

Blow up boundary solutions of some semilinear fractional equations in the unit ball

Mohamed Ben Chrouda^a Mahmoud Ben Fredj^b

November 9, 2015

^a*Dep. of Mathematics, High institut of computer sciences and mathematics, 5000, Monastir, Tunisia*

^b*Dep. of Mathematics, Faculty of Sciences of Monastir, 5019 Monastir, Tunisia*

Abstract

For $\gamma > 0$, we are interested in blow up solutions $u \in \mathcal{C}^+(B)$ of the fractional problem in the unit ball B

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = u^\gamma & \text{in } B \\ u = 0 & \text{in } B^c. \end{cases} \quad (1)$$

We distinguish particularly two orders of singularity at the boundary: solutions exploding at the same rate than $\delta^{1-\frac{\alpha}{2}}$ (δ denotes the Euclidean distance) and those higher singular than $\delta^{1-\frac{\alpha}{2}}$. As a consequence, it will be shown that the classical Keller-Osserman condition can not be readopted in the fractional setting.

Keywords: Fractional Laplacian, semilinear equation, large solution, fractional Dirichlet problem, Green function, Martin kernel.

1 Introduction

Let $N \geq 3$, $0 < \alpha < 2$ and let $(X_t)_{t \geq 0}$ be the standard α -stable process in \mathbb{R}^N . It is determined by its characteristic function which takes the form

$$E^x (e^{i\xi(X_t - X_0)}) = e^{-t|\xi|^\alpha} \quad ; \quad \xi \in \mathbb{R}^N,$$

where E^x is the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^N$. It is a discontinuous Markov process and give rise to equations with the fractional Laplacian $\Delta^{\frac{\alpha}{2}}$.

Nonlocal operators such as $\Delta^{\frac{\alpha}{2}}$ naturally arise in population dynamics, continuum mechanics, game theory and some other fields, we quote for instance [2, 21, 30, 32, 36, 37], one can see also [11] for broader discussion.

In the classical setting, i.e. for $\alpha = 2$, one of the most commonly considered equations in the literature is the following:

$$\Delta u = \varphi(u) \quad \text{in } B, \quad (2)$$

where B is the unit ball of \mathbb{R}^N and $\varphi : [0, \infty[\rightarrow \mathbb{R}$ is a some nonnegative nondecreasing function. Such equations appear naturally in several interesting contexts including in the random systems of branching particles [20, 29]. Solutions of (2) verifying

$$\lim_{x \rightarrow \partial B} u(x) = \infty$$

are called large solutions or blow up (boundary) solutions. Merely for the sake of completeness, we shall recall the pioneering work of [26] and [33]. They proved independently that a necessary and sufficient condition for the existence of a large solution to (2), where φ is a positive nondecreasing function, is

$$\int_1^\infty \left(\int_0^s \varphi(t) dt \right)^{-\frac{1}{2}} ds < \infty,$$

that in the case of $\varphi(u) = u^\gamma$, means $\gamma > 1$.

The aim of the paper is to study (2) substituting the classical Laplacian by one of its fractional powers. Our consideration is motivated by the natural question whether the classical results (particularly the Keller-Osserman condition) in this field may be extended to nonlocal operators.

The behavior of α -harmonic functions contrasts, in some respects, with the one of the classical harmonic functions. Indeed, due to the jumping nature of the α -stable process, roughly speaking, at the exit time one could end up anywhere outside the domain. Put differently, the process typically leaves domains by jumping to the interior of its complement while the continuous paths of Brownian motion leave domains by hitting the boundary. This spans the existence of positive harmonic functions on B blowing up at the boundary. Such functions are called singular α -harmonic functions and they are harmonic for the stable process $(X_t^B)_{t \geq 0}$ killed on leaving B . In the Brownian motion case, such functions do not exist due to the Fatou theorem and nontangential convergence of positive harmonic functions [3, 19, 38]. In this sense, the Martin kernel M_B^α

$$\nu \rightarrow \int_{\partial B} \frac{(1 - |x|^2)^{\frac{\alpha}{2}}}{|x - y|^N} \nu(dy) \quad ; \quad x \in B$$

provides a one-to-one correspondence between positive Random measures ν on ∂B and positive α -harmonic functions on B which supports the fact that singular α -harmonic functions constitute, in some respects, the appropriate class of "harmonic functions". Nevertheless, the usual probabilistic interpretation of singular α -harmonic functions on B as solutions of Dirichlet problem is no longer true. The interested reader is referred to [16, Theorem 3.18] where the authors provide some probabilistic interpretation of these functions. In particular,

$$M_B^\alpha 1(x) := \int_{\partial B} \frac{(1 - |x|^2)^{\frac{\alpha}{2}}}{|x - y|^N} \sigma(dy) \quad ; \quad x \in B,$$

defines a singular α -harmonic on B and it behaves like $\delta(x)^{\frac{\alpha}{2}-1}$ on B . Here, σ denotes the surface area measure on ∂B and $\delta(x) := 1 - |x|$ is the Euclidean distance from x to the boundary ∂B . This entitles us to study the following appropriate reformulated

semilinear Dirichlet problem associated to $\Delta^{\frac{\alpha}{2}}$ in B taking in account the aspects raised above.

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = u^\gamma & \text{in } B \\ u = 0 & \text{in } B^c \\ \lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} u(x) = g(z) \end{cases} \quad (3)$$

where $\gamma > 0$ and g is a nonnegative continuous function on ∂B . It is worth noting that solutions of this problem explode on the boundary at the same rate than $\delta^{\frac{\alpha}{2}-1}$ while if we take $\alpha = 2$ they don't. That is why solutions to the problem (3), behaving like the singular α -harmonic function $M_B^\alpha 1$, in some respects, are not blow up solutions in the strict sens of the word. We rather call them "moderate blow up solutions". In this spirit, we shall consider a second semilinear Dirichlet problem

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = u^\gamma & \text{in } B \\ u = 0 & \text{in } B^c \\ \lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} u(x) = \infty \end{cases} \quad (4)$$

where $\gamma > 0$ as before and we shall call its solutions by "blow up solutions". Note that they explode on the boundary even for $\alpha = 2$. In other words, such solutions, are somehow the ones that "large solution" (or "blow up solution") are in the classical setting.

Recently, a growing and renewed attention is paying to existence of boundary blow up solutions to equations involving fractional powers of the Laplacian in bounded domains. Among the works, we quote [1,12,13,23] (the list is remotely incomplete). In [13], authors proved the existence of a real $\tau_0(\alpha) \in]-1, 0[$ such that for

$$1 + \alpha < \gamma < 1 - \frac{\alpha}{\tau_0(\alpha)}, \quad (5)$$

the problem (1) admits a solution u verifying

$$0 < \liminf_{x \rightarrow \partial B} \delta(x)^{\frac{\alpha}{\gamma-1}} u(x) \leq \limsup_{x \rightarrow \partial B} \delta(x)^{\frac{\alpha}{\gamma-1}} u(x) < \infty.$$

Let us emphasize from the very beginning that (5) turns out to be nearly optimal and it will appear also in our work even though our treated problem is slightly different and the notion of solution is not the same. Indeed, as we will precise later, we deal with distributional solutions (those satisfying the equation when integrated against a suitable set of test functions) and not with viscosity solutions (i.e. every smooth function touching from below or above the continuous solution is required to be a supersolution or a subsolution) as the case in [23] or in [13]. We shall not elaborate further on this but we just want to retain that for fractional equations, the notion of weak solution implies the one of viscosity solution [34].

Let us now describe the main results of this paper more precisely and at the same time give the outline. We first record some tools of potential theory pertaining to the killed process. Next, we prove some technical lemmas and in particular, we take a closer look at the potential $G_B^\alpha(\delta^{-\lambda})$, $\lambda < 1 + \frac{\alpha}{2}$. We shall afterwards characterise nonnegative solutions to problem (3), in the case where $0 < \gamma < \frac{2+\alpha}{2-\alpha}$, in terms of the Green operator G_B^α and

the Martin kernel M_B^α . More precisely, we prove in this case that u is a nonnegative solution of problem (3) if and only if

$$u + G_B^\alpha u^\gamma = M_B^\alpha g.$$

Then, we shall use the previous material to establish our first main result. More specifically, for every nontrivial boundary datum $g \in \mathcal{C}^+(\partial B)$, we prove that problem (3) admits a nonnegative solution $u \in \mathcal{C}(B)$ if and only if $0 < \gamma < \frac{2+\alpha}{2-\alpha}$ and that the solution is unique. The last section will be devoted to the problem (4). Namely we divide the real half-line into four intervals. We prove that problem (4) has a nonnegative solution $u \in \mathcal{C}(B)$ for $1 + \alpha < \gamma < \frac{2+\alpha}{2-\alpha}$ and that it has no solution for $0 < \gamma < 1 + \frac{\alpha}{2}$. We have left open the ranges $1 + \frac{\alpha}{2} \leq \gamma \leq 1 + \alpha$ and $\gamma \geq \frac{2+\alpha}{2-\alpha}$. It is worth noting that the classical Keller-Osserman condition can not be fundamentally readopted in the framework of fractional Laplacian, else, taking $\alpha = 2$, we obtain that (4) has no solution for $\gamma < 2$, which is not consistent with the Brownian motion case, because as we recalled above, this problem admits a solution for every $\gamma \geq 1$ and hence for $1 \leq \gamma < 2$. In other words, the classical Keller-Osserman condition is no longer the range where large solutions exist or do not exist.

Motivated by the considerable advances on extending potential-theoretic properties of Brownian motion to symmetric stable process [7–10, 15–17, 27], there is often an interesting natural generalization of classical results. This work hopefully allows us to better understand how the nonlocal character and the jumping nature of stable processes influences the boundary behavior of solutions. We would like to point out that the approach which we follow is based essentially on some analytic tools from potential theory and it is completely different from that used in [13]. In this paper we do not use probabilistic techniques in an essential way, keeping them only for interpretation.

2 Preliminaries

For every subset F of \mathbb{R}^N , let $\mathcal{B}(F)$ be the set of all Borel measurable functions on F and let $\mathcal{C}(F)$ be the set of all continuous real-valued functions on F . If \mathcal{G} is a set of numerical functions then \mathcal{G}^+ (respectively \mathcal{G}_b) will denote the class of all functions in \mathcal{G} which are nonnegative (respectively bounded). $\mathcal{C}^k(F)$ is the class of all functions that are k times continuously differentiable on F . The uniform norm will be denoted by $\|\cdot\|$.

For every subset A of \mathbb{R}^N we denote its complement by $A^c = \mathbb{R}^N \setminus A$, its closure by \bar{A} and its Euclidean boundary by $\partial A = \bar{A} \cap \bar{A}^c$.

For two nonnegative functions f and g , the notation $f \approx g$ means that there are positive constants c_1 and c_2 such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ in the common domain of definition for f and g .

Let $\alpha \in]0, 2[$ and $N \geq 3$. The fractional Laplacian $\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$ in \mathbb{R}^N is a nonlocal operator defined on the Schwartz class through the Fourier transform

$$\mathcal{F}\left((-\Delta)^{\frac{\alpha}{2}} f\right)(\xi) = (2\pi|\xi|)^\alpha \mathcal{F}(f),$$

where $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i y \cdot \xi} f(y) dy$ up a constant. It does not act by pointwise differentiation but by a global integration with respect to a singular kernel: for every Borel

function u for which the integral exists,

$$\Delta^{\frac{\alpha}{2}}u(x) = c_{N,\alpha} \int_{\mathbb{R}^N} \frac{u(y) - u(x)}{|y - x|^{N+\alpha}} dy; \quad x \in \mathbb{R}^N,$$

in such a way the nonlocal character is emphasized. Here the constant $c_{N,\alpha}$ is depending only on N and α : $c_{N,\alpha} = 2^\alpha \pi^{-\frac{N}{2}} |\Gamma(\frac{-\alpha}{2})|^{-1} \Gamma(\frac{N+\alpha}{2})$. The natural space that we are going to use is a weighted \mathcal{L}^1 -space:

$$\mathcal{L}_\alpha = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ Borel measurable ; } \int_{\mathbb{R}^N} \frac{|u(y)|}{(1 + |y|)^{N+\alpha}} dy < \infty \right\}.$$

In this space, we can define $\Delta^{\frac{\alpha}{2}}$ as a distribution by

$$\langle \Delta^{\frac{\alpha}{2}}u, \varphi \rangle = \int_{\mathbb{R}^N} u(y) \Delta^{\frac{\alpha}{2}}\varphi(y) dy; \quad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N),$$

where $\mathcal{C}_c^\infty(\mathbb{R}^N)$ is the set of all infinitely differentiable functions on \mathbb{R}^N with compact support. In probabilistic terms, the fractional operator $\Delta^{\frac{\alpha}{2}}$ is the infinitesimal generator of α -stable processes which constitute an important class of discontinuous Markov processes and which are widely used in several areas such as mathematical finance, probability, and physics. For more applications and a more complete account on the literature one can see the survey [25].

We denote by (Ω, X_t, P^x) the standard rotation invariant α -stable \mathbb{R}^N -valued Lévy process (homogeneous with independent increment and right continuous sample paths). As usual, E^x is the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^N$. For the sake of brevity, we will refer to this process by "symmetric α -stable process". The limiting classical case $\alpha = 2$ corresponds to the Brownian motion with Laplacian $\Delta = \sum_{i=1}^N \partial_i^2$ as generator.

Let D be a bounded open set and let X^D be the killed process defined by $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and set $X_t^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$ where ∂ is a coffin state added to \mathbb{R}^N .

Unlike the Brownian motion case, there are two different kinds of harmonicity with respect to symmetric α -stable process: function which are harmonic in D with respect to the process X and functions which are harmonic in D with respect to the killed process X^D . The precise definitions are as follows.

Definition 2.1. *Let $u \in \mathcal{L}_\alpha$ be a Borel measurable locally integrable function on \mathbb{R}^N .*

1. *We say that u is α -harmonic in D if $u(x) = E^x[u(X_{\tau_U})]$, $x \in U$, for every bounded open set U with closure \bar{U} contained in D . If, in addition, $u(x) = 0$, for every $x \in D^c$, we say that u is singular α -harmonic.*
2. *The function u is said to be α -superharmonic in D if f is lower semi continuous and $u(x) \geq E^x[u(X_{\tau_U})]$, $x \in U$, for every bounded open set U with closure \bar{U} contained in D .*
3. *The function u is called α -harmonic with respect to X^D if $u(x) = E^x[u(X_{\tau_U}^D)]$, $x \in U$, for every bounded open set U with closure \bar{U} contained in D .*

- Remark 2.2.** 1. Note that a singular α -harmonic function in D is α -harmonic with respect to X^D . Conversely, for a α -harmonic function with respect to X^D , if we extend it to be zero off the domain D then the resulting function is singular α -harmonic function in D .
2. As in the classical case ($\alpha = 2$) the α -harmonicity can be formulated in terms of the fractional power of the Laplacian. Indeed, a function $u \in \mathcal{L}_\alpha$ is α -harmonic in D if and only if it is continuous in D and $\Delta^{\frac{\alpha}{2}}u = 0$ in D in the distributional sense (see e.g., [9, Theorem 3.9]).
3. The following minimum principle for α -superharmonic functions holds [35]. If u is lower semi continuous on \overline{D} , α -superharmonic in D and $u \geq 0$ on D^c then $u \geq 0$ in D .

A point $x \in \partial D$ is called regular for the set D if $P^x(\tau_D = 0) = 1$. The (open) bounded domain D is called regular if all $x \in \partial D$ are regular for D (for instance, $C^{1,1}$ -domains and domains satisfying the exterior cone condition are regular). In this case each function f , say in $\mathcal{C}_b(D^c)$, admits an extension $H_D^\alpha f$ on \mathbb{R}^N such that $H_D^\alpha f$ is α -harmonic in D [28]. In other words, the function $h = H_D^\alpha f$ is the unique solution to the fractional Dirichlet problem

$$\begin{cases} \Delta^{\frac{\alpha}{2}}h = 0 & \text{in } D, \\ h = f & \text{in } D^c. \end{cases}$$

Note that f must be defined on all D^c because the data have to take into account the nonlocal character of the operator. For every $x \in D$, the α -harmonic measure relative to x and D , which will be denoted by $H_D^\alpha(x, \cdot)$, is defined to be the positive Radon measure on D^c given by the mapping $f \mapsto H_D^\alpha f(x)$.

It is worthwhile to remark that the α -harmonic measure for a regular domain D is typically defined (as a function) in the whole \mathbb{R}^N and is supported (as a measure) in the complementary of D contrary to the classical harmonic measure which is defined in \overline{D} and supported on the boundary of D . This can be better interpreted from a probabilistic point of view. It exhibits the fact that the sample paths of the α -stable process fails to be continuous and hence at the exit time they may hit D^c at points of $\overline{D^c}$ and not necessarily at points of ∂D as the case of the Brownian motion.

It is proved in [7] that for D say Lipschitz, $H_D^\alpha(x, \cdot)$ is concentrated on $\overline{D^c}$ and is absolutely continuous with respect to the Lebesgue measure on D^c . Furthermore, the corresponding density function $K_D(x, y)$, $x \in D$, $y \in D^c$, is continuous in $(x, y) \in D \times \overline{D^c}$. In this situation, the solution of the Dirichlet problem can be expressed in term of the Poisson kernel K_D as follows [15]

$$H_D^\alpha f(x) = E^x [f(X_{\tau_D})] = \int_{D^c} K_D(x, y) f(y) dy \quad ; \quad x \in D.$$

The density $K_{B_r}^\alpha$ of the α -harmonic measure for a ball B_r of center 0 and radius r is given by an explicit formula

$$K_{B_r}(x, y) = c_\alpha^N \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^{\frac{\alpha}{2}} \frac{1}{|x - y|^N}, \quad \text{for } |x| < r \text{ and } |y| > r, \quad (6)$$

where $c_\alpha^N := \pi^{-1-N/2}\Gamma(N/2)\sin(\pi\alpha/2)$. The potential operator G_D^α for the process (X_t^D) is defined for every Borel measurable function f for which the following identity exists, by

$$G_D^\alpha f(x) = E^x \left[\int_0^\infty f(X_t^D) dt \right] = E^x \left[\int_0^{\tau_D} f(X_t) dt \right].$$

If $G_D^\alpha f \not\equiv \infty$ then $G_D^\alpha f$ is α -superharmonic. Also for every $f \in \mathcal{B}_b(D)$, $G_D^\alpha f$ is a bounded continuous function on D satisfying $\lim_{x \rightarrow z} G_D^\alpha f(x) = 0$ for every $z \in \partial D$ if we suppose further that D is regular. The proof of these elementary properties follows the line of the corresponding one for the Laplacian performed in [4] or [18]. On the other hand, for every $f \in \mathcal{B}(\mathbb{R}^N)$ such that $G_D^\alpha |f|(x) < \infty$ for some $x \in \mathbb{R}^N$, we have

$$\Delta^{\frac{\alpha}{2}} G_D^\alpha f = -f \quad (7)$$

in the distributional sense ([10, Lemma 5.3]). The kernel $G_D^\alpha(\cdot, \cdot)$ of the operator G_D^α is called the Green function of D and is defined by

$$G_D^\alpha f(x) = \int_D G_D^\alpha(x, y) f(y) dy \quad ; \quad x \in D.$$

We shall record some well known facts of the Green function. $G_D^\alpha(x, y)$ is symmetric in x and y and $G_D^\alpha(\cdot, \cdot)$ is positive and continuous in the extended sense as a mapping from $D \times D$ into $[0, \infty]$. Also $G_D^\alpha(x, y) = 0$ if x or y belongs to D^c . The Green function of the whole space \mathbb{R}^N , which is also called Riesz kernel, is given explicitly by

$$G_{\mathbb{R}^N}^\alpha(x, y) = \frac{C_{N, -\alpha}}{|x - y|^{N-\alpha}}.$$

Also, the explicit formula for the Green function of a ball B_r of center 0 and radius r is well known:

$$G_{B_r}^\alpha(x, y) = \frac{\kappa_{N, \alpha}}{|x - y|^{N-\alpha}} \int_0^{\frac{(r^2 - |x|^2)(r^2 - |y|^2)}{|x - y|^2}} \frac{s^{\frac{\alpha}{2}-1}}{(1+s)^{\frac{N}{2}}} ds \quad ; \quad x, y \in B_r, \quad (8)$$

where $\kappa_{N, \alpha} = \Gamma(N/2) / (2^\alpha \pi^{N/2} [\Gamma(\alpha/2)]^2)$, and the following scaling property holds

$$G_{B_r}^\alpha(x, y) = r^{\alpha-N} G_B^\alpha\left(\frac{x}{r}, \frac{y}{r}\right) \quad ; \quad x, y \in B_r. \quad (9)$$

Let D be a bounded $C^{1,1}$ domain in \mathbb{R}^N . We denote by $\delta(x) := \inf_{z \in \partial D} |x - z|$ the Euclidean distance from $x \in D$ to the boundary of D . For convenience, we shall set $\delta(x) = 0$ for $x \in D^c$. The following sharp estimates on the Green function are established in [17].

$$G_D^\alpha(x, y) \leq c \min \left\{ \frac{1}{|x - y|^{N-\alpha}}, \frac{\delta(x)^{\frac{\alpha}{2}} \delta(y)^{\frac{\alpha}{2}}}{|x - y|^N} \right\} \quad (10)$$

$$G_D^\alpha(x, y) \leq c \frac{\delta(y)^{\frac{\alpha}{2}}}{|x - y|^{N-\frac{\alpha}{2}}} \quad (11)$$

$$G_D^\alpha(x, y) \geq c \frac{\delta(x)^{\frac{\alpha}{2}} \delta(y)^{\frac{\alpha}{2}}}{|x - y|^N} \quad \text{if } |x - y| > \max \left\{ \frac{\delta(x)}{2}, \frac{\delta(y)}{2} \right\}. \quad (12)$$

Here and in the sequel, the letter c with or without subscripts, signifies a positive constant which may change from one location to another (even in the same ligne) depending only on N and α but eventually on another variable which will be specified in the context.

3 Some Technical lemmas

In this section, we shall give two preparatory technical lemmas but each of them turns out to be useful in itself. The first one provides an approximation for the fractional Laplacian value of some important subfamily of functions. The second one gives some useful properties for the potential $G_B^\alpha(\delta^{-\lambda})$.

In the proof of the following lemma we will make use of the Gaussian hypergeometric function. We do not know one reference for all its basic properties but most of the material useful to our purposes can be found in [22, 24, 31].

Lemma 3.1. *Let $-\frac{\alpha}{2} < \beta < 1$ and consider the function u defined on \mathbb{R}^N by*

$$u(x) = \frac{1}{(1 - |x|^2)^\beta} \text{ if } |x| < 1 \quad \text{and} \quad u(x) = 0 \text{ if } |x| \geq 1.$$

Then

$$\Delta^{\frac{\alpha}{2}} u(x) \approx \frac{\alpha + 2\beta - 2}{(1 - |x|^2)^{\alpha+\beta}} \text{ in } B. \quad (13)$$

Proof. For every $0 < t < 1$, let B_t denotes the ball of \mathbb{R}^N of center 0 and radius t . For every $x \in B$, we denote $v(x) := \Delta^{\frac{\alpha}{2}} u(x)$. We first claim that, for every $0 < t < 1$ and every $x \in B_t$,

$$u(x) + G_{B_t}(v)(x) = H_{B_t} u(x),$$

and in particular,

$$u(0) + G_{B_t}(v)(0) = H_{B_t} u(0). \quad (14)$$

In fact, using the integral expression of $\Delta^{\frac{\alpha}{2}} u$, it is not hard to see that v is locally bounded on B . Thus, it follows from (7) that $\Delta^{\frac{\alpha}{2}} G_{B_t}^\alpha(v) = -v$ on B_t and $\lim_{x \rightarrow z} G_{B_t}^\alpha(v)(x) = 0$ for every $z \in \partial B_t$. This yields that the function h defined on \mathbb{R}^N by $h(x) = u(x) + G_{B_t}(v)(x)$ is α -harmonic in B_t and satisfies $h = u$ on B_t^c . Therefore, $h = H_{B_t} u$ on B_t as desired.

By (6) and using the spherical coordinates, for every $0 < t < 1$, we get

$$\begin{aligned} H_{B_t}^\alpha u(0) &= \frac{2 \sin(\frac{\alpha\pi}{2})}{\pi} \int_t^1 \frac{t^\alpha}{s(1-s^2)^\beta (s^2-t^2)^{\frac{\alpha}{2}}} ds \\ &= C(\alpha, \beta) t^\alpha (1-t^2)^{1-\beta-\frac{\alpha}{2}} F(1, 1-\beta, 2-\beta-\frac{\alpha}{2}, 1-t^2) \end{aligned} \quad (15)$$

$$= C(\alpha, \beta) (1-t^2)^{1-\beta-\frac{\alpha}{2}} F(1-\beta-\frac{\alpha}{2}, 1-\frac{\alpha}{2}, 2-\beta-\frac{\alpha}{2}, 1-t^2), \quad (16)$$

where $C(\alpha, \beta)$ is a positive constant. Equalities (15) and (16) are obtained, respectively, by formula (3.228) in [24] (or (33) page 250 in [22]) and by the Euler transformation of the Gaussian hypergeometric function $F(a, b, c, \cdot)$. On the other hand, it follows from

(8) that, for $|y| < 1$,

$$\begin{aligned}
G_B(0, y) &= \kappa_{N,\alpha} (1 - |y|^2)^{\frac{\alpha}{2}} |y|^{-N} \int_0^1 s^{\frac{\alpha}{2}-1} \left(1 + \left(\frac{1 - |y|^2}{|y|^2} \right) s \right)^{-\frac{N}{2}} ds \\
&= \frac{2\kappa_{N,\alpha}}{\alpha} |y|^{-N} (1 - |y|^2)^{\frac{\alpha}{2}} F\left(\frac{N}{2}, \frac{\alpha}{2}, 1 + \frac{\alpha}{2}, 1 - \frac{1}{|y|^2}\right) \\
&= \kappa_{N,\alpha} |y|^{-N} \int_{|y|^2}^1 (1 - s)^{\frac{\alpha}{2}-1} F\left(\frac{N}{2}, \frac{\alpha}{2}, \frac{\alpha}{2}, 1 - \frac{s}{|y|^2}\right) ds \\
&= \kappa_{N,\alpha} |y|^{-N} \int_{|y|^2}^1 (1 - s)^{\frac{\alpha}{2}-1} \left(\frac{s}{|y|^2} \right)^{-\frac{N}{2}} ds \\
&= 2\kappa_{N,\alpha} \int_{|y|}^1 s^{1-N} (1 - s^2)^{\frac{\alpha}{2}-1} ds,
\end{aligned} \tag{17}$$

where the equality (17) can be deduced from formula (3.7) in [31]. Then, using the scaling property (9) and the spherical coordinates, we obtain

$$\begin{aligned}
G_{B_t} v(0) &= t^{\alpha-N} \int_{B_t} G_B(0, \frac{y}{t}) v(y) dy \\
&= C(\alpha) t^{\alpha-N} \int_0^t \lambda^{N-1} v(\lambda) \int_{\frac{\lambda}{t}}^1 s^{1-N} (1 - s^2)^{\frac{\alpha}{2}-1} ds d\lambda \\
&= C(\alpha) \int_0^t \lambda^{N-1} v(\lambda) \int_{\lambda}^t s^{1-N} (t^2 - s^2)^{\frac{\alpha}{2}-1} ds d\lambda \\
&= C(\alpha) \int_0^t (t^2 - s^2)^{\frac{\alpha}{2}-1} s^{1-N} \int_0^s \lambda^{N-1} v(\lambda) d\lambda ds,
\end{aligned}$$

where $C(\alpha)$ is a positive constant. Here, we identify $v(|y|)$ with $v(y)$ since v is radially symmetric on B . So, the equation (14) can be rewritten in the following equivalent form

$$\int_0^t (t^2 - s^2)^{\frac{\alpha}{2}-1} \psi(s) ds = \varphi(t), \quad \forall 0 < t < 1, \tag{18}$$

where $\varphi(t) := H_{B_t} u(0) - u(0)$ and

$$\psi(s) := C(\alpha) s^{1-N} \int_0^s \lambda^{N-1} v(\lambda) d\lambda. \tag{19}$$

taking in account that $H_{B_t} u(0) \rightarrow u(0)$ as $t \rightarrow 0$, we can extend φ continuously by setting $\varphi(0) = 0$. So, by a standard fractional calculus, we can express ψ in terms of φ in (18) as follows

$$\begin{aligned}
\psi(t) &= 2 \frac{\sin(\alpha\pi/2)}{\pi} \left[\varphi(0) t^{1-\alpha} + t \int_0^t \frac{\varphi'(s)}{(t^2 - s^2)^{\frac{\alpha}{2}}} ds \right] \\
&= 2 \frac{\sin(\alpha\pi/2)}{\pi} t \int_0^t \frac{\varphi'(s)}{(t^2 - s^2)^{\frac{\alpha}{2}}} ds.
\end{aligned} \tag{20}$$

Next, using (16) and the fact that $(z\partial_z + c).F(a, b, c + 1, z) = cF(a, b, c, z)$, we easily obtain

$$\varphi'(s) = (\alpha + 2\beta - 2)C(\alpha, \beta) \frac{s^{\alpha-1}}{(1-s^2)^{\beta+\frac{\alpha}{2}}}. \quad (21)$$

Plugging (21) into (20), we get

$$\begin{aligned} \psi(t) &= (\alpha + 2\beta - 2)C(\alpha, \beta) \frac{\sin(\alpha\pi/2)}{\pi} t \int_0^1 \frac{s^{\frac{\alpha}{2}-1}}{(1-s)^{\frac{\alpha}{2}}(1-st^2)^{\beta+\frac{\alpha}{2}}} ds \\ &= (\alpha + 2\beta - 2)C(\alpha, \beta) t F\left(\beta + \frac{\alpha}{2}, \frac{\alpha}{2}, 1, t^2\right). \end{aligned} \quad (22)$$

Multiplying (22) by t^{N-1} , differentiating with respect to t , and combining this together with (19),

$$C(\alpha)v(t) = (\alpha+2\beta-2)C(\alpha, \beta) \left[N F\left(\beta + \frac{\alpha}{2}, \frac{\alpha}{2}, 1, t^2\right) + \frac{\alpha(\alpha+2\beta)}{2} t^2 F\left(1 + \beta + \frac{\alpha}{2}, 1 + \frac{\alpha}{2}, 2, t^2\right) \right].$$

By Euler transformation we get

$$v(t) = \frac{(\alpha + 2\beta - 2)C(\alpha, \beta)}{C(\alpha)} \frac{1}{(1-t^2)^{\alpha+\beta}} \phi(t),$$

with

$$\phi(t) = N(1-t^2)F\left(1 - \beta - \frac{\alpha}{2}, 1 - \frac{\alpha}{2}, 1, t^2\right) + \frac{\alpha(\alpha+2\beta)}{2} t^2 F\left(1 - \beta - \frac{\alpha}{2}, 1 - \frac{\alpha}{2}, 2, t^2\right).$$

Hence, the fact that the function ϕ is (strictly) positive and continuous on the closed interval $[0, 1]$ complete the proof. \square

Using the fact that $\min(s, t) \approx \frac{st}{s+t}$ for $s, t > 0$, it follows from (10) that

$$G_B^\alpha(x, y) \approx \frac{1}{|x-y|^{N-\alpha}} \frac{\delta(x)^{\frac{\alpha}{2}} \delta(y)^{\frac{\alpha}{2}}}{(|x-y|^2 + \delta(x)\delta(y))^{\frac{\alpha}{2}}}.$$

Hence, for every $x, y \in B$ such that $|x-y| \geq r$ we have

$$G_B^\alpha(x, y) \leq c \frac{\delta(x)^{\frac{\alpha}{2}} \delta(y)^{\frac{\alpha}{2}}}{r^N}. \quad (23)$$

Lemma 3.2. *Let $\lambda \in \mathbb{R}$ such that $\lambda < 1 + \frac{\alpha}{2}$ and consider the function h defined on B by*

$$h(x) = \int_B G_B^\alpha(x, y) \delta(y)^{-\lambda} dy.$$

Then h satisfies the following properties.

- (a) $h(x) < \infty$ for every $x \in B$.
- (b) h is continuous on B .
- (c) $\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} h(x) = 0$ for every $z \in \partial B$.

Proof. We prove all the assertions step by step.

- (a) Let $x \in B$ and let $r > 0$ be small enough so that $B(x, r) \subset B$. Using (11) and then splitting integral into two pieces we obtain

$$\begin{aligned} \int_B G_B^\alpha(x, y) \delta(y)^{-\lambda} dy &\leq c \int_B \frac{\delta(y)^{\frac{\alpha}{2}-\lambda}}{|x-y|^{N-\frac{\alpha}{2}}} dy \\ &= c \int_{B(x, r)} \frac{\delta(y)^{\frac{\alpha}{2}-\lambda}}{|x-y|^{N-\frac{\alpha}{2}}} dy + c \int_{B \setminus B(x, r)} \frac{\delta(y)^{\frac{\alpha}{2}-\lambda}}{|x-y|^{N-\frac{\alpha}{2}}} dy. \end{aligned}$$

For $y \in B(x, r)$ it is clear that $\delta(y) \geq \delta(x) - r$ and hence

$$\int_{B(x, r)} \frac{\delta(y)^{\frac{\alpha}{2}-\lambda}}{|x-y|^{N-\frac{\alpha}{2}}} dy \leq w(x) \int_{B(x, r)} \frac{dy}{|x-y|^{N-\frac{\alpha}{2}}} = c r^{\frac{\alpha}{2}} w(x) < \infty,$$

where $w(x) = (\delta(x) - r)^{\frac{\alpha}{2}-\lambda}$ if $\lambda > \frac{\alpha}{2}$, and $w(x) = 1$ otherwise. Now, for every $y \in B \setminus B(x, r)$, $|x-y| > r$ and then, noting that $\delta(y) = 1 - |y|$, we have

$$\begin{aligned} \int_{B \setminus B(x, r)} \frac{\delta(y)^{\frac{\alpha}{2}-\lambda}}{|x-y|^{N-\frac{\alpha}{2}}} dy &\leq \frac{1}{r^{N-\frac{\alpha}{2}}} \int_B \frac{1}{(1-|y|)^{\lambda-\frac{\alpha}{2}}} dy \\ &\leq \frac{c}{r^{N-\frac{\alpha}{2}}} \int_0^1 \frac{t^{N-1}}{(1-t)^{\lambda-\frac{\alpha}{2}}} dt, \end{aligned}$$

which is finite since $\lambda < 1 + \frac{\alpha}{2}$ and hence the first assertion holds.

- (b) Let $x_0 \in B$ and $r > 0$ so that $B(x_0, 2r) \subset B$. Let x such that $|x - x_0| < r$. We break up h into two pieces

$$h(x) = G_B^\alpha(1_{B(x_0, 2r)} \delta^{-\lambda})(x) + G_B^\alpha(1_{B \setminus B(x_0, 2r)} \delta^{-\lambda})(x) =: h_1(x) + h_2(x).$$

The approach is as follows. We check that $|h_1(x) - h_1(x_0)|$ can be made arbitrarily small with a suitably chosen $r > 0$ and that $|h_2(x) - h_2(x_0)| \rightarrow 0$ as $|x - x_0| \rightarrow 0$ for this choose of r . For every $y \in B(x_0, 2r)$, $\delta(y) \geq \delta(x_0) - 2r$. Thus, by (10), we have

$$G_B^\alpha(x, y) \delta^{-\lambda}(y) \leq c (\delta(x_0) - 2r)^{-\lambda} \frac{1}{|x-y|^{N-\alpha}}.$$

Then, recalling that $m(B(x_0, 2r)) = c(2r)^N$, we obtain

$$\begin{aligned} |h_1(x) - h_1(x_0)| &= \int_{B(x_0, 2r)} |G_B^\alpha(x, y) - G_B^\alpha(x_0, y)| \delta^{-\lambda}(y) dy \\ &\leq c (\delta(x_0) - 2r)^{-\lambda} \left(\int_{B(x_0, 2r)} \frac{dy}{|x-y|^{N-\alpha}} + \int_{B(x_0, 2r)} \frac{dy}{|x_0-y|^{N-\alpha}} \right) \\ &\leq c (\delta(x_0) - 2r)^{-\lambda} \left(\int_{B(x, r)} \frac{dy}{|x-y|^{N-\alpha}} + \int_{B(x_0, 2r) \setminus B(x, r)} \frac{dy}{|x-y|^{N-\alpha}} + (2r)^\alpha \right) \\ &\leq c (\delta(x_0) - 2r)^{-\lambda} \left(r^\alpha + \frac{r^N}{r^{N-\alpha}} + r^\alpha \right) \\ &= c (\delta(x_0) - 2r)^{-\lambda} r^\alpha. \end{aligned}$$

It is clear that the last term is arbitrarily small for a careful choose of $r > 0$.

Now, having chosen r , let us turn to the second addend h_2 . For every $y \in B \setminus B(x_0, 2r)$ and every $x \in B(x_0, r)$, we have $|x - y| > r$ and $|x_0 - y| > r$. Then it follows from (23) that

$$\begin{aligned} |G_B^\alpha(x, y) - G_B^\alpha(x_0, y)| \delta(y)^{-\lambda} &\leq c \frac{\delta(x)^{\frac{\alpha}{2}} + \delta(x_0)^{\frac{\alpha}{2}}}{r^N} \delta(y)^{-\lambda + \frac{\alpha}{2}} \\ &\leq \frac{c}{r^N} \delta(y)^{-\lambda + \frac{\alpha}{2}}. \end{aligned}$$

Noting that $\delta(y) = 1 - |y|$ and that $\lambda < 1 + \frac{\alpha}{2}$, we get

$$\int_{B \setminus B(x_0, 2r)} \delta(y)^{-\lambda + \frac{\alpha}{2}} dy \leq \int_B \delta(y)^{-\lambda + \frac{\alpha}{2}} dy = c \int_0^1 \frac{t^{N-1}}{(1-t)^{\lambda - \frac{\alpha}{2}}} dt < \infty.$$

Hence, by dominated convergence theorem,

$$\begin{aligned} \lim_{|x-x_0| \rightarrow 0} |h_2(x) - h_2(x_0)| \\ \leq \lim_{|x-x_0| \rightarrow 0} \int_{B \setminus B(x_0, 2r)} |G_B^\alpha(x, y) - G_B^\alpha(x_0, y)| \delta(y)^{-\lambda} dy = 0. \end{aligned}$$

We can now conclude that h is continuous at x_0 .

- (c) For every $-\frac{\alpha}{2} < \beta < 1 - \frac{\alpha}{2}$, applying G_B^α on both sides of (13) and noting that $\delta(x) \approx 1 - |x|^2$ we immediately obtain

$$G_B^\alpha(\delta^{-(\alpha+\beta)}) \approx \delta^{-\beta} \quad \text{in } B,$$

that is, for every $\frac{\alpha}{2} < \lambda' < 1 + \frac{\alpha}{2}$,

$$G_B^\alpha(\delta^{-\lambda'}) \approx \delta^{\alpha-\lambda'} \quad \text{in } B.$$

Now choose λ' such that $\max(\frac{\alpha}{2}, \lambda) < \lambda' < 1 + \frac{\alpha}{2}$. Then $h \leq G_B^\alpha(\delta^{-\lambda'})$ on B and consequently, for every $z \in \partial B$,

$$\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} h(x) \leq \lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} G_B^\alpha(\delta^{-\lambda'}) = \lim_{x \rightarrow z} \delta(x)^{1+\frac{\alpha}{2}-\lambda'} = 0.$$

□

4 Moderate blow up solutions

For $\gamma > 0$, we consider the following fractional equation

$$\Delta^{\frac{\alpha}{2}} u = u^\gamma. \quad (24)$$

Before we carry on, we make precise the notion of solution that we use in this paper. By a solution of the Eq. (24) in a open set $U \subset \mathbb{R}^N$, we shall mean every real-valued nonnegative function $u \in \mathcal{C}(U) \cap \mathcal{L}_\alpha$ such that

$$\int_{\mathbb{R}^N} u(x) \Delta^{\frac{\alpha}{2}} \varphi(x) dx = \int_{\mathbb{R}^N} u^\gamma(x) \varphi(x) dx$$

holds for every nonnegative function φ belonging to the space $\mathcal{C}_c^\infty(U)$. We first quote from [5] the following lemma which states a straightforward and useful fact.

Lemma 4.1. *Let U be an open set and let $u \in \mathcal{C}(U) \cap \mathcal{L}_\alpha$. The function u is a solution of $\Delta^{\frac{\alpha}{2}}u = u^\gamma$ in U if and only if $u + G_D^\alpha(u^\gamma) = H_D^\alpha u$ holds for every regular open set $D \subset \overline{D} \subset U$.*

Lemma 4.2. *Let D be an open regular bounded domain in \mathbb{R}^N . Let $u, v \in \mathcal{C}(\overline{D}) \cap \mathcal{L}_\alpha$ such that*

$$\Delta^{\frac{\alpha}{2}}u \leq u^\gamma \quad \text{and} \quad \Delta^{\frac{\alpha}{2}}v \geq v^\gamma \quad \text{in } D.$$

If $u \geq v$ on D^c , then $u \geq v$ in \mathbb{R}^N .

Proof. Define $w = u - v$. Arguing by contradiction, we assume that the open set

$$U = \{x \in D; w(x) < 0\}$$

is not empty. We get immediately that $\Delta^{\frac{\alpha}{2}}w \leq (u^\gamma - v^\gamma) \leq 0$ in U , this amounts to say that w is α -superharmonic in U . Furthermore, we have $w \geq 0$ on D^c . Therefore, $w \geq 0$ in U by the minimum principle for α -superharmonic functions (Remark 2.2 (c)) and this yields a contradiction. Consequently, $U = \emptyset$ and hence $u \geq v$ in D . \square

It should be noticed that the comparison principle as stated in the above lemma become more or less a classical result and it is widely used in the literature ([5, 14]...). Also, as we alluded to before, the minimum principle and comparison results require information on the solutions in the whole complement of the domain and not only at the boundary, consistently with the nonlocal character of the operator $\Delta^{\frac{\alpha}{2}}$.

Let D be an open bounded $C^{1,1}$ domain in \mathbb{R}^N . It was shown in [5] that, for every $f \in \mathcal{C}_b^+(D^c)$, there exists one and only one function $u \in \mathcal{C}^+(\overline{D})$ solution of the following semilinear Dirichlet problem

$$\begin{cases} \Delta^{\frac{\alpha}{2}}u = u^\gamma & \text{in } D. \\ u = f & \text{on } D^c \end{cases} \quad (25)$$

As a matter of fact, the continuity of f in the whole complement of D is not vital for the existence of a solution $u \in \mathcal{C}(\overline{D})$ but it guarantees rather the continuity of the solution in the whole space \mathbb{R}^N and this is not important for our purposes. In this context, we need only to assume that $f \in \mathcal{C}(\partial D) \cap \mathcal{B}_b^+(D^c)$ rather than $f \in \mathcal{C}_b^+(D^c)$. In the following proposition, we would like to prove the existence of a solution $u \in \mathcal{C}^+(\overline{D})$ to the same problem but dropping the boundedness of the boundary datum f , thus extending the result in [5].

Proposition 4.3. *For every nonnegative function $f \in \mathcal{C}(\partial D) \cap \mathcal{L}_\alpha$, there exists one and only one function $u \in \mathcal{C}^+(\overline{D})$ solution of the problem (25). Moreover, for every $x \in \mathbb{R}^N$,*

$$u(x) + G_D^\alpha(u^\gamma)(x) = H_D^\alpha f(x).$$

Proof. For every $n \geq 1$, put $f_n = \min(f, n)$. Obviously, $f_n \in \mathcal{C}(\partial D) \cap \mathcal{B}_b^+(D^c)$. Then, as mentioned above, there exists a nonnegative function $u_n \in \mathcal{C}^+(\overline{D})$ solution to the problem (25) with boundary datum f_n instead of f . Furthermore, by Lemma 4.1,

$$u_n(x) + G_D^\alpha(u_n^\gamma)(x) = H_D^\alpha f_n(x) \quad ; \quad x \in \mathbb{R}^N. \quad (26)$$

On account of the comparison principle, (u_n) is nondecreasing. If we put $u := \sup_n u_n$ then obviously, $u \geq 0$. On the other hand, letting n tend to ∞ in (26), we obtain

$$u(x) + G_D^\alpha(u^\gamma)(x) = H_D^\alpha f(x) \quad ; \quad x \in \mathbb{R}^N. \quad (27)$$

We readily observe that u is bounded on D by $\sup_{x \in D} H_D^\alpha f(x)$ and thereby u^γ is also bounded on D . This yields that $G_D^\alpha(u^\gamma) \in \mathcal{C}_0(D)$, the subclass of $\mathcal{C}(\overline{D})$ consisting of functions which vanish continuously on ∂D . Then, from (27), we immediately deduce that $u \in \mathcal{C}(\overline{D})$ and that $u = f$ on D^c . Using (7), we also deduce from (27) that $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ in D . Hence, u is a solution to the problem (25) as desired. The uniqueness of the solution follows immediately from Lemma 4.2. \square

Now we wish to spend a few words on the Martin kernel. Let D be an open bounded $C^{1,1}$ domain in \mathbb{R}^N and let $x_0 \in D$ be a reference point. The Martin kernel of the killed symmetric stable process is defined by

$$M_D^\alpha(x, z) = \lim_{y \rightarrow z} \frac{G_D^\alpha(x, y)}{G_D^\alpha(x_0, y)}, \quad \text{for } x \in D \text{ and } z \in \partial D,$$

and it has properties similar to those of the Martin kernel with respect to the killed Brownian motion [8]: the mapping $(x, z) \mapsto M_D^\alpha(x, z)$ is continuous on $D \times \partial D$. Moreover, for every $z \in \partial D$, $M_D^\alpha(\cdot, z)$ is a positive singular α -harmonic in D and for every $z, w \in \partial D$ such that $z \neq w$ we have $\lim_{x \rightarrow w} M_D^\alpha(x, z) = 0$. It is proved also that the formula

$$h(x) = \int_{\partial D} M_D^\alpha(x, z) d\nu(z)$$

realizes a one-to-one correspondence between nonnegative singular α -harmonic functions on D and positive Radon measures on ∂D . The Martin kernel of the unit ball B is given explicitly by

$$M_B^\alpha(x, y) = \frac{(1 - |x|^2)^{\alpha/2}}{|x - y|^N} \quad ; \quad x \in B, z \in \partial B.$$

For every nonnegative continuous function g on ∂B , we denote

$$M_B^\alpha g(x) := \begin{cases} \int_{\partial B} M_B^\alpha(x, y) g(y) d\sigma(y) & \text{if } x \in B \\ 0 & \text{if } x \notin B, \end{cases}$$

where σ is the surface area measure on ∂B . An elementary calculation leads to the following estimates.

$$M_B^\alpha 1(x) \approx \delta(x)^{\frac{\alpha}{2}-1} \quad (28)$$

Furthermore, it follows from [27, Theorem 3.18] that $M_B^\alpha g$ is the unique positive solution h of the following boundary value problem:

$$\begin{cases} \Delta^{\frac{\alpha}{2}} h = 0 & \text{in } B \\ h = 0 & \text{on } B^c \\ \lim_{x \rightarrow z \in \partial B} \delta(x)^{1-\frac{\alpha}{2}} h(x) = g(z). \end{cases} \quad (29)$$

For $g \in \mathcal{C}^+(\partial B)$, we shall investigate the existence and the uniqueness of a solution to the following problem:

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = u^\gamma \text{ in } B \\ u = 0 \text{ on } B^c \\ \lim_{x \rightarrow z \in \partial B} \delta(x)^{1-\frac{\alpha}{2}} u(x) = g(z). \end{cases} \quad (30)$$

Clearly, solutions of this problem explode at the boundary. The explosion is driven by the function g and the explosion rate has to be controlled by $\delta(x)^{1-\frac{\alpha}{2}}$.

To put on readiness, we shall give a useful characterization of solutions to problem (30) in terms of the Green operator and the Martin kernel. This characterization will be the key ingredient to prove the uniqueness as it is stated in the forthcoming main theorem.

Proposition 4.4. *Let $0 < \gamma < \frac{2+\alpha}{2-\alpha}$. A function $u \in \mathcal{B}^+(\mathbb{R}^N)$ is a solution to the problem (30) if and only if $u + G_B^\alpha u^\gamma = M_B^\alpha g$ on \mathbb{R}^N .*

Proof. Let $u \in \mathcal{B}^+(\mathbb{R}^N)$. Suppose first that u is a solution to the problem (30) and consider $w := u + G_B^\alpha(u^\gamma)$. Since $M_B^\alpha g$ is the unique solution of the problem (29), it will be sufficient to show that w is also a solution of this problem. Taking in account that, for every $z \in \partial B$, $\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} u(x) = g(z)$, we can extend $\delta(x)^{1-\frac{\alpha}{2}} u$ continuously on \overline{B} and consequently there exists some constant $C > 0$ such that $u \leq C \delta^{\frac{\alpha}{2}-1}$ on B . So, for every $x \in B$,

$$G_B^\alpha(u^\gamma)(x) \leq C^\gamma G_B^\alpha(\delta^{\gamma(\frac{\alpha}{2}-1)})(x).$$

Then, since $\gamma(1-\frac{\alpha}{2}) < 1+\frac{\alpha}{2}$, it follows from assertion (a) in Lemma 3.2 that $G_B^\alpha(u^\gamma)(x) < \infty$ which yields, using (7), that $\Delta^{\frac{\alpha}{2}} G_B^\alpha(u^\gamma) = -u^\gamma$ in B . Thus $\Delta^{\frac{\alpha}{2}} w = 0$ in B . Also, we deduce using assertion (c) in Lemma 3.2 that $\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} G_B^\alpha(u^\gamma)(x) = 0$ and hence

$$\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} w(x) = \lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} u(x) = g(z).$$

Conversely, suppose that u satisfies $u + G_B^\alpha(u^\gamma) = M_B^\alpha g$ on \mathbb{R}^N . Obviously, $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ in B , $u = 0$ on B^c and $u \leq \|g\| M_B^\alpha 1 \leq c \|g\| \delta^{\frac{\alpha}{2}-1}$. Then, in view of the assertion (c) in Lemma 3.2, we obtain

$$\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} G_B^\alpha(u^\gamma)(x) \leq (c\|g\|)^\gamma \lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} G_B^\alpha(\delta^{\gamma(\frac{\alpha}{2}-1)})(x) = 0.$$

This entails that

$$\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} u(x) = \lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} M_B^\alpha g(x) = g(z).$$

Now it remains to prove that u is continuous in B . Define $v(x) = \|g\| (\delta(x))^{\gamma(\frac{\alpha}{2}-1)} - u^\gamma(x)$ for $x \in B$. Obviously, v is nonnegative and

$$G_B^\alpha(u^\gamma) + G_B^\alpha v = \|g\| G_B^\alpha(\delta^{\gamma(\frac{\alpha}{2}-1)}) \quad \text{on } B.$$

By the assertion (b) in Lemma 3.2, $G_B^\alpha(\delta^{\gamma(\frac{\alpha}{2}-1)}) \in \mathcal{C}(B)$. Then, seeing that $G_B^\alpha(u^\gamma)$ and $G_B^\alpha v$ are lower semi-continuous on B , we immediately conclude that $G_B^\alpha(u^\gamma) \in \mathcal{C}(B)$ and hence the continuity of u on B holds since $u + G_B^\alpha(u^\gamma) = M_B^\alpha g$ and $M_B^\alpha g \in \mathcal{C}(B)$. Consequently, u is a solution to the problem (30) and the proof is finished. \square

Theorem 4.5. *Let $\gamma > 0$ and let $g \in \mathcal{C}^+(\partial D)$ not identically vanishing.*

1. *If $0 < \gamma < \frac{2+\alpha}{2-\alpha}$, then problem (30) admits one and only one nonnegative solution.*
2. *If $\gamma \geq \frac{2+\alpha}{2-\alpha}$, then problem (30) has no nonnegative solution.*

Proof. 1. Let $0 < \gamma < \frac{2+\alpha}{2-\alpha}$. For every $n \geq 2$, we denote by B_n the ball of \mathbb{R}^N of center 0 and radius $1 - \frac{1}{n}$. In virtue of proposition 4.3, invoking that $M_B^\alpha g \in \mathcal{C}(\partial B_n) \cap \mathcal{L}_\alpha$, for every $n \geq 1$, there exists a unique $u_n \in \mathcal{C}^+(\overline{B}_n)$ solution of the problem

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u_n = u_n^\gamma & \text{in } B_n \\ u_n = M_{B_n}^\alpha g & \text{on } B_n^c. \end{cases}$$

Moreover,

$$u_n(x) + G_{B_n}^\alpha(u_n^\gamma)(x) = M_{B_n}^\alpha g(x) \quad ; \quad x \in \mathbb{R}^N. \quad (31)$$

For every $n \geq 2$, we readily observe that $u_n \leq M_B^\alpha g$ which yields using Lemma 4.2 that $u_{n+1} \leq u_n$. Denote $u = \inf_n u_n$. On account of (28), for every $x, y \in \mathbb{R}^N$ we have

$$G_{B_n}^\alpha(x, y) u_n^\gamma(y) \leq G_B^\alpha(x, y) (M_B^\alpha g)^\gamma(y) \leq c \|g\|^\gamma G_B^\alpha(x, y) \delta(y)^{\gamma(\frac{\alpha}{2}-1)}.$$

Bearing in mind the hypothesis $0 < \gamma < \frac{2+\alpha}{2-\alpha}$, we deduce from assertion (a) in Lemma 3.2 that

$$\int_B G_B^\alpha(x, y) \delta(y)^{\gamma(\frac{\alpha}{2}-1)} dy < \infty.$$

Thus, by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} G_{B_n}^\alpha(u_n^\gamma)(x) = G_B^\alpha(u^\gamma)(x).$$

Consequently, by letting n tend to ∞ in (31), we obtain

$$u(x) + G_B^\alpha(u^\gamma)(x) = M_B^\alpha g(x) \quad x \in \mathbb{R}^N,$$

and hence u is a nonnegative solution to problem (30) in virtue of Proposition 4.4.

Let us now prove the uniqueness. Let u and v be two solutions of the problem (30). We denote $w := u - v$ and h the nonnegative function defined on B by

$$h(x) := \begin{cases} \frac{u^\gamma(x) - v^\gamma(x)}{u(x) - v(x)} & \text{if } u(x) \neq v(x) \\ 0 & \text{if } u(x) = v(x) \end{cases}$$

Since, by proposition 4.4, $u + G_B^\alpha(u^\gamma) = M_B^\alpha g = v + G_B^\alpha(v^\gamma)$, we immediately deduce that $w + G_B^\alpha(hw) = 0$ which is equivalent to $w + G_B^\alpha(hw^+) = G_B^\alpha(hw^-)$ since $G_B^\alpha(h|w|) < \infty$. Here and according to the notational custom, $w^+ = \max(w, 0)$ and $w^- = \max(-w, 0)$. In particular, $G_B^\alpha(hw^+) \leq G_B^\alpha(hw^-)$ on $\{w > 0\} = \{w^+ > 0\}$. Then, by the dominated principle [6], we must have $G_B^\alpha(hw^+) \leq G_B^\alpha(hw^-)$ on B and hence $G_B^\alpha(hw) \leq 0 = w + G_B^\alpha(hw)$, which yields that $w \geq 0$. Consequently, $0 \leq w \leq w + G_B^\alpha(hw) = 0$ and so $w = 0$ as desired.

2. Let $\gamma \geq \frac{2+\alpha}{2-\alpha}$. Arguing by contradiction, we suppose that the problem (30) admits a nonnegative solution u . Seeing that the function $\delta^{1-\frac{\alpha}{2}}u$ is a continuous extension of g on \overline{B} , we can find $c > 0$ so that

$$\delta(x)^{1-\frac{\alpha}{2}}u(x) \leq c \quad \text{for every } x \in B.$$

Thus $H_{B_t}^\alpha u(0) \leq cH_{B_t}^\alpha(\delta^{\frac{\alpha}{2}-1})(0)$ for every $0 < t < 1$ where B_t , as before, is the ball of center 0 and radius t . Since $\delta(y) \approx 1 - |y|^2$, it follows from (16) that the function $t \rightarrow H_{B_t}^\alpha(\delta^{\frac{\alpha}{2}-1})(0)$ is bounded on $[0, 1]$. But, in virtue of Lemma 4.1, for every $0 < t < 1$ we have

$$u(0) + G_{B_t}^\alpha(u^\gamma)(0) = H_{B_t}^\alpha u(0).$$

Thus, letting t tend to 1, we obtain

$$u(0) + G_B^\alpha(u^\gamma)(0) = u(0) + \lim_{t \uparrow 1} G_{B_t}^\alpha(u^\gamma)(0) = \lim_{t \uparrow 1} H_{B_t}^\alpha u(0) < \infty. \quad (32)$$

On the other hand, let $z \in \partial B$ such that $g(z) > 0$. Since, by hypothesis, $\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}}u(x) = g(z)$, there exist $r, \eta > 0$ small enough such that

$$\delta(x)^{1-\frac{\alpha}{2}}u(x) \geq \eta \quad \text{for every } x \in B \cap B(z, r).$$

Then, using (12), we get

$$\begin{aligned} G_B^\alpha(u^\gamma)(0) &= \int_B G_B^\alpha(0, y) u^\gamma(y) dy \\ &\geq \int_{\frac{1}{2} < |y| < 1} G_B^\alpha(0, y) u^\gamma(y) dy \\ &\geq c \eta^\gamma \int_{B \cap B(z, r)} \frac{\delta(y)^{\gamma(\frac{\alpha}{2}-1) + \frac{\alpha}{2}}}{|y|^N} dy \\ &= \infty \end{aligned}$$

contradicting (32). Hence, problem (30) has no solution. □

5 Blow up solutions

For $\gamma > 0$, we shall investigate the existence of a nonnegative solution of the following problem:

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = u^\gamma \text{ in } B \\ u = 0 \text{ in } B^c \\ \lim_{x \rightarrow z \in \partial B} \delta(x)^{1-\frac{\alpha}{2}} u(x) = \infty. \end{cases} \quad (33)$$

More precisely, we divide the positive real half line into different intervals, where (33) admits no solutions and at least one solution. In the later case, we give a simple equivalent to the solution. Our proof does not require heavy computations. We would point out that solutions to problem (33) are relatively with coarse singularity at the boundary and hence, they are, in some respects, the fractional counterpart of large solutions in the classical case $\alpha = 2$.

Theorem 5.1. 1. If $1 + \alpha < \gamma < \frac{2+\alpha}{2-\alpha}$, then the problem (33) admits a nonnegative solution u . Furthermore,

$$u \approx \delta^{1-\gamma} \quad \text{in } B. \quad (34)$$

2. If $0 < \gamma < 1 + \frac{\alpha}{2}$, then problem (33) has no nonnegative solution.

Proof. 1. Let $1 + \alpha < \gamma < \frac{2+\alpha}{2-\alpha}$ and put $\beta := \frac{\alpha}{\gamma-1}$. It is obvious that β satisfies $1 - \frac{\alpha}{2} < \beta < 1$. Let ϑ be the function defined on \mathbb{R}^N as follows.

$$\vartheta(x) = \frac{1}{(1 - |x|^2)^\beta} \quad \text{if } |x| < 1 \quad \text{and} \quad \vartheta(x) = 0 \quad \text{if } |x| \geq 1.$$

Then, by Lemma 3.1, there exists some constant $C > 1$ such that, for every $x \in B$,

$$\frac{1}{C} \frac{1}{(1 - |x|^2)^{\alpha+\beta}} \leq \Delta^{\frac{\alpha}{2}} \vartheta(x) \leq C \frac{1}{(1 - |x|^2)^{\alpha+\beta}}. \quad (35)$$

Let $K > 1$ be so that $C = K^{\gamma-1}$ and consider the functions $v := K \vartheta$ and $w := \frac{1}{K} \vartheta$. Clearly, $w \leq v$ and

$$\Delta^{\frac{\alpha}{2}} v \leq v^\gamma \quad \text{and} \quad \Delta^{\frac{\alpha}{2}} w \geq w^\gamma \quad \text{in } B.$$

For every $n \geq 2$, we denote by B_n the ball of \mathbb{R}^N of center 0 and radius $1 - \frac{1}{n}$. Since $v \in \mathcal{C}(\partial B_n) \cap \mathcal{L}_\alpha$, by proposition 4.3 there exists a unique $u_n \in \mathcal{C}^+(\overline{B}_n)$ solution of the problem

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u_n = u_n^\gamma & \text{in } B_n \\ u_n = v & \text{on } B_n^c. \end{cases}$$

Moreover, using Lemma 4.2, it is easy to see that

$$w \leq u_{n+1} \leq u_n \leq v.$$

Put $u := \inf_n u_n$, then we immediately deduce from these inequalities that $u \in \mathcal{L}_\alpha$, $u = 0$ on B^c , u satisfies (34) and that $\lim_{x \rightarrow z \in \partial B} \delta(x)^{1-\frac{\alpha}{2}} u(x) = \infty$ for every $z \in \partial B$. So, it remains to prove that $u \in C(B)$ and $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ in B . Let D be a regular open set such that $\overline{D} \subset B$ and let n_0 be the smallest integer $n \geq 2$ such that $\overline{D} \subset B_n$. Then for every $n \geq n_0$ and every $x \in \mathbb{R}^N$,

$$u_n(x) + G_D^\alpha(u_n^\gamma)(x) = H_D^\alpha u_n(x) \quad (36)$$

since $\Delta^{\frac{\alpha}{2}} u_n = u_n^\gamma$ in D and $u_n \in \mathcal{C}(D) \cap \mathcal{L}_\alpha$. Whence, by letting n tend to ∞ in (36), we obtain

$$u(x) + G_D^\alpha(u^\gamma)(x) = H_D^\alpha u(x)$$

which entails that $u \in \mathcal{C}(D)$ because $H_D^\alpha u$ and $G_D^\alpha(u^\gamma)$ are in $\mathcal{C}(D)$. Since $D \subset B$ is arbitrary, we conclude that $u \in \mathcal{C}(B)$ and, by Lemma 4.1, that $\Delta^{\frac{\alpha}{2}} u = u^\gamma$ in B . This ends up the first part of the proof.

2. Let $0 < \gamma < 1 + \frac{\alpha}{2}$ and assume that the problem (33) admits a nonnegative solution u . We first claim that

$$G_B^\alpha(u^\gamma)(x) = \infty \text{ for every } x \in B. \quad (37)$$

In fact, assume that $G_B^\alpha(u^\gamma)(x_0) < \infty$ for some $x_0 \in B$ and consider the function h defined on \mathbb{R}^N by

$$h(x) := u(x) + G_B^\alpha(u^\gamma)(x) \text{ if } x \in B \text{ and } h(x) = 0 \text{ if } x \notin B.$$

Obviously, h is singular α -harmonic in B since, by (7), $\Delta^{\frac{\alpha}{2}} G_B^\alpha(u^\gamma) = -u^\gamma$ in B . Then, it follows from [27] that $\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} h(x) < \infty$ for some $z \in \partial B$. But this leads to a contradiction because $\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} h(x) \geq \lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} u(x)$ and $\lim_{x \rightarrow z} \delta(x)^{1-\frac{\alpha}{2}} u(x) = \infty$ by hypothesis. So, the claim is checked. On the other hand, u is Lebesgue integrable since it is in \mathcal{L}_α and then the function δu must be bounded on \overline{B} which yields the existence of some constant $c > 0$ such that $u \leq c \delta^{-1}$ on B and hence

$$G_B^\alpha(u^\gamma) \leq c^\gamma G_B^\alpha(\delta^{-\gamma}) \text{ on } B.$$

Now, minding the hypothesis $0 < \gamma < 1 + \frac{\alpha}{2}$, we deduce from assertion (a) in Lemma 3.2 that $G_B^\alpha(\delta^{-\gamma})(x) < \infty$ for every $x \in B$ and consequently $G_B^\alpha(u^\gamma)(x) < \infty$ contradicting the claim (37). Hence, under the condition $0 < \gamma < 1 + \frac{\alpha}{2}$, the problem (33) has no nonnegative solution. \square

The restriction " $\gamma > \alpha + 1$ " seems to be technical but nearly optimal for the existence of solutions to problem (33). We remember that this exponent " $\alpha + 1$ " appeared in [13] too. On the other hand, taking $\alpha = 2$, in virtue of Theorem 5.1, there exists no solution to problem (33) for $0 < \gamma < 2$ and this is not consistent with the classical Keller-Osserman condition. In fact, it is well known that, in the old Laplacian case and taking in account the Keller-Osserman condition, (large) solutions exist for every $\gamma > 1$ and hence for $1 < \gamma < 2$. Put differently, the classical Keller-Osserman condition, in the fractional setting, constitute no longer the appropriate range where large solutions exists or do not exist as in the Laplacian framework.

References

- [1] N. Abatangelo. Large s -harmonic functions and boundary blow-up solutions for the fractional laplacian. *arXiv preprint arXiv:1310.3193*, 2013.
- [2] L. Abdelouhab, J. L. Bona, M. Felland, and J-C. Saut. Nonlocal models for nonlinear, dispersive waves. *Physica D: Nonlinear Phenomena*, 40(3):360–392, 1989.
- [3] S. Axler, P. Bourdon, and R. Wade. *Harmonic function theory*, volume 137. Springer Science & Business Media, 2001.
- [4] R. F. Bass. *Probabilistic techniques in analysis*. Probability and its Applications (New York). Springer-Verlag, New York, 1995.

- [5] M. Ben Chrouda and M. Ben Fredj. Nonnegative entire bounded solutions to some semilinear equations involving the fractional laplacian. *arXiv preprint arXiv:1506.08294*, 2015.
- [6] J. Bliedtner and W. Hansen. *Potential theory*. Universitext. Springer-Verlag, Berlin, 1986. An analytic and probabilistic approach to balayage.
- [7] K. Bogdan. The boundary Harnack principle for the fractional Laplacian. *Studia Math.*, 123(1):43–80, 1997.
- [8] K. Bogdan. Representation of α -harmonic functions in Lipschitz domains. *Hiroshima Math. J.*, 29(2):227–243, 1999.
- [9] K. Bogdan and T. Byczkowski. Potential theory for the α -stable Schrödinger operator on bounded Lipschitz domains. *Studia Math.*, 133(1):53–92, 1999.
- [10] K. Bogdan and T. Byczkowski. On the Schrödinger operator based on the fractional Laplacian. *Bull. Polish Acad. Sci. Math.*, 49(3):291–301, 2001.
- [11] L. Caffarelli. Non-local diffusions, drifts and games. In *Nonlinear Partial Differential Equations*, pages 37–52. Springer, 2012.
- [12] R. Chemmam, H. Mâagli, and S. Masmoudi. On a new kato class and positive solutions of dirichlet problems for the fractional laplacian in bounded domains. *Nonlinear Analysis: Theory, Methods & Applications*, 74(5):1555–1576, 2011.
- [13] H. Chen, P. Felmer, and A. Quaas. Large solutions to elliptic equations involving fractional laplacian. *RN*, 1:2, 2013.
- [14] H. Chen, P. Felmer, and A. Quaas. Large solutions to elliptic equations involving fractional laplacian. *RN*, 1:2, 2013.
- [15] Z-Q. Chen and R. Song. Estimates on Green functions and Poisson kernels for symmetric stable processes. *Math. Ann.*, 312(3):465–501, 1998.
- [16] Z. Q. Chen and R. Song. Martin boundary and integral representation for harmonic functions of symmetric stable processes. *journal of functional analysis*, 159(1):267–294, 1998.
- [17] Z-Q. Chen and R. Song. General gauge and conditional gauge theorems. *Ann. Probab.*, 30(3):1313–1339, 2002.
- [18] K. L. Chung and Z. X. Zhao. *From Brownian motion to Schrödinger’s equation*, volume 312 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1995.
- [19] J. L. Doob. *Classical potential theory and its probabilistic counterpart*, volume 262 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1984.

- [20] E. B Dynkin. A probabilistic approach to one class of nonlinear differential equations. *Probability theory and related fields*, 89(1):89–115, 1991.
- [21] A. Elgart and B. Schlein. Mean field dynamics of boson stars. *arXiv preprint math-ph/0504051*, 2005.
- [22] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. Tables of integral transforms. *Part, on Notes Left by Harry Bateman and Compiled by the Staff of the Bateman Manuscript Project*, 2, 1954.
- [23] P. Felmer and A. Quaas. Boundary blow up solutions for fractional elliptic equations. *Asymptotic Analysis*, 78(3):123–144, 2012.
- [24] I. S. Gradshteyn and I. M. Ryzhik. Table of integrals, series, and products. *Seventh edition*, Academic Press, 2007.
- [25] A. Janicki and A. Weron. *Simulation and chaotic behavior of alpha-stable stochastic processes*, volume 178. CRC Press, 1993.
- [26] J. B. Keller. On solutions of $\Delta u = f(u)$. *Communications on Pure and Applied Mathematics*, 10(4):503–510, 1957.
- [27] P. Kim. Relative fatou’s theorem for $(-\Delta)^{\frac{\alpha}{2}}$ -harmonic functions in bounded κ -fat open sets. *Journal of Functional Analysis*, 234(1):70–105, 2006.
- [28] N. S. Landkof. *Foundations of modern potential theory*. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.
- [29] J. Le Gall. Probabilistic approach to a class of semilinear partial differential equations. *Contemp. Math.*, 446:255, 2007.
- [30] E. H. Lieb and H-T. Yau. The chandrasekhar theory of stellar collapse as the limit of quantum mechanics. *Communications in Mathematical Physics*, 112(1):147–174, 1987.
- [31] E. R. Love. Some integral equations involving hypergeometric functions. *Proceedings of the Edinburgh Mathematical Society (Series 2)*, 15(03):169–198, 1967.
- [32] A. J. Majda, D. W. McLaughlin, and E. G. Tabak. A one-dimensional model for dispersive wave turbulence. *Journal of Nonlinear Science*, 7(1):9–44, 1997.
- [33] R. Osserman. On the inequality $\Delta u \geq f(u)$. *Pacific Journal of Mathematics*, 7(4):1641–1647, 1957.
- [34] R. Servadei and E. Valdinoci. Weak and viscosity solutions of the fractional laplace equation. In *Publicacions matemàtiques*, volume 58, pages 0133–154, 2014.
- [35] L. Silvestre. Regularity of the obstacle problem for a fractional power of the laplace operator. *Communications on pure and applied mathematics*, 60(1):67–112, 2007.

- [36] L. Vlahos, H. Isliker, Y. Kominis, and K. Hizanidis. Normal and anomalous diffusion: A tutorial. *arXiv preprint arXiv:0805.0419*, 2008.
- [37] H. Weitzner and G-M. Zaslavsky. Some applications of fractional equations. *Communications in nonlinear science and numerical simulation*, 8(3):273–281, 2003.
- [38] J-M. G. Wu. Comparisons of kernel functions boundary harnack principle and relative fatou theorem on lipschitz domains. In *Annales de l'institut Fourier*, volume 28, pages 147–167, 1978.