(x,0)dx + \int_{-1}^{1} g_2(w_1(x,0), w_2(x,0))\phi_2(x,0)dx.\end{aligned}$$

It is clear that critical points of $I_\lambda$ in $\mathcal{W}^1_{0,L}(\mathcal{C})$ corresponds to the critical points of $J_\lambda$ in $X \times X$. So as noted in the introduction, we will look for solutions $w = (w_1, w_2)$ of $(P_E)_\lambda$.

In this paper we establish the multiplicity results for fractional systems with critical exponential nonlinearities. In section 2 we derive some delicate technical estimates to study the structure of Nehari manifold using the associate fibering maps. In section 3 we show the existence of two solutions that arise out of minimizing the functional over the Nehari manifold. Here we show the existence of second solution using mountain-pass arguments over positive cones and compactness analysis of Palais-Smale sequences. We show the following multiplicity result.

**Theorem 1.2.** *There exists a* $\Lambda_0 > 0$ *such that* $(P_\lambda)$ *has at least two solutions for every* $\lambda \in (0, \Lambda)$.

Similarly we can deal with the superlinear case, i.e.

$$(P)\begin{cases} (-\Delta)^{1/2}u &= h_1(u,v), \text{ in } (-1,1),\\ (-\Delta)^{1/2}v &= h_2(u,v), \text{ in } (-1,1),\\ u, v &> 0 \text{ in}(-1,1),\\ u = v &= 0 \text{ in } \mathbb{R}\backslash(-1,1)\end{cases}$$

where $h_1(u,v)$ and $h_2(u,v)$ satisfy the following conditions:

(h1) $h_1, h_2 \in C^1(\mathbb{R}^2), h_1(s,t) = h_2(s,t) = 0$ for $s \le 0, t \le 0$, $h_1(s,t) = \frac{\partial H}{\partial s}(s,t)$, $h_2(s,t) = \frac{\partial H}{\partial t}(s,t) > 0$ for $s > 0, t > 0$ and for any $\epsilon > 0$, $\lim\limits_{(s,t)\to\infty} h_1(s,t)e^{-(1+\epsilon)(s^2+t^2)} = 0$ and $\lim\limits_{(s,t)\to\infty} h_2(s,t)e^{-(1+\epsilon)(s^2+t^2)} = 0$.

(h2) There exists $\mu > 2$ such that for all $s, t > 0$,

$$0 \leq \mu H(s,t) \leq s h_1(s,t) + t h_2(s,t).$$

(h3) There exist positive constants $s_0 > 0, t_0 > 0$ and $M_1 > 0, M_2 > 0$ such that $H(s,t) \leq M_1 h_1(s,t) + M_2 h_2(s,t)$ for all $s > s_0$ and $t > t_0$.

(h4) $\lim\limits_{s,t\to\infty} (s h_1(s,t) + t h_2(s,t))\, e^{-(s^2+t^2)} = \infty$.

(h5) $\limsup\limits_{s,t\to 0} \dfrac{2H(s,t)}{s^2+t^2} < \lambda_1$, where $\lambda_1 = 2 \min\limits_{w \in \mathcal{H}^1_{0,L}(\mathcal{C})} \left\{ \int_C |\nabla w_1|^2 \; ; \; \int_{-1}^{1} |w_1(x,0)|^2 dx = 1 \right\}$.

An example of a function satisfying above assumptions is

$$H(s,t) = \begin{cases} (s^4 + t^4) e^{s^2+t^2} & \text{if } s > 0,\ t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The variational functional associated to the problem $(P)$ is given as

$$J(u,v) = \frac{1}{2} \int_{-1}^{1} |(-\Delta)^{\frac{1}{4}} u|^2 dx + \frac{1}{2} \int_{-1}^{1} |(-\Delta)^{\frac{1}{4}} v|^2 dx - \int_{-1}^{1} H(u,v) dx.$$

Following very closely the variational approach in Subsection 3.1, we prove in Section 4:

**Theorem 1.3.** *Suppose $(h1) - (h5)$ are satisfied. Then the problem $(P)$ has a solution.*

## 2 Nehari manifold and fibering maps

In this section, we consider the Nehari manifold associated to the problem $(P_E)_\lambda$ as

$$\mathcal{N}_\lambda = \{ w \in \mathcal{W}^1_{0,L}(\mathcal{C}) \mid \langle I'_\lambda(w), w \rangle = 0 \}.$$

Thus $w = (w_1, w_2) \in \mathcal{N}_\lambda$ if and only if

$$\|w\|^2 = \lambda \int_{-1}^{1} f(x) |w_1(x,0)|^p |w_2(x,0)|^q dx + \int_{-1}^{1} g_1(w_1(x,0), w_2(x,0)) w_1(x,0) dx + \int_{-1}^{1} g_2(w_1(x,0), w_2(x,0)) w_2(x,0) dx. \tag{2.2}$$

Now for every $w = (w_1, w_2) \in \mathcal{W}^1_{0,L}(\mathcal{C})$, we define the fiber map $\Phi_w : \mathbb{R}^+ \to \mathbb{R}$ as $\Phi_w(t) = I_\lambda(tw)$. Thus $tw \in \mathcal{N}_\lambda$ if and only if

$$\Phi'_w(t) = t\|w\|^2 - \lambda t^{p+q-1}\int_{-1}^{1} f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx \\ - \int_{-1}^{1} g_1(tw(x,0))w_1(x,0)dx - \int_{-1}^{1} g_2(tw(x,0))w_2(x,0)dx = 0.$$

In particular, $w \in \mathcal{N}_\lambda$ if and only if

$$\|w\|^2 - \lambda \int_{-1}^{1} f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx - \int_{-1}^{1} g_1(w(x,0))w_1(x,0)dx \\ - \int_{-1}^{1} g_2(w(x,0))w_2(x,0)dx = 0.$$

Also

$$\Phi''(w)(1) = \|w\|^2 - (p+q-1)\lambda \int_{-1}^{1} f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx \\ - \int_{-1}^{1} \left\{(\alpha+\beta+2w_1^2+2w_2^2)(\alpha+\beta-1+2w_1^2+2w_2^2) + 4(w_1^2+w_2^2)\right\}|w_1|^\alpha|w_2|^\beta e^{w_1^2+w_2^2}dx. \tag{2.3}$$

We split $\mathcal{N}_\lambda$ into three parts as

$$\mathcal{N}^\pm_\lambda = \{w \in \mathcal{N}_\lambda \mid \Phi''(w)(1) \gtrless 0\} \text{ and } \mathcal{N}^0_\lambda = \{w \in \mathcal{N}_\lambda \mid \Phi''(w)(1) = 0\}.$$

In order to prove Theorem 1.2, we need following version of Lemma 3.1 of [26].

**Lemma 2.1.** *Let $\Gamma \subset \mathcal{W}^1_{0,L}(\mathcal{C})$ such that for any $w \in \Gamma$,*
$\|w\|^2 \le \frac{1}{2-p-q}\int_{-1}^{1}(\alpha+\beta-(p+q)+2w_1^2+2w_2^2)(\alpha+\beta+2+2w_1^2+2w_2^2)|w_1|^\alpha|w_2|^\beta e^{w_1^2+w_2^2}dx.$
*Then there exists a positive $\Lambda_0$ such that*

$$\Gamma_0 := \inf_{w\in\Gamma\setminus\{0\}}\left\{\int_{-1}^{1}(\alpha+\beta-2+2w_1^2+2w_2^2)(\alpha+\beta+2w_1^2+2w_2^2)|w_1|^\alpha|w_2|^\beta e^{w_1^2+w_2^2}dx \\ - (2-p-q)\lambda\int_{-1}^{1} f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx\right\} > 0, \tag{2.4}$$

*for every $\lambda \in (0, \Lambda_0)$.*

*Proof.* We divide the proof into three steps:

**Step 1:** $\inf_{w\in\Gamma\setminus\{0\}}\|w\| > 0$.

Suppose not. Then there exists $\{w_k\}$ in $\mathcal{W}^1_{0,L}(\mathcal{C})\setminus\{0\}$ such that $\|w_k\|\to 0$ and

$$\|w\|^2 \le \frac{1}{2-p-q}\int_{-1}^{1}(\alpha+\beta-p-q+2w_1^2+2w_2^2)(\alpha+\beta+2+2w_1^2+2w_2^2)|w_1|^{\alpha}|w_2|^{\beta}e^{w_1^2+w_2^2}dx. \tag{2.5}$$

Now, using Theorem 1.1, we get

$$\begin{aligned}
&\int_{-1}^{1}(\alpha+\beta-p-q+2w_{k_1}^2+2w_{k_2}^2)(\alpha+\beta+2+2w_{k_1}^2+2w_{k_2}^2)|w_{k_1}|^{\alpha}|w_{k_2}|^{\beta}e^{w_{k_1}^2+w_{k_2}^2}dx\\
&\qquad=(\alpha+\beta-p-q)(\alpha+\beta+2)\int_{-1}^{1}|w_{k_1}|^{\alpha}|w_{k_2}|^{\beta}e^{w_{k_1}^2+w_{k_2}^2}dx\\
&\qquad+2(2\alpha+2\beta+2-p-q)\int_{-1}^{1}(w_{k_1}^2+w_{k_2}^2)|w_{k_1}|^{\alpha}|w_{k_2}|^{\beta}e^{w_{k_1}^2+w_{k_2}^2}dx\\
&\qquad+4\int_{-1}^{1}(w_{k_1}^2+w_{k_2}^2)^2|w_{k_1}|^{\alpha}|w_{k_2}|^{\beta}e^{w_{k_1}^2+w_{k_2}^2}dx\\
&\qquad\le C_1\|w_k\|^{\alpha+\beta}+C_2\|w_k\|^{\alpha+\beta+2}+C_3\|w_k\|^{\alpha+\beta+4}.
\end{aligned}$$

Hence from equation (2.5) and the last inequality, we get $\|w_k\|^{\alpha+\beta}\ge C>0$, which is a contradiction as $\alpha+\beta>2$. Hence the claim is proved.

**Step 2:** Let $C_* = \inf_{w\in\Gamma\setminus\{0\}}\int_{-1}^{1}(\alpha+\beta-2+2w_1^2+2w_2^2)(\alpha+\beta+2w_1^2+2w_2^2)|w_1|^{\alpha}|w_2|^{\beta}e^{w_1^2+w_2^2}dx$. Then $C_*>0$.

Proof follows from step 1 and the definition of $\Gamma$.

**Step 3:** Let $\lambda\in(0,\Lambda_0)$ for $\Lambda_0=\frac{C_*^{1-a}}{2-p-q}$. Then equation (2.4) holds. Indeed, let $\alpha_0=\frac{pr}{r-1}>1$ and $\beta_0=\frac{qr}{r-1}>1$ and $a=1-\frac{1}{r}$. Then we have:

$$\begin{aligned}
\lambda\int_{-1}^{1}f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx &\le \lambda C_f\left(\int_{-1}^{1}|w_1(x,0)|^{\alpha_0}|w_2(x,0)|^{\beta_0}dx\right)^a\\
&\le\frac{(\lambda C_f)}{C_*^{1-a}}\int_{-1}^{1}(\alpha+\beta-2+2w_1^2+2w_2^2)(\alpha+\beta+2w_1^2+2w_2^2)|w_1|^{\alpha}|w_2|^{\beta}e^{w_1^2+w_2^2}dx
\end{aligned}$$

where $C_f=\|f\|_{L^{\frac{1}{1-a}}((-1,1))}$. Thus if $(\lambda C_f)<\Lambda_0=\frac{C_*^{1-a}}{2-p-q}$. Then equation (2.4) holds. □

Now we discuss the behavior of $\Phi_w$ with respect to $\int_{-1}^{1}f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx$.

**Case 1:** $\int_{-1}^{1}f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx\le 0$.

We define

$$\Psi_w(t)=t^{2-p-q}\|w\|^2-t^{1-p-q}\int_{-1}^{1}g_1(tw(x,0))w_1(x,0)dx-t^{1-p-q}\int_{-1}^{1}g_2(tw(x,0))w_2(x,0)dx.$$

It is clear that $\Psi_w(0)=0$ and $tw\in\mathcal{N}_\lambda$ if and only if $\Psi_w(t)=\lambda\int_{-1}^{1}f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx$. Observe that $\lim_{t\to\infty}\Psi_w(t)\to-\infty$, $\lim_{t\to\infty}\Psi'_w(t)\to$

$-\infty$, $\Psi_w''(t) \leq 0$ for any $t > 0$ and $\lim\limits_{t\to 0^+} \Psi_w'(t) > 0$. Hence there exists a unique $t_*(w) > 0$ such that $\Psi_w(t)$ is increasing in $(0, t_*)$, decreasing in $(t_*, \infty)$. Hence for all values of $\lambda$ there exists a unique $t^-(w) > t_*(w)$ such that $\Psi_w(t^-) = \lambda \int_{-1}^1 f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx \leq 0$. Since $\Psi_w'(t) < 0$ for $t > t_*$, using that the relation $\Phi_{tw}''(1) = t^{p+q+1}\Psi_w'(t)$ is valid for $t = t^-$, we get $t^- w \in \mathcal{N}_\lambda^-$.

**Case 2:** $\int_{-1}^1 f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx > 0$.

As discussed in case 1, we have $\Psi_w'(t_*) = 0$ which implies $t_* w \in \Gamma \setminus \{0\}$. Now

$$\Psi_w(t_*) \geq \frac{1}{(2-p-q)t_*^{p+q}} \int_{-1}^1 (\alpha+\beta-2+2|t_*w_1|^2+2|t_*w_2|^2)(\alpha+\beta+2|t_*w_1|^2+2|t_*w_2|^2) \times |t_*w_1|^\alpha |t_*w_2|^\beta e^{|t_*w_1|^2+|t_*w_2|^2} dx.$$

Now from Lemma 2.1, $\Psi_w(t_*) - \lambda \int_{-1}^1 f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx > 0$. Therefore there exists unique $t^+(w) < t_*(w) < t^-(w)$ such that $\Psi_w'(t^+) > 0$, $\Psi_w'(t^-) < 0$, and $\Psi_w(t^+) = \Psi_w(t^-) = \lambda \int_{-1}^1 f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx$. As a consequence $t^+ w \in \mathcal{N}_\lambda^+$ and $t^- w \in \mathcal{N}_\lambda^-$. Moreover since $\Phi_w'(t^-) = \Phi_w'(t^+) = 0$, $\Phi_w'(t) < 0$ on $(0, t^+)$ and $\Phi_w'(t) > 0$ on $(t^+, t^-)$, $I_\lambda(t^+ w) = \min\limits_{0<t<t^-} I_\lambda(tw)$ and $I_\lambda(t^- w) = \max\limits_{t>t_*} I_\lambda(tw)$. From above discussion we have the following lemma.

**Lemma 2.2.** *For any $w \in \mathcal{W}_{0,L}^1(\mathcal{C})$,*

*(i) If $\int_{-1}^1 f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx \leq 0$, there exists a unique $t^-(w) > 0$ such that $t^- w \in \mathcal{N}_\lambda^-$ for every $\lambda > 0$.*

*(ii) If $\int_{-1}^1 f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx > 0$, there exists unique $t^+(w) < t_*(w) < t^-(w)$ such that $t^+ w \in \mathcal{N}_\lambda^+$ and $t^- w \in \mathcal{N}_\lambda^-$ for every $\lambda_0$ satisfying Lemma 2.1. Moreover $I_\lambda(t^+ w) < I_\lambda(tw)$ for any $t \in [0, t^-]$ such that $t \neq t^+$ and $I_\lambda(t^- w) = \max\limits_{t>t_*} I_\lambda(tw)$ .*

Now, the next Lemma shows that $\mathcal{N}_\lambda$ is a manifold.

**Lemma 2.3.** *Let $\Lambda_0$ as in Lemma 2.1. Then $\mathcal{N}_\lambda^0 = \{0\}$ for all $\lambda \in (0, \Lambda_0)$.*

*Proof.* We prove this lemma by contradiction. Let $w \in \mathcal{N}_\lambda^0$ such that $w \not\equiv 0$. Then from (2.3) we get

$$\|w\|^2 = (p+q-1)\lambda \int_{-1}^1 f(x)|w_1(x,0)|^p|w_2(x,0)|^q + \int_{-1}^1 \left\{(\alpha+\beta+2w_1^2+2w_2^2)(\alpha+\beta-1+2w_1^2+2w_2^2)+4(w_1^2+w_2^2)\right\}|w_1|^\alpha|w_2|^\beta e^{w_1^2+w_2^2}. \tag{2.6}$$

Using equation (2.2), it is easy to show that $w \in \Gamma$. Now from equation (2.2) and (2.6), we get

$$\begin{aligned}(2-p-q)\lambda\int_{-1}^{1} & f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx\\ &\geq \int_{-1}^{1}(\alpha+\beta-p-q+2w_1^2+2w_2^2)(\alpha+\beta+2w_1^2+2w_2^2)|w_1|^\alpha|w_2|^\beta e^{w_1^2+w_2^2}dx\\ &\geq \int_{-1}^{1}(\alpha+\beta-2+2w_1^2+2w_2^2)(\alpha+\beta+2w_1^2+2w_2^2)|w_1|^\alpha|w_2|^\beta e^{w_1^2+w_2^2}dx.\end{aligned}$$

Hence if $\lambda\in(0,\Lambda_0)$, we get a contradiction. □

**Lemma 2.4.** *$I_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$.*

*Proof.* Let $w\in\mathcal{N}_\lambda$. Using equations (2.2), and the fact that $(\alpha+\beta)G(u,v)<g_1(u,v)u+g_2(u,v)v$, we get

$$\begin{aligned}I_\lambda(w) &= \left(\frac{1}{2}-\frac{1}{\alpha+\beta}\right)\|w\|^2-\lambda\left(\frac{1}{p+q}-\frac{1}{\alpha+\beta}\right)\int_{-1}^{1}f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx\\ &- \int_{-1}^{1}\left(G(w(x,0))-\frac{1}{\alpha+\beta}(g_1(w(x,0))w_1(x,0)+g_2(w(x,0))w_2(x,0)\right)dx\\ &\geq \left(\frac{1}{2}-\frac{1}{\alpha+\beta}\right)\|w\|^2-\lambda\left(\frac{1}{p+q}-\frac{1}{\alpha+\beta}\right)\int_{-1}^{1}f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx.\end{aligned}$$

So using the fact that $\alpha+\beta>2>p+q$ and Hölder's inequality the lemma is proved. □

The following lemma shows that minimizers of $I_\lambda$ on $\mathcal{N}_\lambda$ are critical points of $I_\lambda$.

**Lemma 2.5.** *Let $w$ be a local minimizer of $I_\lambda$ in any decompositions of $\mathcal{N}_\lambda$ such that $w\notin\mathcal{N}_\lambda^0$. Then $w$ is a critical point of $I_\lambda$.*

*Proof.* If $w$ minimizes $I_\lambda$ in $\mathcal{N}_\lambda$. Then by Lagrange multiplier theorem, we get

$$I_\lambda^{'}(w)=\nu C_\lambda^{'}(w),\ \text{where } C_\lambda(u)=\langle I_\lambda^{'}(u),u\rangle=0. \tag{2.7}$$

Now

$$\langle I_\lambda^{'}(w),w\rangle=\nu\langle C_\lambda^{'}(w),w\rangle=\nu\left(\Phi_w^{''}(1)+\langle I_\lambda^{'}(w),w\rangle\right)=0.$$

But $\Phi_w^{''}(1)\neq 0$ as $w\notin N_\lambda^0$. Thus $\nu=0$. Hence by (2.7) we get $I_\lambda'(w)=0$. □

**Lemma 2.6.** *Let $\Lambda_0$ be such that equation (2.4) hold. Then given $w\in\mathcal{N}_\lambda\setminus\{0\}$, there exist $\epsilon>0$ and a differentiable function $\xi:\boldsymbol{B}(0,\epsilon)\subset\mathcal{W}_{0,L}^1(\mathcal{C})\to\mathbb{R}$ such that $\xi(0)=1$, the function $\xi(v)(w-v)\in\mathcal{N}_\lambda$. Moreover, for all $v\in\mathcal{W}_{0,L}^1(\mathcal{C})$, $\langle\xi'(0),v\rangle=\frac{A}{H}$, where $A$ and $H$ are defined as*

$$
\begin{aligned}
A =& I''_\lambda(w)(w,v) + \langle I'(w), v\rangle = \\
& 2\int_{\mathcal{C}} \nabla w_1 \nabla v_1 dxdy + 2\int_{\mathcal{C}} \nabla w_2 \nabla v_2 dxdy \\
& - \frac{\lambda p^2}{p+q}\int_{-1}^{1} f(x)|w_1(x,0)|^{p-2} w_1(x,0)|w_2(x,0)|^q v_1(x,0)dx \\
& - \frac{\lambda q^2}{p+q}\int_{-1}^{1} f(x)|w_1(x,0)|^{p}|w_2(x,0)|^{q-2} w_2(x,0) v_1(x,0)dx \\
& - \int_{-1}^{1} \left\{(\alpha + 2|w_1|^2 + 2|w_2|^2)(\alpha + 2|w_1|^2) + 4w_1^2\right\}|w_1|^{\alpha-2} v_1 |w_2|^\beta e^{|w_1|^2+|w_2|^2} dx \\
& - \int_{-1}^{1} \left\{(\beta + 2|w_1|^2 + 2|w_2|^2)(\beta + 2|w_2|^2) + 4w_2^2\right\}|w_1|^{\alpha}|w_2|^{\beta-2} w_2 v_2 e^{|w_1|^2+|w_2|^2} dx,
\end{aligned}
$$

$$
\begin{aligned}
H =& I''_\lambda(w)(w,w) = (2-p-q)\|w\|^2 \\
& - \int_{-1}^{1} \left\{(\alpha+\beta-p-q+2|w_1|^2+2|w_2|^2)(\alpha+\beta+2|w_1|^2+2|w_2|^2) + 4(w_1^2+w_2^2)\right\} \\
& \qquad \times |w_1|^\alpha |w_2|^\beta e^{|w_1|^2+|w_2|^2} dx.
\end{aligned}
$$

*Proof.* For fixed $w \in \mathcal{N}_\lambda \setminus \{0\}$, define $\mathcal{F}_w : \mathbb{R} \times \mathcal{W}^1_{0,L}(\mathcal{C}) \to \mathbb{R}$ as follows

$$
\begin{aligned}
\mathcal{F}_w(t,v) &= \langle I'_\lambda(t(w-v)), (w-v)\rangle \\
&= t\|w-v\|^2 - t^{p+q-1}\lambda \int_{-1}^{1} f(x)|w_1(x,0)-v_1(x,0)|^p|w_2(x,0)-v_2(x,0)|^q dx \\
&- \int_{-1}^{1} g_1(t(w(x,0)-v(x,0)))(w_1(x,0)-v_1(x,0))dx \\
&- \int_{-1}^{1} g_2(t(w(x,0)-v(x,0)))(w_2(x,0)-v_2(x,0))dx.
\end{aligned}
$$

Then $\mathcal{F}_w(1,0) = 0$, $\frac{\partial}{\partial t}\mathcal{F}_w(1,0) \neq 0$ as $w \notin \mathcal{N}^0_\lambda$. So we can apply implicit function theorem to get a differentiable function $\xi : \mathcal{B}(0,\epsilon) \subseteq \mathcal{W}^1_{0,L}(\mathcal{C}) \to \mathbb{R}$ such that $\xi(0) = 1$ and $\langle \xi'(0), v\rangle = \frac{A}{H}$ for the choice of $A$ and $H$ defined in the Lemma. Moreover $\mathcal{F}_w(\xi(v), v) = 0$, for all $v \in \mathcal{B}(0,\epsilon)$ which implies $\xi(v)(w-v) \in \mathcal{N}_\lambda$. □

We define $\theta_\lambda := \inf\left\{I_\lambda(w) | w \in \mathcal{N}^+_\lambda\right\}$ and we prove the following lemma.

**Lemma 2.7.** *There exists a constant $C > 0$ such that $\theta_\lambda < -\left(\frac{(\alpha+\beta-p-q)(\alpha+\beta-2)}{2(p+q)}\right) C$.*

*Proof.* Let $v \in \mathcal{W}^1_{0,L}(\mathcal{C})$ be such that $\int_{-1}^{1} f(x)|v_1(x,0)|^p|v_2(x,0)|^q dx > 0$. Then from Lemma

2.2(ii), we get a $w \in \mathcal{N}_\lambda^+$.

$$\begin{aligned} I_\lambda(w) = \left(\frac{p+q-2}{2(p+q)}\right)\|w\|^2 \ &+ \ \frac{1}{p+q}\left(\int_{-1}^{1} g_1(w(x,0))w_1(x,0)dx\right. \\ &+ \left.\int_{-1}^{1} g_2(w(x,0))w_2(x,0)dx\right) - \int_{-1}^{1} G(w(x,0))dx. \quad (2.8)\end{aligned}$$

Also using (2.3), we get

$$(2-p-q)\|w\|^2 \geq \int_{-1}^{1}(\alpha+\beta-p-q+2w_1^2+2w_2^2)(\alpha+\beta+2w_1^2+2w_2^2)|w_1|^\alpha|w_2|^\beta e^{w_1^2+w_2^2}dx. \quad (2.9)$$

Now using (2.9) in (2.8), we get

$$\begin{aligned} &I_\lambda(w) \leq \\ &\frac{-1}{2(p+q)}\int_{-1}^{1}\left\{(\alpha+\beta-p-q+2w_1^2+2w_2^2)(\alpha+\beta-2+2w_1^2+2w_2^2)\right\}|w_1|^\alpha|w_2|^\beta e^{w_1^2+w_2^2}dx \\ &\leq -\frac{(\alpha+\beta-2)(\alpha+\beta-p-q)}{2(p+q)}\int_{-1}^{1}|w_1|^\alpha|w_2|^\beta e^{w_1^2+w_2^2}dx \\ &\leq -\frac{(\alpha+\beta-2)(\alpha+\beta-p-q)}{2(p+q)}C \end{aligned}$$

where $C = \int_{-1}^{1}|w_1|^\alpha|w_2|^\beta$. □

In the next proposition we show the existence of a Palais-Smale sequence.

**Proposition 1.** *There exists* $0 < \Lambda_0' \leq \Lambda_0$ *such that for any* $\lambda \in (0,\Lambda_0')$*, then there exists a minimizing sequence* $\{w_k\} = (w_{k_1}, w_{k_2}) \subset \mathcal{N}_\lambda$ *such that*

$$I_\lambda(w_k) = \theta_\lambda + o_k(1) \text{ and } I_\lambda^{'}(w_k) = o_k(1). \quad (2.10)$$

*Proof.* From Lemma 2.4, $I_\lambda$ is bounded below on $\mathcal{N}_\lambda$. So by Ekeland variational principle, there exists a minimizing sequence $\{w_k\} \in \mathcal{N}_\lambda$ such that

$$\begin{aligned} I_\lambda(w_k) &\leq \theta_\lambda + \frac{1}{k}, \\ I_\lambda(v) &\geq I_\lambda(w_k) - \frac{1}{k}\|v-w_k\| \ \text{ for all } \ v \in \mathcal{N}_\lambda. \end{aligned}$$

for all $v \in \mathcal{N}_\lambda$. Using equation (2.11) and Lemma 2.7, it is easy to show that $w_k \not\equiv 0$. From Lemma 2.4, we have that $\sup_k \|w_k\| < \infty$. Next we claim that $\|I_\lambda'(w_k)\| \to 0$ as $k \to 0$. Now, using the Lemma 2.6 we get the differentiable functions $\xi_k : \mathcal{B}(0,\epsilon_k) \to \mathbb{R}$ for some $\epsilon_k > 0$ such that $\xi_k(v)(w_k - v) \in \mathcal{N}_\lambda$, for all $v \in \mathcal{B}(0,\epsilon_k)$. For fixed $k$, choose $0 < \rho < \epsilon_k$. Let $w \in \mathcal{W}_{0,L}^1(\mathcal{C})$ with $w \not\equiv 0$ and let $v_\rho = \frac{\rho w}{\|w\|}$. We set $\eta_\rho = \xi_k(v_\rho)(w_k - v_\rho)$. Since $\eta_\rho \in \mathcal{N}_\lambda$, we get from equation (2.2)

$$I_\lambda(\eta_\rho) - I_\lambda(w_k) \geq -\frac{1}{k}\|\eta_\rho - w_k\|.$$

Now by mean value Theorem, and using $\langle I'_\lambda(\eta_\rho), (w_k - v_\rho)\rangle = 0$, we get

$$-\rho\langle I'_\lambda(w_k), \frac{w}{\|w\|}\rangle + (\xi_k(v_\rho) - 1)\langle I'_\lambda(w_k) - I'_\lambda(\eta_\rho), (w_k - v_\rho)\rangle \geq \frac{-1}{k}\|\eta_\rho - w_k\| + o_k(\|\eta_\rho - w_k\|).$$

Thus

$$\langle I'_\lambda(w_k), \frac{w}{\|w\|}\rangle \leq \frac{1}{k\rho}\|\eta_\rho - w_k\| + \frac{o_k(\|\eta_\rho - w_k\|)}{\rho} + \frac{(\xi_k(v_\rho) - 1)}{\rho}\langle I'_\lambda(w_k) - I'_\lambda(\eta_\rho), (w_k - v_\rho)\rangle. \tag{2.11}$$

Since $\|\eta_\rho - w_k\| \leq \rho|\xi_k(v_\rho)| + |\xi_k(v_\rho) - 1|\|w_k\|$ and $\lim\limits_{\rho\to 0^+} \dfrac{|\xi_k(v_\rho) - 1|}{\rho} = \|\xi'_k(0)\|$, taking limit $\rho \to 0^+$ in (2.11), we get

$$\langle I'_\lambda(w_k), \frac{w}{\|w\|}\rangle \leq \frac{C}{k}(1 + \|\xi'_k(0)\|)$$

for some constant $C > 0$, independent of $w$. So if we can show that $\|\xi'_k(0)\|$ is bounded then we are done. First, we observe that from the last inequality in the proof of Lemma 2.4, the boundednes of $\{w_k\}$ and Lemma 2.7, we obtain that $\|w_k\| \leq C\lambda$. Hence from Lemma 2.6, taking $\Lambda_0$ small enough and Hölder's inequality, for some $M > 0$, we get $\langle \xi'(0), v\rangle = \frac{M\|v\|}{H}$. So to prove the claim we only need to prove that denominator $H$ in the expression of $\langle \xi'(0), v\rangle$ is bounded away from zero. Suppose not. Then there exists a subsequence still denoted $\{w_k\}$ such that

$$\begin{aligned}(2 - p - q)\|w_k\|^2 - \int_{-1}^{1} &(\alpha + \beta - p - q + 2|w_{k_1}|^2 + 2|w_{k_2}|^2)(\alpha + \beta + 2|w_{k_1}|^2 + 2|w_{k_2}|^2) \\ &\times |w_{k_1}|^\alpha |w_{k_2}|^\beta e^{|w_{k_1}|^2 + |w_{k_2}|^2} dx = o_k(1).\end{aligned} \tag{2.12}$$

Therefore $w_k \in \Gamma \setminus \{0\}$ for all $k$ large. Now using the fact that $w_k \in \mathcal{N}_\lambda$, equation (2.12) contradicts (2.4) for $\lambda \in (0, \Lambda_0)$. Hence the proof of the lemma is now complete. □

## 3 Existence and multiplicity results

In this section we show the existence and multiplicity of solutions by minimizing $I_\lambda$ on nonempty decomposition of $\mathcal{N}_\lambda$ and generalized mountain pass theorem for suitable range of $\lambda$.

**Theorem 3.1.** *Let $\lambda$ be such that Lemma 2.1 and Proposition 1 hold. Then there exists a function $\tilde{w} \in \mathcal{N}_\lambda^+$ such that $I_\lambda(\tilde{w}) = \inf\limits_{w\in\mathcal{N}_\lambda^+} I_\lambda(w)$.*

*Proof.* Using Lemma 2.4 and Proposition 1, we get a minimizing sequence $\{w_k\}$ in $\mathcal{N}_\lambda^+$ satisfying equation (2.10). From Lemma 2.4, it follows that $\{w_k\}$ is bounded in $\mathcal{W}_{0,L}^1(\mathcal{C})$. So up to subsequence, $w_{k_1} \rightharpoonup \tilde{w}_1, w_{k_2} \rightharpoonup \tilde{w}_2$ in $\mathcal{W}_{0,L}^1(\mathcal{C}), w_{k_1}(x,0) \to \tilde{w}_1(x,0), w_{k_2}(x,0) \to \tilde{w}_2(x,0)$ pointwise a.e. $(-1,1)$ and $w_{k_1}(x,0) \to \tilde{w}_1(x,0), w_{k_2}(x,0) \to \tilde{w}_2(x,0)$ in $L^r(-1,1)$ for all $r > 1$. Now Hölder's inequality implies that

$$\int_{-1}^{1} f(x)|w_{k_1}(x,0)|^p|w_{k_2}(x,0)|^q dx \to \int_{-1}^{1} f(x)|\tilde{w}_1(x,0)|^p|\tilde{w}_2(x,0)|^q dx.$$

Also using compactness of $w \mapsto \int_{-1}^1 f(x)|w_1(x,0)|^p|w_2(x,0)|^q dx$ and the fact that $w_k \in \mathcal{N}_\lambda$, we get

$$\int_{-1}^{1} g_1(w_k(x,0)w_{k_1}(x,0))dx < \infty \text{ and } \int_{-1}^{1} g_2(w_k(x,0)w_{k_2}(x,0))dx < \infty. \tag{3.1}$$

So from the Vitali's convergence theorem,

$$\int_{-1}^{1} g_1(w_{k_1}(x,0), w_{k_2}(x,0))\phi_1(x,0)dx \to \int_{-1}^{1} g_1(\tilde{w}_1(x,0), \tilde{w}_2(x,0))\phi_1(x,0)dx,$$
$$\int_{-1}^{1} g_2(w_{k_1}(x,0), w_{k_2}(x,0))\phi_2(x,0)dx \to \int_{-1}^{1} g_2(\tilde{w}_1(x,0), \tilde{w}_2(x,0))\phi_2(x,0)dx.$$

Hence $\tilde{w}$ solves $(P_E)_\lambda$ and $\tilde{w} \in \mathcal{N}_\lambda$. Next we show that $\tilde{w} \in \mathcal{N}_\lambda^+$. Now using Lemma 2.7 and (3.1) , we get $\int_{-1}^{1} f(x)|\tilde{w}_1(x,0)|^p|\tilde{w}_2(x,0)|^q dx > 0$. Hence by Lemma 2.2(ii), there exists $t^+(\tilde{w})$ such that $t^+\tilde{w} \in \mathcal{N}_\lambda^+$. Our claim is that $t^+(\tilde{w}) = 1$. Suppose $t^+(\tilde{w}) < 1$ then $t^-(\tilde{w}) = 1$. So $\tilde{w} \in \mathcal{N}_\lambda^-$. Now from Lemma 2.2 (ii), $I_\lambda(t^+(\tilde{w})\tilde{w}) < I_\lambda(\tilde{w}) \leq \theta_\lambda$, which is impossible as $t^+(\tilde{w})\tilde{w} \in \mathcal{N}_\lambda^+$. This completes the proof of the theorem. □

**Theorem 3.2.** *Let $\lambda$ be such that Lemma 2.1 and Proposition 1 hold. Then, the function $\tilde{w} \in \mathcal{N}_\lambda^+$ is a local minimum of $I_\lambda(w)$ in $\mathcal{W}_{0,L}^1(\mathcal{C})$.*

*Proof.* Since $\tilde{w} \in \mathcal{N}_\lambda^+$, we have $t^+(\tilde{w}) = 1 < t_*(\tilde{w})$. Hence since $\Psi_w''(t) < 0$ for $t > 0$, $w \mapsto t_*(w)$ is continuous and given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $1+\epsilon < t_*(\tilde{w}-w)$ for all $\|w\| < \delta$. Also, from Lemma 2.6 we have, for $\delta > 0$ small enough, we obtain a $C^1$ map $t : \mathbf{B}(0,\delta) \to \mathbb{R}^+$ such that $t(w)(\tilde{w}-w) \in \mathcal{N}_\lambda$, $t(0) = 1$. Therefore, for $\delta > 0$ small enough we have $t^+(\tilde{w}-w) = t(w) < 1+\epsilon < t_*(\tilde{w}-w)$ for all $\|w\| < \delta$. Since $t_*(\tilde{w}-w) > 1$, we obtain $I_\lambda(\tilde{w}) \leq I_\lambda(t^+(\tilde{w}-w)(\tilde{w}-w)) \leq I_\lambda(\tilde{w}-w)$ for all $\|w\| < \delta$. This shows that $\tilde{w}$ is a local minimizer for $I_\lambda$ in $\mathcal{W}_{0,L}^1(\mathcal{C})$.

□

### 3.1 Existence of a second solution

Throughout this section, we fix $\lambda \in (0, \Lambda_0)$ and $w = (\tilde{w}_1, \tilde{w}_2)$ as the local minimum of $I_\lambda$ obtained in the previous section. Using min-max methods and Mountain pass lemma around a closed set, we prove the existence of a second solution $(\tilde{z}_1, \tilde{z}_2)$ of $(P_E)_\lambda$ such that $\tilde{z}_1 \geq \tilde{w}_1$ and $\tilde{z}_2 \geq \tilde{w}_2$ in $\mathcal{C}$.

**Definition 3.1.** Let $\mathcal{F} \subset \mathcal{W}^1_{0,L}(\mathcal{C})$ be a closed set. We say that a sequence $\{w_k\} \subset \mathcal{W}^1_{0,L}(\mathcal{C})$ is a Palais-Smale sequence for $I_\lambda$ at level $\rho$ around $\mathcal{F}$, and we denote $(PS)_{\mathcal{F},\rho}$, if

$$\lim_{k\to+\infty} dist\Big(w_k, \mathcal{F}\Big) = 0, \quad \lim_{k\to+\infty} I_\lambda(w_k) = \rho, \quad \lim_{k\to+\infty} \|I'_\lambda(w_k)\| = 0.$$

We have the following version of compactness Lemma based on the Vitali's convergence theorem[1, 13]:

**Lemma 3.1.** *For any $(PS)_{F,\rho}$ sequence $\{w_k\} \subset \mathcal{W}^1_{0,L}(\mathcal{C})$ of $I_\lambda$ for any closed set $\mathcal{F}$. Then there exists $w_0 \in \mathcal{W}^1_{0,L}(\mathcal{C})$ such that, up to a subsequence, $g_1(w_k(x,0)) \to g_1(w_0(x,0))$ , $g_2(w_k(x,0)) \to g_2(w_0(x,0))$ in $L^1(-1,1)$ and $G(w_k(x,0)) \to G(w_0(x,0))$ in $L^1(-1,1)$.*

Let $T = \{(z_1, z_2) : z_1 \geq \tilde{w}_1, z_2 \geq \tilde{w}_2$ a.e. in $\mathcal{C}\}$. We note that $\lim_{t\to+\infty} I(\tilde{w}_1 + tz_1, \tilde{w}_2 + tz_2) = -\infty$ for any $(z_1, z_2) \in \mathcal{W} \setminus \{0\}$. Hence, we may fix $(\overline{w}_1, \overline{w}_2) \in \mathcal{W} \setminus \{0\}$ such that $I_\lambda(\tilde{w}_1 + \overline{w}_1, \tilde{w}_2 + \overline{w}_2) < 0$. We define the mountain pass level

$$\rho_0 = \inf_{\gamma\in\Upsilon} \sup_{t\in[0,1]} I_\lambda(\gamma(t)), \tag{3.2}$$

where $\Upsilon = \{\gamma : [0,1] \to \mathcal{W}^1_{0,L}(\mathcal{C}) : \gamma([0,1]) \subset C,\ \gamma(0) = (\overline{w}_1, \overline{w}_2), \gamma(1) = (\tilde{w}_1 + \overline{w}_1, \tilde{w}_2 + \overline{w}_2)\}$. It follows that $\rho_0 \geq I_\lambda(\tilde{w}_1, \tilde{w}_2)$. If $\rho_0 = I_\lambda(\tilde{w}_1, \tilde{w}_2)$, we obtain that $\inf\{I_\lambda(z_1, z_2) : \|(z_1, z_2) - (\tilde{w}_1, \tilde{w}_2)\| = R\} = I_\lambda(\tilde{w}_1, \tilde{w}_2)$ for all $R \in (0, R_0)$ for some $R_0$ small. We now let $\mathcal{F} = T$ if $\rho_0 > I_\lambda(\tilde{w}_1, \tilde{w}_2)$, and $\mathcal{F} = T \cap \{\|(z_1, z_2) - (\tilde{w}_1, \tilde{w}_2)\| = \frac{R_0}{2}\}$ if $\rho_0 = I_\lambda(\tilde{w}_1, \tilde{w}_2)$.
Now we need the following version of the "sequence of Moser functions concentrated on the boundary" as defined in [2, 17] for the scalar case.

**Lemma 3.2.** *There exists a sequence $\{\phi_k\} \subset H^1_{0,L}(\mathcal{C})$ satisfying*

1. *$\phi_k \geq 0$, $supp(\phi_k) \subset B(0,1) \cap \mathbb{R}^2_+$,*
2. *$\|\phi_k\| = 1$,*
3. *$\phi_k$ is constant on $x \in B(0, \frac{1}{k}) \cap \mathbb{R}^2_+$, and $\phi_k^2 = \frac{1}{\pi} \log k + O(1)$ for $x \in B(0, \frac{1}{k}) \cap \mathbb{R}^2_+$.*

*Proof.* Let

$$\psi_k(x,y) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log k} & 0 \leq \sqrt{x^2+y^2} \leq \frac{1}{k}, \\ \frac{\log \frac{1}{\sqrt{x^2+y^2}}}{\sqrt{\log k}} & \frac{1}{k} \leq \sqrt{x^2+y^2} \leq 1, \\ 0 & \sqrt{x^2+y^2} \geq 1. \end{cases}$$

Then $\int_{\mathbb{R}^2} |\nabla\psi_k|^2 dxdy = 1$ and $\int_{\mathbb{R}^2} |\psi_k|^2 dxdy = O\left(\frac{1}{\log k}\right)$. Let $\tilde{\psi}_k = \psi_k|_{\mathbb{R}^2_+}$ and $\phi_k = \frac{\tilde{\psi}_k}{\|\tilde{\psi}_k\|}$. Then $\phi_k \geq 0$ and $\|\phi_k\| = 1$. Also $\int_{\mathbb{R}^2_+} |\nabla\tilde{\psi}_k|^2 dxdy = \frac{1}{2}$ and $\int_{\mathbb{R}^2_+} |\tilde{\psi}_k|^2 dxdy = O\left(\frac{1}{\log k}\right)$. Therefore $\phi_k^2 = \frac{1}{\pi}\log k + O(1)$. □

We have the following upper bound on $\rho_0$.

**Lemma 3.1.** *With $\rho_0$ defined as in (3.2), we have $\rho_0 < I_\lambda(\tilde{w}_1, \tilde{w}_2) + \frac{\pi}{2}$.*

*Proof.* We prove this lemma by contradiction. Define $\overline{\psi}_k(x) = \frac{1}{\sqrt{2}}\phi_k(x)$, where $\phi_k$ are defined in Lemma 3.2 then we have the following

1. $\|(\bar{\psi}_k, \bar{\psi}_k)\|^2 = 1$,
2. $\int_{\mathbb{R}^2_+} |\bar{\psi}_k|^2 dxdy = O(\frac{1}{\log k})$,
3. $\bar{\psi}_k^2 = \frac{1}{2\pi}\log k + O(1)$.

Now, suppose $\rho_0 \geq I_\lambda(\tilde{w}_1, \tilde{w}_2) + \frac{\pi}{2}$ and we derive a contradiction. This means that for some $t_k,\ s_k > 0$:

$$I_\lambda(\tilde{w}_1 + t_k\psi_k, \tilde{w}_2 + s_k\psi_k) = \sup_{t,s>0} I_\lambda(\tilde{w}_1 + t\psi_k, \tilde{w}_2 + s\psi_k) \geq I_\lambda(\tilde{w}_1, \tilde{w}_1) + \frac{\pi}{2}, \quad \forall k.$$

Since $I_\lambda(\tilde{w}_1 + tz_1, \tilde{w}_1 + sz_2) \to -\infty$ as $t, s \to +\infty$, we obtain that $(t_k, s_k)$ is bounded in $\mathbb{R}^2$. Then, using $\|(\bar{\psi}_k, \bar{\psi}_k)\| = 1$, we obtain

$$\begin{aligned} &\frac{t_k^2 + s_k^2}{4} + t_k \int_{-1}^{1} \nabla\tilde{w}_1 \nabla\bar{\psi}_k dx + s_k \int_{-1}^{1} \nabla\tilde{w}_2 \nabla\bar{\psi}_k dx \\ &\quad \geq \frac{\lambda}{p+q}\left(\int_{-1}^{1} f(x)|\tilde{w}_1 + t_k\bar{\psi}_k|^p|\tilde{w}_2 + s_k\bar{\psi}_k|^q dx - \int_{-1}^{1} f(x)|\tilde{w}_1|^p|\tilde{w}_2|^q dx\right) \\ &\quad + \int_{-1}^{1} G(\tilde{w}_1 + t_k\bar{\psi}_k, \tilde{w}_2 + s_k\bar{\psi}_k)dx - \int_{-1}^{1} G(\tilde{w}_1, \tilde{w}_2)dx + \frac{\pi}{2} \end{aligned}$$

from which together with the fact that $(\tilde{w}_1, \tilde{w}_2)$ is a solution for problem $(P_E)_\lambda$ we obtain

$$\begin{aligned} &\frac{t_k^2 + s_k^2}{4} + \frac{\lambda p}{p+q} t_k \int_{-1}^{1} f(x)|\tilde{w}_1|^{p-2}\tilde{w}_1\bar{\psi}_k|\tilde{w}_2|^q dx + \frac{\lambda q}{p+q} s_k \int_{-1}^{1} f(x)|\tilde{w}_1|^p|\tilde{w}_2|^{q-2}\tilde{w}_2\bar{\psi}_k dx \\ &\quad \geq \frac{\lambda}{p+q}\left(\int_{-1}^{1} f(x)|\tilde{w}_1 + t_k\bar{\psi}_k|^p|\tilde{w}_2 + s_k\bar{\psi}_k|^q dx - \int_{-1}^{1} f(x)|\tilde{w}_1|^p|\tilde{w}_2|^q dx\right) \\ &\quad + \int_{-1}^{1} G(\tilde{w}_1 + t_k\bar{\psi}_k, \tilde{w}_2 + s_k\bar{\psi}_k)dx - \int_{-1}^{1} G(\tilde{w}_1, \tilde{w}_2)dx \\ &\quad - t_k \int_{-1}^{1} g_1(\tilde{w}_1, \tilde{w}_2)\bar{\psi}_k dx - s_k \int_{-1}^{1} g_2(\tilde{w}_1, \tilde{w}_2)\bar{\psi}_k dx + \frac{\pi}{2}. \end{aligned}$$

Now using that $G$ is convex, we get

$$\frac{t_k^2+s_k^2}{4}+\frac{\lambda p}{p+q}t_k\int_{-1}^{1}f(x)|\tilde{w}_1|^{p-2}\tilde{w}_1\bar{\psi}_k|\tilde{w}_2|^q dx+\frac{\lambda q}{p+q}s_k\int_{-1}^{1}f(x)|\tilde{w}_1|^p|\tilde{w}_2|^{q-2}\tilde{w}_2\bar{\psi}_k dx$$
$$-\frac{\lambda}{p+q}\left(\int_{-1}^{1}f(x)|\tilde{w}_1+t_k\bar{\psi}_k|^p|\tilde{w}_2+s_k\bar{\psi}_k|^q dx-\int_{-1}^{1}f(x)|\tilde{w}_1|^p|\tilde{w}_2|^q dx\right)\geq\frac{\pi}{2}.$$

From the Taylor expansion, we have for some $\theta_1,\ \theta_2\in(0,1)$

$$\frac{\lambda p}{p+q}t_k\int_{-1}^{1}f(x)|\tilde{w}_1|^{p-2}\tilde{w}_1\bar{\psi}_k|\tilde{w}_2|^q dx+\frac{\lambda q}{p+q}s_k\int_{-1}^{1}f(x)|\tilde{w}_1|^p|\tilde{w}_2|^{q-2}\tilde{w}_2\bar{\psi}_k dx$$
$$-\frac{\lambda}{p+q}\left(\int_{-1}^{1}f(x)|\tilde{w}_1+t_k\bar{\psi}_k|^p|\tilde{w}_2+s_k\bar{\psi}_k|^q dx-\int_{-1}^{1}f(x)|\tilde{w}_1|^p|\tilde{w}_2|^q dx\right)$$
$$=\lambda\frac{p(p-1)}{p+q}t_k^2\int_{1}^{1}f(x)|\tilde{w}_1+\theta_1 t_k\bar{\psi}|^{p-2}|\tilde{w}_2|^q\bar{\psi}_k^2 dx$$
$$+\lambda\frac{q(q-1)}{p+q}s_k^2\int_{1}^{1}f(x)|\tilde{w}_1|^p|\tilde{w}_2|^q|\tilde{w}_2+\theta_2\bar{\psi}_k|^{q-2}\bar{\psi}_k^2 dx=(t_k^2+s_k^2)O\left(\frac{1}{\log(k)}\right).$$

Hence we obtain that

$$t_k^2+s_k^2\geq 2\pi(1-O\left(\frac{1}{\log(k)}\right)).$$

Since $(t_k,s_k)$ is the critical point for $I_\lambda(\tilde{w}_1+t\bar{\psi}_k,\tilde{w}_2+s\bar{\psi}_k)$, we have

$$\frac{d}{dt}I_\lambda(\tilde{w}_1+t\bar{\psi}_k,\tilde{w}_2+s\bar{\psi}_k)|_{(t,s)=(t_k,s_k)}=0,\ \frac{d}{ds}I_\lambda(\tilde{w}_1+t\bar{\psi}_k,\tilde{w}_2+s\bar{\psi}_k)|_{(t,s)=(t_k,s_k)}=0.$$

Therefore

$$\frac{t_k^2+s_k^2}{2}+t_k\int_{-1}^{1}\nabla\tilde{w}_1\nabla\bar{\psi}_k dx+s_k\int_{-1}^{1}\nabla\tilde{w}_2\nabla\bar{\psi}_k dx$$
$$=\frac{\lambda p}{p+q}\int_{-1}^{1}f(x)|\tilde{w}_1+t_k\bar{\psi}_k|^{p-2}(\tilde{w}_1+t_k\bar{\psi}_k)|\tilde{w}_2+s_k\bar{\psi}_k|^q t_k\bar{\psi}_k dx$$
$$+\frac{\lambda q}{p+q}\int_{-1}^{1}f(x)|\tilde{w}_1+t_k\bar{\psi}_k|^p|\tilde{w}_2+s_k\bar{\psi}_k|^{q-2}(\tilde{w}_2+s_k\bar{\psi}_k)s_k\bar{\psi}_k dx$$
$$+\int_{-1}^{1}g_1(\tilde{w}_1+t_k\bar{\psi}_k,\tilde{w}_2+s_k\bar{\psi}_k)t_k\bar{\psi}_k dx+\int_{-1}^{1}g_2(\tilde{w}_1+t_k\bar{\psi}_k,\tilde{w}_2+s_k\bar{\psi}_k)s_k\bar{\psi}_k dx.$$

Without loss of generality, we can assume that $f$ is positive in the neighborhood of 0, then we get

$$t_k^2+s_k^2+t_k\int_{-1}^{1}\nabla\tilde{w}_1\nabla\bar{\psi}_k dx+s_k\int_{-1}^{1}\nabla\tilde{w}_2\nabla\bar{\psi}_k dx\geq C\int_{-\frac{1}{k}}^{\frac{1}{k}}e^{t_k^2\psi_k^2+s_k^2\bar{\psi}_k^2}(t_k+s_k)\bar{\psi}_k dx$$
$$\geq\frac{C}{\sqrt{\pi}}e^{\left(\frac{t_k^2+s_k^2}{2\pi}-1\right)}(t_k+s_k)\sqrt{\log k}.$$

This and the fact that $t_k^2+s_k^2 \geq 2\pi(1-O\left(\frac{1}{\log(k)}\right))$ imply that $t_k^2+s_k^2 \to \infty$ as $k \to \infty$, which is a contradiction. □

We use the following version of Lions' higher integrability Lemma. The proof follows the same steps as of Lemma 3.4 in [22].

**Lemma 3.2.** *Let $\{w_k\}$ be a sequence in $\mathcal{W}^1_{0,L}(\mathcal{C})$ such that $\|w_k\|=1$, for all $k$ and $w_{k_1} \rightharpoonup w_1$ in $H^1_{0,L}(\mathcal{C})$ and $w_{k_2} \rightharpoonup w_2$ in $H^1_{0,L}(\mathcal{C})$ for some $(w_1,w_2) \neq (0,0)$ in $\mathcal{W}^1_{0,L}(\mathcal{C})$. Then, for $0<p<\pi(1-\|(w_1,w_2)\|^2)^{-1}$,*

$$\sup_{k\geq 1}\int_{-1}^{1} e^{p(w_{k_1}^2+w_{k_2}^2)}dx < \infty.$$

Now, we prove our main result.

**Theorem 3.3.** *For $\lambda \in (0,\Lambda_0)$, problem $(P_E)_\lambda$ has a second nontrivial solution $\hat{w}=(\hat{w}_1,\hat{w}_2)$ such that $\hat{w}_1 \geq \tilde{w}_1 > 0$ and $\hat{w}_2 \geq \tilde{w}_2 > 0$ in $\mathcal{C}$.*

*Proof.* Let $\{w_k\}$ be a Palais-Smale sequence for $I_\lambda$ at the level $\rho_0$ around $\mathcal{F}$ obtained by applying Ekeland Variational principle on $\mathcal{F}$. Then it is easy to verify that the sequence $\{w_k\}$ is bounded in $\mathcal{W}^1_{0,L}(\mathcal{C})$. So upto a subsequence $w_{k_1} \rightharpoonup \hat{w}_1$ and $w_{k_2} \rightharpoonup \hat{w}_2$ in $H^1_{0,L}(\mathcal{C})$. It can be shown that $(\hat{w}_1,\hat{w}_2)$ is a solution of $(P_E)_\lambda$ . It remains to show that $(\hat{w}_1,\hat{w}_2) \not\equiv (\tilde{w}_1,\tilde{w}_2)$. We prove it by contradiction. Suppose it is not so. Then we have only the two following cases:
**Case 1:** $\rho_0 = I_\lambda(\tilde{w}_1,\tilde{w}_2)$.
In this case, using Lemma 3.1 we have

$$\begin{aligned} I_\lambda(\tilde{w}_1,\tilde{w}_2)+o(1) &= I_\lambda(w_{k_1},w_{k_2}) \\ &= \frac{1}{2}\|w_k\|^2 - \frac{\lambda}{p+q}\int_{-1}^{1} f(x)|w_{k_1}|^p|w_{k_2}|^q dx - \int_{-1}^{1} G(w_{k_1},w_{k_2})dx. \end{aligned}$$

Thus from Lemma 3.1, we have $\|(w_k-\tilde{w})\| = o(1)$, which contradicts the fact that $w_k \in \mathcal{F}$.
**Case 2:** $\rho_0 \neq I_\lambda(\tilde{w}_1,\tilde{w}_2)$.
In this case $\rho_0 - I_\lambda(\tilde{w}_1,\tilde{w}_2) \in (0,\frac{\pi}{2})$ and $I_\lambda(w_{k_1},w_{k_2}) \to \rho_0$.
Let $\beta_0 = \frac{\lambda}{p+q}\int_{-1}^{1} f(x)|w_{k_1}(x,0)|^p|w_{k_2}(x,0)|^q dx + \int_{-1}^{1} G(\tilde{w}_1,\tilde{w}_2)dx$. Then from Lemma 3.1,

$$\frac{1}{2}\|(w_{k_1},w_{k_2})\|^2 \to (\rho_0+\beta_0) \quad \text{as } k\to\infty. \tag{3.3}$$

By Lemma 3.1, for $\epsilon>0$ small, we have

$$(1+\epsilon)(\rho_0 - I_\lambda(\tilde{w}_1,\tilde{w}_2))) < \frac{\pi}{2}.$$

Hence, from (3.3) we have

$$
\begin{aligned}
(1+\epsilon)\|(w_{k_1},w_{k_2})\|^2 &< \pi\frac{\rho_0+\beta_0}{\rho_0+\beta_0-\frac{1}{2}\|(\tilde{w}_1,\tilde{w}_2)\|^2}\\
&< \pi\Big(1-\frac{1}{2}(\frac{\|(\tilde{w}_1,\tilde{w}_2)\|^2}{\rho_0+\beta_0})\Big)^{-1}\\
&< \pi\Big(1-\|\frac{w_{k_1}}{\sqrt{2(\rho_0+\beta_0)}}\|^2-\|\frac{w_{k_2}}{\sqrt{2(\rho_0+\beta_0)}}\|^2\Big)^{-1}.
\end{aligned}
$$

Now, choose $p$ such that

$$
(1+\epsilon)\|(w_{k_1},w_{k_2})\|^2 \le p < \pi(1-\|\frac{w_{k_1}}{\sqrt{2(\rho_0+\beta_0)}}\|^2-\|\frac{w_{k_2}}{\sqrt{2(\rho_0+\beta_0)}}\|^2)^{-1}.
$$

Since $\frac{w_{k_1}}{\|(w_{k_1},w_{k_2})\|} \rightharpoonup \frac{\tilde{w}_1}{\sqrt{2(\rho_0+\beta_0)}}$ and $\frac{w_{k_2}}{\|(w_{k_1},w_{k_2})\|} \rightharpoonup \frac{\tilde{w}_2}{\sqrt{2(\rho_0+\beta_0)}}$ weakly in $H^1_{0,L}(-1,1)$, by Lemma 3.2, we have

$$
\sup_k \int_\Omega \exp\Big(p\big[(\frac{w_{k_1}}{\|(w_{k_1},w_{k_2})\|})^2+(\frac{w_{k_2}}{\|(w_{k_1},w_{k_2})\|})^2\big]\Big)dx < \infty. \tag{3.4}
$$

Now, from (3.4), together with Hölder inequality we have as $k\to\infty$

$$
\int_\Omega g_1(w_{k_1},w_{k_2})w_{k_1}dx \to \int_\Omega g_1(\tilde{w}_1,\tilde{w}_2)\tilde{w}_1dx; \quad \int_\Omega g_2(w_{k_1},w_{k_2})w_{k_2}dx \to \int_\Omega g_2(\tilde{w}_1,\tilde{w}_2)\tilde{w}_2dx.
$$

Hence,

$$
\begin{aligned}
o(1) &= I'_\lambda(w_{k_1},w_{k_2})(w_{k_1},0)\\
&= \int_{-1}^1 |\nabla w_{k_1}|^2dx - \lambda\frac{p}{p+q}\int_{-1}^1 f(x)|w_{k_1}|^p|w_{k_2}|^q dx - \int_{-1}^1 g_1(w_{k_1},w_{k_2})w_{k_1}dx\\
&= I'_\lambda(\tilde{w}_1,\tilde{w}_2)(\tilde{w}_1,0)+o(1)
\end{aligned}
$$

from which we deduce $w_{k_1}\to\tilde{w}_1$. Similarly, we obtain $w_{k_2}\to\tilde{w}_2$. This is a contradiction to the assumption that $\rho_0 \neq I_\lambda(\tilde{w}_1,\tilde{w}_2)$. □

## 4 Superlinear problems

In this section we consider existence of solution for the problem $(P)$ We use the idea of harmonic extension to solve the problem $(P)$. The extension problem corresponding to the problem $(P)$ can be considered as

$$(P_E)\begin{cases} -\Delta w_1 = -\Delta w_2 &= 0,\ w_1>0, w_2>0 \text{ in } \mathcal{C}=(-1,1)\times(0,\infty),\\ w_1=w_2 &= 0 \text{ on } \{-1,1\}\times(0,\infty),\\ \frac{\partial w_1}{\partial \nu} &= h_1(w_1,w_2) \text{ on } (-1,1)\times\{0\},\\ \frac{\partial w_2}{\partial \nu} &= h_2(w_1,w_2) \text{ on } (-1,1)\times\{0\}.\end{cases}$$

where $\frac{\partial w_1}{\partial \nu} = \lim_{y\to 0^+}\dfrac{\partial w_1}{\partial y}(x,y)$ and $\frac{\partial w_2}{\partial \nu} = \lim_{y\to 0^+}\dfrac{\partial w_2}{\partial y}(x,y)$. As discussed in the introduction, the problem in $(P)$ is equivalent to solving $(P_E)$ on $\mathcal{W}^1_{0,L}(\mathcal{C})$. The variational functional, $I:\mathcal{W}^1_{0,L}(\mathcal{C})\to\mathbb{R}$ related to the problem $(P_E)$ is given as

$$I(w_1,w_2)=\frac{1}{2}\int_{\mathcal{C}}|\nabla w_1|^2dxdy+\frac{1}{2}\int_{\mathcal{C}}|\nabla w_2|^2dxdy-\int_{-1}^{1}H(w_1(x,0),w_2(x,0))dx.$$

Any function $w\in\mathcal{W}^1_{0,L}(\mathcal{C})$ is called the weak solution of $(P_E)$ if for any $\phi=(\phi_1,\phi_2)\in\mathcal{W}^1_{0,L}(\mathcal{C})$

$$\int_{\mathcal{C}}(\nabla w_1.\nabla\phi_1+\nabla w_2.\nabla\phi_2)\,dxdy=\int_{-1}^{1}(h_1(w(x,0))\phi_1(x,0)+h_2(w(x,0))\phi_2(x,0))\,dx \quad (4.5)$$

It is clear that critical points of $I$ in $\mathcal{W}^1_{0,L}(\mathcal{C})$ corresponds to the critical points of $J$ in $H_0^{\frac{1}{2}}(\mathbb{R})$. Thus if $(w_1,w_2)$ solves $(P_E)$ then $(u,v)=\text{trace}(w_1,w_2)=(w_1(x,0),w_2(x,0))$ is the solution of problem $(P)$ and vice versa.
We will use the mountain pass lemma to show the existence of a solution in the critical case.

**Lemma 4.1.** *Assume that the conditions $(h1)-(h5)$ hold. Then $I$ satisfies the mountain pass geometry around* $0$.

*Proof.* Using assumption $(h_4)$ , we get

$$H(s,t)\geq C_1|s|^{\mu_1}+C_2|t|^{\mu_2}-C_3$$

for some $C_1,C_2,C_3>0$ and $\mu_1,\mu_2>2$. Hence for function $w\in\mathcal{W}^1_{0,L}(\mathcal{C})\cap C^\infty$ with support in $[-1,1]\times(0,1)$, we get

$$I(tw)\leq\frac{1}{2}t^2\|w\|^2-C_1t^{\mu_1}\int_{-1}^{1}|w_1(x,0)|^{\mu_1}dx-C_2t^{\mu_2}\int_{-1}^{1}|w_2(x,0)|^{\mu_2}dx+C_3.$$

Hence $I(tw)\to-\infty$ as $t\to\infty$. Next we will show that there exists $\alpha,\rho>0$ such that $I(w)>\alpha$ for all $\|w\|<\rho$. From $(h1)$ and $(h5)$, for $\epsilon>0$ $r>2$ there exists $C_1>0$ such that

$$|H(s,t)|\leq\frac{\lambda_1-\epsilon}{2}(s^2+t^2)+C_1(|s|^r+|t|^r)e^{(1+\epsilon)(s^2+t^2)}.$$

Hence, using the Hölder's inequality, (1.1) and for $\rho$ small enough, we get for $\ell > 1$

$$\int_{-1}^{1} |H(w(x,0))|dx \leq \frac{\lambda_1 - \epsilon}{2}\|w(x,0)\|^2_{L^2(-1,1)} + C_1\|w_1(x,0)\|^r_{L^{r\ell}(-1,1)} + C_2\|w_2(x,0)\|^r_{L^{r\ell}(-1,1)}.$$

Now using Sobolev embedding and choosing $\|w\| = \rho$ for sufficiently small $\rho$, we get

$$I(w) \geq \frac{1}{2}\left(1 - \frac{\lambda_1 - \epsilon}{\lambda_1}\right)\rho^2 - C_3\rho^r.$$

Hence we get $\alpha > 0$ such that $I(w) > \alpha$ for all $\|w\| = \rho$ for sufficiently small $\rho$. □

Next we show the boundedness of Palais-Smale sequences ($(PS)$ sequences for short).

**Lemma 4.2.** *Every Palais-Smale sequence of $I$ is bounded in $H^1_{0,L}(\mathcal{C})$.*

*Proof.* Let $\{w_k\} = \{(w_{k_1}, w_{k_2})\}$ be a $(PS)_c$ sequence, that is

$$\frac{1}{2}\|w_k\|^2 - \int_{-1}^{1} H(w_k(x,0))dx = c + o(1) \text{ and} \tag{4.6}$$

$$\|w_k\|^2 - \int_{-1}^{1} h_1(w_k(x,0))w_{k_1}(x,0)dx - \int_{-1}^{1} h_2(w_k(x,0))w_{k_2}(x,0)dx = o(\|w_k\|). \tag{4.7}$$

Therefore,

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|w_k\|^2 - \frac{1}{\mu}\int_{-1}^{1}\left(\mu H(w_k) - h_1(w_k)w_{k_1} - h_2(w_k)w_{k_2}\right)dx = c + o(\|w_k\|).$$

Using assumption $(h2)$, we get $\|w_k\| \leq C$ for some $C > 0$. □

We have the following version of compactness Lemma:

**Lemma 4.3.** *For any $(PS)_c$ sequence $\{w_k\}$ of $I$, there exists $\ddot{w} \in \mathcal{W}^1_{0,L}(\mathcal{C})$ such that, up to subsequence, $h_1(w_k(x,0)) \to h_1(\ddot{w}(x,0))$ in $L^1(-1,1)$, $h_2(w_k(x,0)) \to h_2(\ddot{w}(x,0))$ in $L^1(-1,1)$ and $H(w_k(x,0)) \to H(\ddot{w}(x,0))$ in $L^1(-1,1)$.*

Define $\Gamma = \{\gamma \in C([0,1]; \mathcal{W}^1_{0,L}(\mathcal{C})) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$ and the corresponding mountain pass level as $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t))$.

**Lemma 4.4.** $c < \frac{\pi}{2}$.

*Proof.* We prove by contradiction. Suppose $c \geq \pi 2$. Then we have

$$\sup_{t,s\geq 0} I(s\bar{\psi}_k, t\bar{\psi}_k) = I(s_k\bar{\psi}_k, t_k\bar{\psi}_k) \geq \frac{\pi}{2} \tag{4.8}$$

where functions $\phi_k$ are given by Lemma 3.1. From equation (4.8) and since $H \geq 0$ on $\mathbb{R}^2_+$, we get

$$s_k^2 + t_k^2 \geq 2\pi.$$

Now as $(s_k, t_k)$ is point of maximum we get $\frac{\partial}{\partial s}I(s\bar{\psi}_k, t\bar{\psi}_k)|_{s=s_k} = 0$ and $\frac{\partial}{\partial t}I(s\bar{\psi}_k, t\bar{\psi}_k)|_{t=t_k} = 0$. Therefore by $(h4)$,

$$\begin{aligned}\frac{s_k^2}{2} + \frac{t_k^2}{2} &= s_k^2\|\bar{\psi}_k\|^2 + t_k^2\|\bar{\psi}_k\|^2 \\ &= \int_{-1}^{1} \left(h_1(s_k\bar{\psi}_k(x,0), t_k\bar{\psi}_k(x,0))s_k + h_2(s_k\bar{\psi}_k(x,0), t_k\bar{\psi}_k(x,0))t_k\right) \bar{\psi}_k(x,0)dx \\ &\geq 2\sqrt{\log(k)}(s_k h_1(\frac{s_k\sqrt{\log(k)}}{\sqrt{2\pi}}, \frac{t_k\sqrt{\log(k)}}{\sqrt{2\pi}}) + t_k h_2(\frac{s_k\sqrt{\log(k)}}{\sqrt{2\pi}}, \frac{t_k\sqrt{\log(k)}}{\sqrt{2\pi}})e^{-\log(k)} \\ &\to \infty \quad \text{as } k \to \infty\end{aligned}$$

which contradicts the boundedness of $(t_k)_{k\in I\!N}$ and $(s_k)_{k\in I\!N}$. □

Next we prove Theorem 1.3 using the above Lemmas.

**Proof of Theorem 1.3:** Using the variational Ekeland principle and Lemma 4.2, there exists a bounded $(PS)_c$ sequence. So there exists $\ddot{w} \in \mathcal{W}^1_{0,L}(\mathcal{C})$ such that, upto a subsequence, $w_k \rightharpoonup \ddot{w}$ in $\mathcal{W}^1_{0,L}(\mathcal{C})$ and $w_k(x,0) \to \ddot{w}(x,0)$ pointwise. We first prove that $\ddot{w}$ solves the problem, then we show that $\ddot{w}$ is non zero. From Lemma 4.2 and equation (4.6) together with $h_1$, $h_2 \geq 0$, we get for some constant $C > 0$,

$$\int_{-1}^{1} h_1(w_k(x,0))w_{k_1}(x,0)dx \leq C, \ \int_{-1}^{1} h_2(w_k(x,0))w_{k_2}(x,0)dx \leq C, \ \int_{-1}^{1} H(w_k(x,0))dx \leq C.$$

Now from Lemma 4.3, we get $h_1(w_k(x,0)) \to h_1(\ddot{w}(x,0))$ in $L^1(-1,1)$. So for $\psi \in C_c^\infty$ the equation (4.5) holds. Hence from density of $C_c^\infty$ in $\mathcal{W}^1_{0,L}(\mathcal{C})$, $\ddot{w}$ is weak solution of $(P_E)$.
Next we claim that $\ddot{w} \not\equiv 0$. Suppose not. Then from Lemma 4.3, we get $H(w_k(x,0)) \to 0$ in $L^1(\mathbb{R})$. Hence from equations (4.6) and (4.7), we get $\frac{1}{2}\|w_k\|^2 \to c$ as $k \to \infty$. Hence, from Lemma 4.4 $\sup_k \|w_k\|^2 \leq \pi - \epsilon$ for some $\epsilon > 0$. Let $0 < \delta < \frac{\epsilon}{\pi}$ and $q = \frac{\pi}{(1+\delta)(1+\epsilon)(\pi-\epsilon)} > 1$. Using Moser- Trudinger inequality (1.1), we get

$$\begin{aligned}\int_{-1}^{1} |h_1(w_k)w_{k_1}|^q dx &\leq A\int_{-1}^{1} |w_{k_1}(x,0)|^{2q}dx + C\int_{-1}^{1} e^{q(1+\epsilon)(1+\delta)(w_{k_1}^2 + w_{k_2}^2)}dx \\ &\leq C_1\|w_k\|^{2q} + C\int_{-1}^{1} e^{q(1+\delta)(1+\epsilon)\|w_k\|^2\frac{w_{k_1}^2 + w_{k_2}^2}{\|w_k\|^2}}dx < \infty.\end{aligned}$$

Similarly $\int_{-1}^{1} |h_1(w_k)w_{k_1}|^q dx < \infty$. Therefore by

$$\int_{-1}^{1} h_1(w_k(x,0))w_{k_1}(x,0)dx \to 0, \ \int_{-1}^{1} h_2(w_k(x,0))w_{k_2}(x,0)dx \to 0$$

and from equation (4.6), we get $\lim_k \|w_k\|^2 = 0$, which contradicts $c \geq \alpha$. Hence $\ddot{w}$ is a non-trivial solution of the problem $(P_E)$.

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## References

[1] Adimurthi, *Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 17 (1990), 393–413.

[2] Adimurthi and S. L. Yadava, *Critical exponent problem in* $\mathbb{R}^2$ *with Neumann boundary conditions*, Commun. in Partial Differential Equations 15(4),(1990), 461–501.

[3] K. Adriouch and A. EL Hamidi, *The Nehari manifold for systems of nonlinear elliptic equations*, Nonlinear Anal. 64 (2006), 2149–2164.

[4] C.O. Alves, D.C. de M. Filho and M.A.S. Souto, *On systems of elliptic equations involving subcritical or critical Sobolev exponents*, Nonlinear Anal. 42 (2000), 771–787.

[5] D. Applebaum, *Levy processes - From probability fo finance and quantum groups*, Notices Amer. Math. Soc. 51,(2004), 1336–1347.

[6] Y. Bozhkov and E. Mitidieri, *Existence of multiple solutions for quasilinear systems via fibering method*, J. Differ. Equat. 190 (2003), 239–267.

[7] K. J. Brown and T. F. Wu, *A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function*, J. Math. Anal. Appl. 337 (2008), 1326–1336.

[8] X. Cabré and J.G. Tan, *Positive solutions of nonlinear problem involving the square root of the Laplacian*, Adv. Math. 224(2), (2010), 2052–2093.

[9] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32, (2007), 1245–1260.

[10] C.Y. Chen and T.F. Wu, *The Nehari manifold for indefinite semilinear elliptic systems involving critical exponent*, Applied Mathematics and Computation 218 (2012), 10817–10828.

[11] W. Chen and S. Deng, *The Nehari manifold for a fractional p-Laplacian system involving concave convex nonlinearities*, Nonlinear Analysis: Real World Applications 27 (2016), 80–92.

[12] P. Clément, D.G. de Figueiredo and E. Mitidieri, *Positive solutions of semilinear elliptic systems*, Commun. Partial Differ. Equat. 17 (1992), 923–940.

[13] D. G. de Figueiredo, O. H. Miyagaki and B. Ruf, *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations, 3(2) (1995), 139–153.

[14] D. G. de Figueiredo, J. M. Do Ò and B. Ruf, *Critical and subcritical elliptic systems in dimension two*, Indiana Univ. Math. J. 53(4) (2004), 1037–1054.

[15] D.G. de Figueiredo and P. Felmer, *On superquadratic elliptic systems*, Trans. Amer. Math. Soc. 343 (1994), 99–116.

[16] A. El Hamidi, *Existence results to elliptic systems with nonstandard growth conditions*, J. Math. Anal. Appl. 300 (2004), 30–42.

[17] J. Giacomoni, P.K. Mishra and K. Sreenadh, *Fractional elliptic equations with critical exponential nonlinearity*, Advances in Nonlinear Analysis, online doi:10.1515/anona-2015-0081.

[18] X. He, M. Squassina and W. Zou, *The Nehari manifold for fractional systems involving critical nonlinearities*, preprint available on arxiv, http://arxiv.org/abs/1509.02713

[19] A. Iannizzotto and M. Squassina, *1/2-laplacian problems with exponential nonlinearity*, J. Math. Anal. Appl. 414(6) (2014), 372-385.

[20] H. Kozono, T. Sato and H.Wadade, Upper bound of the best constant of a Trudinger-Moser inequality and its application to Gagliardo-Nirenberg inequality, Indian Univ. Math. J. 55 (2006), 1951–1974.

[21] L. Martinazzi, *Fractional Adams Moser-Trudinger inequalities,* priprint available at Arxiv 1506.00489v1.

[22] N. Megrez, K. Sreenadh and B Khalidi, *Multiplicity of positive solutions for a gradient sysytem with an exponential nonlinearity*, Electronic Journal Differential Equations 236 (2012), 1–16.

[23] G. Molica Bisci and R. Servadei, *Lower semicontinuity of functionals of fractional type and applications to nonlocal equations with critical Sobolev exponent*, Adv. Differential Equations, 20, (2015), 635–660.

[24] G. Molica Bisci and R. Servadei, *A Brezis-Nirenberg spitting approach for nonlocal fractional equations*, Nonlinear Anal., 119, (2015), 341–353.

[25] T. Ozawa, *On critical cases of Sobolev's inequalities*, J. Funct. Anal., 127, (1995), 259–269.

[26] S. Prashanth and K. Sreenadh, *Multiplicity of solutions to a nonhomogeneous elliptic equation in* $\mathbb{R}^2$ , Differential and Integral Equations 18(6) (2005), 681–698.

[27] M. Squassina, *An eigenvalue problem for elliptic systems*, New York J. Math. 6 (2000), 95–106.

[28] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. 367 (2015), 67-102.

[29] J. Tan, *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations 36(1-2) (2011), 21–41.

[30] J. Vélin, *Existence results for some nonlinear elliptic system with lack of compactness*, Nonlinear Analysis, 52 (2003), 1017–1034.

[31] T. F. Wu, *The Nehari manifold for a semilinear elliptic system involving sign-changing weight functions*, Nonlinear Analysis 68 (2008), 1733–1745.

[32] X. Yu, *The Nehari manifold for elliptic equation involving the square root Laplacian*, Journal of Differential Equations, 252 (2012), 1283–1308.

[33] J. Zhang, X. Liu and H. Jiao, *Multiplicity of positive solutions for a fractional Laplacian equations involving critical nonlinearity*, preprint available at arXiv:1502.02222v1 .