Explicit estimates for solutions to higher order elliptic PDEs via Morse index

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Abstract

In this paper, we establish L^∞ and L^p estimates for solutions of some polyharmonic elliptic equations via the Morse index. As far as we know, it seems to be the first time that such explicit estimates are obtained for polyharmonic problems.

Keywords: polyharmonic equation, Morse index, elliptic estimates.

1. Introduction

Consider the following polyharmonic equations (P_k) $: (-\Delta)^k u = f(x,u)$ in Ω with the Dirichlet boundary conditions

$$u = \frac{\partial u}{\partial \nu} = \ldots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0 \quad \text{on} \;\; \partial\Omega; \tag{1.1}$$

or the Navier boundary conditions

$$u = \Delta u = \ldots = \Delta^{k-1} u = 0 \quad \text{on} \;\; \partial\Omega. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^N$ $(N > 2k)$ is a bounded domain with smooth boundary and f is a $C^1(\Omega \times \mathbb{R})$ function that we will specify later. Define

$$\Lambda_u(\phi) := \int_\Omega |D^k\phi|^2 - f'(x,u)\phi^2 \quad \text{for} \;\; \phi \in \Sigma_k \tag{1.3}$$

where

$$D^k = \begin{cases} \nabla \Delta^{\frac{k-1}{2}} & \text{for } k \text{ odd;} \\ \Delta^{\frac{k}{2}} & \text{for } k \text{ even} \end{cases}$$

and

$$\Sigma_k := \begin{cases} H_0^k(\Omega) & \text{if we work with (1.1);} \\ \left\{\phi \in H^k(\Omega), \phi = \Delta\phi = ... = \Delta^{[\frac{k-1}{2}]}\phi = 0 \text{ on } \;\partial\Omega\right\} & \text{if we work with (1.2).} \end{cases}$$

The Morse index of a classical solution u of (P_k), denoted by $i(u)$ is defined as the maximal dimension of all subspaces of Σ_k such that $\Lambda_u(\phi) < 0$ in $\Sigma \setminus \{0\}$. We say that u is stable if its Morse index is equal to zero. Our aim here is to get some explicit estimates of u using its Morse index $i(u)$.

We begin by presenting some assumptions on the nonlinearity f:

(H_1) (superlinearity) There exists $\mu > 0$ such that

$$f'(x,s)s^2 \geq (1+\mu)f(x,s)s > 0, \quad \text{for } |s| > s_0, \;\; x \in \Omega.$$

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(H_2) (subcritical growth) There exists $0 < \theta < 1$ such that

$$\frac{2N}{N-2k}F(x,s) \geq (1+\theta)f(x,s)s, \quad \text{for } |s| > s_0, \ \ x \in \Omega,$$

where $F(x,s) = \displaystyle\int_0^t f(x,t)dt$.

(H_3) There is a constant $C \geq 0$ such that

$$|\nabla_x F(x,s)| \leq C(F(x,s)+1), \quad x \in \Omega.$$

We say that f satisfies (H_i) in $\mathbb{R}_+$, if we have the assumption (H_i) only for s large enough.

For the second order case, i.e. $k = 1$, Bahri and Lions obtained in [1] the estimates of solutions in $H_0^1(\Omega)$ for superlinear and subcritical growth f, by using the blow-up technique and the Morse index of the solutions. Motivated by [1], based on some local interior estimates and careful boundary estimates, Yang obtained in [5] the first explicit estimates of L^p or L^∞ norm for solutions to (P_1) via the Morse index. More precisely, Yang proved that

Theorem A. *Let f satisfy (H_1)-(H_3), then there exist positive constant C, α and β such that any $u \in C^2(\Omega) \cap C(\overline{\Omega})$, solution of (P_1) satisfies*

$$\int_\Omega |f(x,u)|^{p_0} dx \leq C(i(u)+1)^\alpha, \quad \|u\|_{L^\infty(\Omega)} \leq C(i(u)+1)^\beta,$$

where

$$p_0 = 1 + \frac{(1+\theta)(N-2)}{(1-\theta)N + 2(1+\theta)}, \quad \alpha = \left(\frac{3}{2} + \frac{3}{2+\mu}\right)\frac{(2+\mu)^2}{3\mu+\mu^2}$$

and

$$\beta = \frac{2\alpha}{p_0 N(2-p_0)}\left[\frac{2}{N(2-p_0)} - \frac{1}{p_0}\right]^{-1}.$$

Hajlaoui, Harrabi and Mtiri revised in [3] the results of [5], they obtained similar L^∞-estimate for solution to (P_1). The proof in [3] is more transparent, and it allows them to get a slightly better estimate for large dimension N :

Theorem B. *Let f satisfy (H_1)-(H_3), then there exist positive constant C, α' and β' such that any classical solution u of (P_1) satisfies*

$$\int_\Omega |\nabla u|^2 dx \leq C(i(u)+1)^{\alpha'}, \quad \|u\|_{L^\infty} \leq C(i(u)+1)^{\beta'},$$

where

$$\alpha' = \frac{4}{\mu} + 3 \quad \text{and} \quad \beta' = \frac{3\mu+4}{3\mu\theta} \times \frac{3N^2(1-\theta) + N(7\theta-4) - 2\theta + 12}{N(N-2)^2}.$$

In this paper, we will try to handle the polyharmonic equations. Let

$$(E_k) \qquad \begin{cases} (-\Delta)^k u = f(x,u) & \text{in } \ \Omega; \\ u \text{ satisfies (1.1)}, & \text{if } k \text{ is odd;} \\ u \text{ satisfies (1.2)}, & \text{if } k \text{ is even.} \end{cases}$$

To simplify the presentation, we will concentrate on the cases $k = 2$ and $k = 3$, even we believe that the results should hold true for general $k \in \mathbb{N}$. We will provide some L^p and L^∞ estimates in polynomial growth function of the Morse index, for classical solutions of (E_2) and (E_3), provided suitable conditions on f. As far as we know, it seems to be the first time that some explicit estimates are obtained for polyharmonic problems via the Morse index.

As in [3], we shall employ a cut-off function with compact support to derive a variant of the Pohozaev identity. This device allows us to avoid the spherical integrals raised in [5], which are very difficult to control, especially for the polyharmonic situations. Furthermore, under (H_1)-(H_3), the local L^2-estimate of ∇u and Δu via the Morse index seem also difficult to derive for the polyharmonic equation than for (P_1) the second order case. As in [3], we need to exhibit the explicit dependence on $i(u)$ (see Lemma 2.3 and lemma 3.3 below). The following are our main results.

Theorem 1.1. *If u is a classical solution of (E_2) with $f \geq 0$ satisfying (H_1)-(H_3) in $\mathbb{R}_+$; or if u is a classical solution of (E_3) with f satisfying (H_1)-(H_3), then there exists a positive constant C independent of u such that*

$$\int_\Omega |f(x,u)|^{p_k} dx \leq C(i(u)+1)^{\alpha_k}$$

where

$$p_k = \frac{2N}{N(1-\theta)+2k(1+\theta)} \quad \text{and} \quad \alpha_k = \frac{4k(\mu+1)}{\mu} \quad \text{where } k = 2 \text{ or } 3 \text{ respectively.}$$

By setting up a standard boot-strap iteration, as f has subcritical growth, we can proceed similarly as in the proof of Theorem 2.2 in [5] and claim that

Theorem 1.2. *If u is a classical solution of (E_2) with $f \geq 0$ satisfying (H_1)-(H_3) in $\mathbb{R}_+$; or if u is a classical solution of (E_3) with f satisfying (H_1)-(H_3), then there exists a positive constant C independent of u such that (for $k = 2$ or 3 respectively),*

$$\|u\|_{L^\infty(\Omega)} \leq C(i(u)+1)^{\beta_k}, \quad \text{where } \beta_k = \frac{2k\alpha_k}{p_k N(2-p_k)}\left[\frac{2k}{N(2-p_k)} - \frac{1}{p_k}\right]^{-1}, \quad \alpha_k = \frac{4k(\mu+1)}{\mu},$$

and p_k is defined in Theorem 1.1.

By assumptions (H_1) and (H_2) in $\mathbb{R}$ (resp. in $\mathbb{R}_+$), there exist two positive constants C_1 and C_2 such that for $|s|$ large enough (resp. for s large enough),

$$\frac{(N-2k)(1+\theta)}{2N} f(x,s)s - C_1 \leq F(x,s) \leq \frac{1}{2+\mu} f(x,s)s + C_1, \tag{1.4}$$

$$f(x,s)s \geq C_1(|s|^{2+\mu} - 1) \tag{1.5}$$

and

$$|f(x,s)| \leq C_2\left(|s|^{\frac{N(1-\theta)+2k(1+\theta))}{(N-2k)(1+\theta)}} + 1\right). \tag{1.6}$$

This paper is organized as follows : We give the proof of Theorem 1.1 for $k = 2$ and $k = 3$ respectively in sections 2 and 3. In the following, C denotes always a generic positive constant independent of the solution u, even their value could be changed from one line to another one.

2. Proof for $k = 2$

Here we will prove Theorem 1.1 for $k = 2$.

2.1. Preliminaries

Let $y \in \mathbb{R}^N$ and $R > 0$. Throughout the paper, we denote by $B_R(y)$ the open ball of center y and radius R and $\partial\Omega_R(y) := \partial\Omega \cap B_R(y)$. For $x \in B_R(y) \cap \Omega$, let $n := x - y$. We denote also

$$u_{j_i \cdots j_k} := \frac{\partial^k u}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_k}}.$$

First of all, we have the following Pohozaev identity.

Lemma 2.1. *Let u be a classical solution to (E_2). Let $\psi \in C_c^2(B_R(y))$. Then*

$$\begin{aligned}
&\frac{2N}{N-4}\int_\Omega F(x,u)\psi dx + \frac{2}{N-4}\int_\Omega \nabla_x F(x,u)\cdot n\psi dx - \int_\Omega (\Delta u)^2 \psi dx \\
= &-\frac{4}{N-4}\int_\Omega \Delta u \nabla^2 u(\nabla\psi, n) dx + \frac{1}{N-4}\int_\Omega (\nabla\psi\cdot n)(\Delta u)^2 dx \\
&-\frac{4}{N-4}\int_\Omega (\nabla u\cdot\nabla\psi)\Delta u dx - \frac{2}{N-4}\int_\Omega (\nabla u\cdot n)\Delta u\Delta\psi dx \\
&-\frac{2}{N-4}\int_\Omega F(x,u)\nabla\psi\cdot n dx - \frac{2}{N-4}\int_{\partial\Omega_R(y)} \frac{\partial\Delta u}{\partial\nu}(\nabla u\cdot n)\psi d\sigma.
\end{aligned}$$

The proof is classical by multiplying the equation by $(n\cdot\nabla u)\psi$ and integration by parts, so we omit it.

To establish a global estimate, we will cover the domain Ω by small balls and obtain local estimates. To be more precise, consider

$$\Omega_{1,R} := \left\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega) > \frac{R}{2}\right\} \text{ and } \Omega_{2,R} := \left\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega) \le \frac{R}{3}\right\}, \quad \forall\, R>0.$$

The main difficulty is the estimates of u near the boundary, that is, in $\Omega_{2,R}$. We need to choose carefully the balls as in [5]. Indeed, we will take balls with center lying in

$$\Gamma(R) := \left\{x\in\mathbb{R}^N\backslash\Omega:\ \operatorname{dist}(x,\partial\Omega) = \frac{R}{20}\right\}, \tag{2.1}$$

The domain $\Omega\backslash\Omega_{2,R}$ will be covered by balls with center lying in $\Omega_{1,R}$. The following lemma is devoted to the control of the boundary term for $y\in\Gamma(R)$ in the above Pohozaev identity.

Lemma 2.2. *There exists $R_1>0$ depending on Ω such that if $f(x,u)\ge 0$ and u is a classical solution of (E_2), then for any $0<R\le R_1$ and $y\in\Gamma(R)$, there holds*

$$\int_{\partial\Omega_R(y)} \frac{\partial\Delta u}{\partial\nu}(\nabla u\cdot n)\psi d\sigma \ge 0,$$

for any nonnegative function $\psi\in C_c^2(B_R(y))$.

Proof. As in the proof of Lemma 2.2 of [5], there exists $R_1>0$ such that if $0<R\le R_1$ and $y\in\Gamma(R)$ then $\nu\cdot n\le 0$ for any $x\in\partial\Omega_R(y)$.

As $f(x,u)\ge 0$, the maximum principle implies that $-\Delta u\ge 0$ in Ω as $\Delta u = 0$ on $\partial\Omega$, hence $u\ge 0$. Therefore $\frac{\partial\Delta u}{\partial\nu}\ge 0$ on $\partial\Omega$ and $\nabla u\cdot n = (n\cdot v)\frac{\partial u}{\partial\nu}\ge 0$ on $\partial\Omega$, so we obtain the claim. □

Consequently, we get

Proposition 2.1. *There exists $R_0>0$ small who satisfies the following property: Let u be a classical solution of (E_2) with $f\ge 0$ verifying (H_1)-(H_3) in $\mathbb{R}_+$. Then for any $0<R\le R_0$, $y\in\Gamma(R)$ and $0\le\psi\in C_c^4(B_R(y))$, there holds*

$$\begin{aligned}
&\int_\Omega f(x,u)u\psi dx + \int_\Omega(\Delta u)^2\psi dx\\
&\le CR\|\nabla\psi\|_\infty\int_{A_{R,\psi}(y)} f(x,u)udx + CR^2\int_{A_{R,\psi}(y)}|\nabla^2(u\nabla\psi)|^2dx\\
&\quad + C\Big(1+R\|\nabla\psi\|_\infty\Big)\|\Delta u\|^2_{L^2(A_{R,\psi}(y))} + C\Big(R^2\|\nabla(\Delta\psi)\|^2_\infty + \|\Delta\psi\|^2_\infty\Big)\|u\|^2_{L^2(A_{R,\psi}(y))}\\
&\quad + CR^2\Big(\|\Delta\psi\|^2_\infty + \frac{1}{R^2}\|\nabla\psi\|^2_\infty + \|\nabla^2\psi\|^2_\infty\Big)\|\nabla u\|^2_{L^2(A_{R,\psi}(y))} + CR^N,
\end{aligned} \tag{2.2}$$

where

$$A_{R,\psi}(y) = B_R(y)\cap\Omega\cap\{\nabla\psi\ne 0\}.$$

Moreover, for $y\in\Omega_{1,R}$, the above inequality holds true if we replace R by $\frac{R}{2}$.

Proof. Let $y\in\Gamma(R)$ with $R<R_1$ and $0\le\psi\in C_c^4(B_R(y))$. Using Lemmas 2.1–2.2, (H_1)-(H_3) and (1.4), we obtain

$$\begin{aligned}
&(1+\theta)\int_\Omega f(x,u)u\psi dx - \int_\Omega(\Delta u)^2\psi dx\\
&\le \frac{4}{N-4}\int_{A_{R,\psi}(y)}|\Delta u||\nabla^2 u(\nabla\psi,n)|dx + \frac{1}{N-4}\int_{A_{R,\psi}(y)}(\Delta u)^2|\nabla\psi\cdot n|dx\\
&\quad + \frac{4}{N-4}\int_{A_{R,\psi}(y)}|\Delta u||\nabla u\cdot\nabla\psi|dx + \frac{2}{N-4}\int_{A_{R,\psi}(y)}|\Delta u||\nabla u\cdot n||\Delta\psi|dx\\
&\quad + \frac{1}{(N-4)}\int_{A_{R,\psi}(y)} f(x,u)u|\nabla\psi\cdot n|dx + CR\int_{B_R(y)\cap\Omega} f(x,u)u\psi dx + CR^N.
\end{aligned} \tag{2.3}$$

A direct calculation implies that

$$\nabla^2 u(\nabla\psi, n) = \sum_{ij} u_{ij}\psi_i n_j = \sum_{ij}(u\psi_i)_{ij} n_j - u\nabla(\Delta\psi)\cdot n - \Delta\psi(\nabla u\cdot n) - \nabla^2\psi(\nabla u, n).$$

By the Cauchy-Schwarz inequality, there exists $C > 0$ such that

$$\begin{aligned}\int_{A_{R,\psi}(y)} |\Delta u||\nabla^2 u(\nabla\psi, n)|dx \leq C\int_{A_{R,\psi}(y)} |\Delta u|^2 dx + CR^2\int_{A_{R,\psi}(y)} u^2|\nabla(\Delta\psi)|^2 dx \\ + CR^2\int_{A_{R,\psi}(y)} |\nabla^2(u\nabla\psi)|^2 dx \\ + CR^2\int_{A_{R,\psi}(y)} |\nabla u|^2\Big(\|\Delta\psi\|_\infty^2 + \|\nabla^2\psi\|_\infty^2\Big)dx.\end{aligned} \tag{2.4}$$

On the other hand, recall that $u = \Delta u = 0$ on $\partial\Omega$ and $\psi \in C_c^4(B_R(y))$, multiplying the equation (E_2) by $u\psi$ and integrating by parts, we get readily

$$\begin{aligned}\int_\Omega (\Delta u)^2\psi dx - \int_\Omega f(x,u)u\psi dx &\leq C\int_{A_{R,\psi}(y)} |\Delta u|\Big[|\nabla u\cdot\nabla\psi| + |u||\Delta\psi|\Big]dx \\ &\leq C\int_{A_{R,\psi}(y)} \Big[(\Delta u)^2 + |\nabla u\cdot\nabla\psi|^2 + (\Delta\psi)^2u^2\Big]dx.\end{aligned} \tag{2.5}$$

Remark that

$$\begin{aligned}\frac{\theta}{2}\int_\Omega(\Delta u)^2\psi dx + \frac{\theta}{2}\int_\Omega f(x,u)u\psi dx = (1+\theta)\int_\Omega f(x,u)u\psi dx - \int_\Omega(\Delta u)^2\psi dx \\ + \left(1+\frac{\theta}{2}\right)\left[\int_\Omega(\Delta u)^2\psi dx - \int_\Omega f(x,u)u\psi dx\right].\end{aligned}$$

Fix $R_0 \in (0, R_1)$ such that $CR_0 < 1$. Combining (2.3)-(2.5), using again Cauchy-Schwarz inequality, there holds clearly (2.2). The proof for $y \in \Omega_{1,R}$ is completely similar , so we omit it. □

Remark 2.1. *The key point in* (2.2) *is that the integral over the ball $B_R(y)\cap\Omega$ is now controlled by the integrals over the annuli type domain $A_{R,\psi}(y)$ when we work with suitable cut-off function ψ.*

Let $R > 0$, $y \in \Omega_{1,R}\cup\Gamma(R)$, $0 < a < b$. Denote

$$A := A_a^b = \{x\in\mathbb{R}^N;\ a < |x-y| < b\},\quad A_\rho := A_{a+\rho}^{b-\rho}\ \text{ for }\ 0 < \rho < \frac{b-a}{4}. \tag{$*$}$$

We will use also the following classical estimates.

Lemma 2.3. *There exists a constant $C > 0$ depending only on N such that for any $u \in H^2(\Omega)\cap H_0^1(\Omega)$ and $0 < \rho < \min(1, \frac{b-a}{4})$, we have*

$$\|\nabla u\|_{L^2(A_\rho\cap\Omega)}^2 \leq C\left(\frac{1}{\rho^2}\|u\|_{L^2(A\cap\Omega)}^2 + \|\Delta u\|_{L^2(A\cap\Omega)}^2\right).$$

Remark 2.2. *If f satisfies (H_1), using* (1.5)*, there holds*

$$\|u\|_{L^2(A\cap\Omega)}^2 \leq C\left(\int_{A\cap\Omega} f(x,u)udx\right)^{\frac{2}{2+\mu}} + C.$$

2.2. Estimation via Morse index

Let u be a solution to (E_2) with $f \geq 0$ and finite Morse index $i(u)$. For $y \in \Gamma(R)\cup\Omega_{1,R}$, denote

$$A_j =: A_{a_j}^{b_j}\quad\text{with}\ \ a_j = \frac{2(j+i(u))}{4(i(u)+1)}R,\ \ b_j = \frac{2(j+i(u))+1}{4(i(u)+1)}R,\quad 1\leq j\leq i(u)+1. \tag{2.6}$$

Fix a cut-off function $\Phi \in C^{\infty}(\mathbb{R})$ such that $\Phi = 1$ in $[0,1]$ and $\mathrm{supp}(\Phi) \subset (-\frac{1}{2}, \frac{3}{2})$. Let

$$\phi_j(x) := \Phi\left(\frac{4(i(u)+1)|x-y|}{R} - 2j - 2i(u)\right).$$

Then for any $1 \le j \le i(u)+1$, $\phi_j \in C_c^{\infty}(B_R(y))$,

$$\phi_j(x) = 1 \ \text{ in } \ A_j, \quad \|\nabla \phi_j\|_\infty \le \frac{C}{R}(1+i(u)) \quad \text{and} \quad \|\Delta \phi_j\|_\infty \le \frac{C}{R^2}(1+i(u))^2. \tag{2.7}$$

We prove the following lemma.

Lemma 2.4. *Let f satisfy (H_1) and let u be a smooth solution to (E_2) with Morse index $i(u) < \infty$. Then for any $0 < R \le R_0$, $y \in \Gamma(R) \cup \Omega_{1,R}$, there exists $j_0 \in \{1, 2, ..., 1+i(u)\}$ verifying*

$$\int_{A_{j_0}\cap\Omega} (\Delta u)^2 dx + \int_{A_{j_0}\cap\Omega} f(x,u)u dx \le C\left(\frac{1+i(u)}{R}\right)^{\frac{4\mu+8}{\mu}}. \tag{2.8}$$

Proof. First, for $\epsilon \in (0,1)$ and $\eta \in C^2(\mathbb{R}^N)$,

$$\begin{aligned}\int_\Omega [\Delta(u\eta)]^2 dx &= \int_\Omega (u\Delta\eta + 2\nabla u \nabla \eta + \eta \Delta u)^2 dx \\ &\le \left(1+\frac{\epsilon}{2}\right)\int_\Omega (\Delta u)^2\eta^2 dx + \frac{C}{\epsilon}\int_\Omega u^2(\Delta\eta)^2 dx + \frac{C}{\epsilon}\int_\Omega |\nabla u|^2|\nabla\eta|^2 dx.\end{aligned}$$

Using $\Delta(u^2) = 2|\nabla u|^2 + 2u\Delta u$, there holds

$$\int_\Omega |\nabla u|^2|\nabla \eta|^2 dx \le \frac{1}{2}\int_\Omega u^2\Delta(|\nabla\eta|^2)dx + \int_\Omega |u||\Delta u||\nabla\eta|^2 dx. \tag{2.9}$$

Take $\eta = \zeta^m$ with $m > 2$, $\zeta \ge 0$ and apply Young's inequality, we get

$$\begin{aligned}\int_\Omega |u||\Delta u||\nabla\zeta^m|^2 dx &= m^2\int_\Omega |u||\Delta u||\nabla\zeta|^2\zeta^{2m-2}dx \\ &\le \epsilon^2\int_\Omega (\Delta u)^2\zeta^{2m}dx + C_{\epsilon,m}\int_\Omega u^2|\nabla\zeta|^4\zeta^{2m-4}dx.\end{aligned} \tag{2.10}$$

Here $C_{\epsilon,m}$ denotes a constant depending only on ϵ and m. Therefore

$$\int_\Omega [\Delta(u\zeta^m)]^2 dx \le (\epsilon+1)\int_\Omega (\Delta u)^2\zeta^{2m}dx + C_{\epsilon,m}\int_\Omega u^2\Big[|\Delta\zeta|^2 + |\nabla\zeta|^4 + |\Delta(|\nabla\zeta|^2)|\Big]\zeta^{2m-4}dx. \tag{2.11}$$

Consider now the family of functions $\{u\phi_j^m\}_{1\le j\le i(u)+1}$, $m > 2$. With the definition of ϕ_j, it's easy to see that different ϕ_j are supported by disjoint sets for different j, so they are linearly independent as $u > 0$ in Ω. Therefore, there must exist $j_0 \in \{1, 2, ..., 1+i(u)\}$ such that $\Lambda_u(u\phi_{j_0}^m) \ge 0$ where Λ is the quadratic form given by (1.3). Combining $\Lambda_u(u\phi_{j_0}^m) \ge 0$ with (2.7) and (2.11), we obtain

$$\int_\Omega f'(x,u)u^2\phi_{j_0}^{2m}dx - (1+\epsilon)\int_\Omega (\Delta u)^2\phi_{j_0}^{2m}dx \le \frac{C_\epsilon}{R^4}(1+i(u))^4\int_\Omega u^2\phi_{j_0}^{2m-4}dx. \tag{2.12}$$

Moreover, multiply the equation (E_2) by $u\eta^2$ and integrate by parts, we get, using (2.9)

$$\begin{aligned}&\int_\Omega \Big[(\Delta u)^2\eta^2 - f(x,u)u\eta^2\Big]dx \\ = &-4\int_\Omega \eta\Delta u\nabla u\cdot\nabla\eta dx - 2\int_\Omega \eta u\Delta u\Delta\eta dx - 2\int_\Omega u\Delta u|\nabla\eta|^2 dx \\ \le &\ \epsilon\int_\Omega (\Delta u)^2\eta^2 dx + C_\epsilon\int_\Omega u^2(\Delta\eta)^2 dx + C_\epsilon\int_\Omega |\nabla u|^2|\nabla\eta|^2 dx - 2\int_\Omega u\Delta u|\nabla\eta|^2 dx \\ \le &\ \epsilon\int_\Omega (\Delta u)^2\eta^2 dx + C_\epsilon\int_\Omega u^2\Big[(\Delta\eta)^2 + |\Delta(|\nabla\eta|^2)\Big]dx + C_\epsilon\int_\Omega |u\Delta u||\nabla\eta|^2 dx.\end{aligned}$$

Take now $\eta = \phi_{j_0}^m$ with $m = 2 + \frac{2}{\mu} > 2$, there holds as for (2.10),

$$\int_\Omega |u\Delta u||\nabla\eta|^2 dx \le \epsilon \int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx + C_\epsilon \int_\Omega u^2 \phi_{j_0}^{2(m-2)} |\nabla \phi_{j_0}|^4 dx.$$

By (2.7), we deduce then

$$(1-2\epsilon)\int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx - \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^4}(1+i(u))^4 \int_\Omega u^2 \phi_{j_0}^{2m-4} dx. \tag{2.13}$$

Let $\epsilon < \frac{1}{2}$, multiplying (2.13) by $\frac{1+2\epsilon}{1-2\epsilon}$, using (2.12) and (H_1), we get

$$\epsilon \int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx + \left(\mu - \frac{4\epsilon}{1-2\epsilon}\right)\int_\Omega f(x,u)u\phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^4}(1+i(u))^4 \int_\Omega u^2 \phi_{j_0}^{2m-4} dx + C_\epsilon.$$

Fix now $\epsilon < \min(2, \frac{\mu}{4+2\mu})$, there holds

$$\int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx + \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx \le \frac{C}{R^4}(1+i(u))^4 \int_\Omega u^2 \phi_{j_0}^{2m-4} dx + C.$$

Therefore, using (1.5) and $R \le R_0$, for any $\epsilon' > 0$,

$$\begin{aligned}
\int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx + \int_\Omega uf(x,u)\phi_{j_0}^{2m} dx &\le C_{\epsilon'}\left(\frac{1+i(u)}{R}\right)^{\frac{4\mu+8}{\mu}} + C + \epsilon' \int_\Omega |u|^{\mu+2}\phi_{j_0}^{(m-2)(\mu+2)} dx\\
&\le C_{\epsilon'}\left(\frac{1+i(u)}{R}\right)^{\frac{4\mu+8}{\mu}} + C\epsilon' \int_\Omega f(x,u)u\phi_{j_0}^{(m-2)(\mu+2)} dx\\
&= C_{\epsilon'}\left(\frac{1+i(u)}{R}\right)^{\frac{4\mu+8}{\mu}} + C\epsilon' \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx.
\end{aligned}$$

For the last line, we used $(m-2)(\mu+2) = 2m$. Take $\epsilon' > 0$ small enough, the estimate (2.8) is proved. □

2.3. Proof of Theorem 1.1 completed

Now, we are in position to prove Theorem 1.1 for $k = 2$. Fix

$$R = R_0, \quad \rho := \frac{R}{10(i(u)+1)}, \quad A_{j_0,\rho} := A_{a_{j_0}+\rho}^{b_{j_0}-\rho} \subset A_{j_0} \text{ be as in } (*).$$

According to Lemmas 2.3, 2.4 and Remark 2.2, there exists a positive constant C independent of $y \in \Gamma(R) \cup \Omega_{1,R}$ such that

$$\|\Delta u\|_{L^2(A_{j_0,\rho}\cap\Omega)}^2 + \|\nabla u\|_{L^2(A_{j_0,\rho}\cap\Omega)}^2 \le C(1+i(u))^{\frac{4\mu+8}{\mu}}. \tag{2.14}$$

Here, a_{j_0} and b_{j_0} are defined in (2.6) with j_0 given by Lemma 2.4.

Consider a cut-off function $\xi_{j_0} \in C_c^4(B_{b_{j_0}-\rho}(y))$ verifying $\xi_{j_0}(x) \equiv 1$ in $B_{a_{j_0}+\rho}(y)$, with

$$\|\nabla \xi_{j_0}\|_\infty \le \frac{C}{R}(1+i(u)), \quad \|\Delta \xi_{j_0}\|_\infty \le \frac{C}{R^2}(1+i(u))^2.$$

Applying Proposition 2.1 with $\psi = \xi_{j_0}$, as $A_{R,\psi}(y) \subset A_{j_0,\rho} \cap \Omega$, we get

$$\begin{aligned}
&\int_\Omega f(x,u)u\xi_{j_0} dx + \int_\Omega (\Delta u)^2 \xi_{j_0} dx\\
\le\;& C(1+i(u))\int_{A_{j_0,\rho}\cap\Omega}\Big[(\Delta u)^2 + f(x,u)u\Big]dx + C\int_{A_{j_0,\rho}\cap\Omega} |\nabla^2(u\nabla\xi_{j_0})|^2 dx\\
&+ C(1+i(u))^6\|u\|_{L^2(A_{j_0,\rho}\cap\Omega)}^2 + C(1+i(u))^4\|\nabla u\|_{L^2(A_{j_0,\rho}\cap\Omega)}^2 + CR^N.
\end{aligned} \tag{2.15}$$

Since $u\nabla\xi_{j_0} = 0$ on $\partial\Omega$, by standard elliptic theory, there exists $C_\Omega > 0$ depending only on Ω such that

$$\begin{aligned}\int_\Omega |\nabla^2(u\nabla\xi_{j_0})|^2 dx &\le C_\Omega \int_\Omega |\Delta(u\nabla\xi_{j_0})|^2 dx\\ &= C_\Omega \int_{A_{j_0,\rho}\cap\Omega} |\Delta(u\nabla\xi_{j_0})|^2 dx \\ &\le C \int_{A_{j_0,\rho}\cap\Omega} \Big[u^2|\nabla(\Delta\xi_{j_0})|^2 + |\nabla u|^2|\nabla^2\xi_{j_0}|^2 + (\Delta u)^2|\nabla\xi_{j_0}|^2\Big] dx.\end{aligned} \tag{2.16}$$

From (2.15), (2.16), we get the following inequality

$$\begin{aligned}&\int_\Omega f(x,u)u\xi_{j_0} dx + \int_\Omega (\Delta u)^2\xi_{j_0} dx\\ \le\ & C(1+i(u))\int_{A_{j_0,\rho}\cap\Omega}\Big[(\Delta u)^2 + f(x,u)u\Big]dx + C(1+i(u))^2\|\Delta u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)}\\ &+ C(1+i(u))^6\|u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} + C(1+i(u))^4\|\nabla u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} + CR^N.\end{aligned} \tag{2.17}$$

On the other hand, using Remark 2.2 and Lemma 2.4, there holds

$$\|u\|^2_{L^2(A_{j_0}\cap\Omega)} \le C\left(\int_{A_{j_0}\cap\Omega} f(x,u)udx\right)^{\frac{2}{2+\mu}} + C \le C(1+i(u))^{\frac{8}{\mu}}. \tag{2.18}$$

Combining (2.8), (2.14), (2.17) and (2.18), one obtains

$$\int_\Omega f(x,u)u\xi_{j_0} dx + \int_\Omega (\Delta u)^2\xi_{j_0} dx \le C(1+i(u))^{\frac{8\mu+8}{\mu}}.$$

As $\frac{R}{2} < a_{j_0}$ and $R = R_0$, we get then for any $y \in \Gamma(R)\cup\Omega_{1,R}$,

$$\int_{B_{\frac{R_0}{2}}(y)\cap\Omega}\Big[|\Delta u|^2 + f(x,u)u\Big]dx \le C(1+i(u))^{\frac{8\mu+8}{\mu}}.$$

By covering argument and (1.6), we get finally

$$\int_\Omega f(x,u)^{p_2} dx \le C\int_\Omega f(u)udx + C \le C(1+i(u))^{\alpha_2},$$

where $p_2 = \frac{2N}{N(1-\theta)+4(1+\theta)}$ and $\alpha_2 = \frac{8(\mu+1)}{\mu}$. So we are done. □

3. Proof of Theorem 1.1 for $k = 3$

In this section, we consider the equation (E_3). We will proceed as for (E_2) and keep the same notations, but we replace the Navier boundary conditions by the Dirichlet boundary conditions and we have no more the sign condition for f.

3.1. Preliminaries

We make some preparations here. For $\psi \in C^m$ for $m \ge 1$, to simplify the notation, we define

$$[\psi]_m(x) = \sum_{|\beta_1|+\ldots+|\beta_p|=m, |\beta_i|\ge 1}\prod_{i=1}^p |\partial_{\beta_i}\psi(x)|$$

and the semi-norms

$$|\psi|_{m,\infty} = \sum_{\alpha_1+\ldots+\alpha_p=m, \alpha_i\ge 1}\prod_{i=1}^p \|\nabla^{\alpha_i}\psi\|_\infty, \forall\, m \ge 1.$$

Obviously, for any $\psi \in C^m$, we have $\|[\psi]_m\|_\infty \le C_m|\psi|_{m,\infty}$.

Lemma 3.1. *Let $m \geq 3$. For any $\epsilon > 0$, there exists $C_{\epsilon,m} > 0$ such that for any $u \in H_0^3(\Omega)$ and $\zeta \in C^6(\overline{\Omega})$, there holds*

$$\int_\Omega \Big[(\Delta u)^2|\nabla\zeta^m|^2 + |\nabla u|^2|\nabla^2\zeta^m|^2\Big]dx \leq \epsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx. \tag{3.1}$$

Proof. Using the equality $\Delta(u^2) = 2u\Delta u + 2|\nabla u|^2$, we have

$$\int_\Omega |\nabla u|^2|\nabla\zeta|^4\zeta^{2m-4}dx \leq \frac{1}{2}\int_\Omega u^2\Delta\left(|\nabla\zeta|^4\zeta^{2m-4}\right)dx + \int_\Omega |u||\Delta u||\nabla\zeta|^4\zeta^{2m-4}dx.$$

Applying Young's inequality, we get, for any $\epsilon > 0$

$$\int_\Omega |u\Delta u||\nabla\zeta|^4\zeta^{2m-4}dx \leq \epsilon\int_\Omega (\Delta u)^2|\nabla\zeta|^2\zeta^{2m-2}dx + C_\epsilon\int_\Omega u^2|\nabla\zeta|^6\zeta^{2m-6}dx.$$

So we get

$$\int_\Omega |\nabla u|^2|\nabla\zeta|^4\zeta^{2m-4}dx \leq \epsilon\int_\Omega (\Delta u)^2|\nabla\zeta|^2\zeta^{2m-2}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx. \tag{3.2}$$

On the other hand, direct integrations by parts yield (recall that $u \in H_0^3(\Omega)$)

$$\begin{aligned}
\int_\Omega (\Delta u)^2|\nabla\eta|^2dx &= -\int_\Omega \nabla u\nabla(\Delta u)|\nabla\eta|^2dx - 2\int_\Omega \Delta u\nabla^2\eta(\nabla\eta,\nabla u)dx \\
&= -\int_\Omega \nabla u\nabla(\Delta u)|\nabla\eta|^2dx + 2\int_\Omega u\nabla^2\eta(\nabla\eta,\nabla(\Delta u))dx \\
&\quad + 2\int_\Omega u\Delta u|\nabla^2\eta|^2dx + 2\int_\Omega u\Delta u\nabla\eta\cdot\nabla(\Delta\eta)dx \\
&= -\int_\Omega \nabla u\nabla(\Delta u)|\nabla\eta|^2dx + 2\int_\Omega u\nabla^2\eta(\nabla\eta,\nabla(\Delta u))dx \\
&\quad + \int_\Omega \Big[\Delta(u^2) - 2|\nabla u|^2\Big]|\nabla^2\eta|^2dx + 2\int_\Omega u\Delta u\nabla\eta\cdot\nabla(\Delta\eta)dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_\Omega \Big[(\Delta u)^2|\nabla\eta|^2 + 2|\nabla u|^2|\nabla^2\eta|^2\Big]dx = &-\int_\Omega \nabla u\nabla(\Delta u)|\nabla\eta|^2dx + 2\int_\Omega u\nabla^2\eta(\nabla\eta,\nabla(\Delta u))dx \\
&+ \int_\Omega u^2\Delta(|\nabla^2\eta|^2)dx + 2\int_\Omega u\Delta u\nabla\eta\cdot\nabla(\Delta\eta)dx.
\end{aligned} \tag{3.3}$$

Consider $\eta = \zeta^m$. For any $\epsilon > 0$, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&-\int_\Omega \nabla u\nabla(\Delta u)|\nabla\eta|^2dx + 2\int_\Omega u\nabla^2\eta(\nabla\eta,\nabla(\Delta u))dx \\
&\leq \epsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega |\nabla u|^2|\nabla\zeta|^4\zeta^{2m-4}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx
\end{aligned}$$

and

$$2\int_\Omega u\Delta u\nabla\eta\cdot\nabla(\Delta\eta)dx \leq \epsilon\int_\Omega |\Delta u|^2|\nabla\zeta^m|^2dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.$$

Inserting the two above estimates in (3.3), one gets

$$\begin{aligned}
&(1-\epsilon)\int_\Omega (\Delta u)^2|\nabla\zeta^m|^2dx + \int_\Omega |\nabla u|^2|\nabla^2\zeta^m|^2dx \\
&\leq \epsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega |\nabla u|^2|\nabla\zeta|^4\zeta^{2m-4}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.
\end{aligned}$$

Take another small enough ϵ in (3.2), there holds

$$(1-2\epsilon)\int_\Omega (\Delta u)^2|\nabla\zeta^m|^2dx + \int_\Omega |\nabla u|^2|\nabla^2\zeta^m|^2dx \le \epsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.$$

The proof is completed. □

Using Lemma 3.1, we obtain also

Lemma 3.2. *Let $m \ge 3$. For any $0<\epsilon<1$, there exists $C_\epsilon>0$ such that for any $u \in H_0^3(\Omega)$ and $\zeta \in C^6(\overline{\Omega})$,*

$$\int_\Omega \Big[|\nabla u|^2(\Delta\zeta^m)^2 + |\nabla^2 u|^2|\nabla\zeta^m|^2\Big]dx \le \epsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.$$

Proof. From (2.9), we obtain

$$\begin{aligned}\int_\Omega |\nabla^2 u|^2|\nabla\zeta^m|^2dx &\le \frac{1}{2}\int_\Omega |\nabla u|^2\Delta(|\nabla\zeta^m|^2)dx + m^2\int_\Omega |\nabla u\cdot\nabla(\Delta u)||\nabla\zeta|^2\zeta^{2m-2}dx\\ &\le \int_\Omega |\nabla u|^2\Big[C_\epsilon|\nabla\zeta|^4\zeta^{2m-4} + \nabla\zeta^m\nabla(\Delta\zeta^m)\Big]dx + \int_\Omega |\nabla u|^2|\nabla^2\zeta^m|^2dx\\ &\quad + \epsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx.\end{aligned} \tag{3.4}$$

Rewrite

$$C_\epsilon|\nabla\zeta|^4\zeta^{2m-4} + \nabla\zeta^m\nabla(\Delta\zeta^m) = \zeta^{2m-4}\nabla\zeta\cdot\Psi$$

with a smooth function Ψ. In the spirit of (2.9), we have

$$\begin{aligned}\int_\Omega |\nabla u|^2\zeta^{2m-4}\nabla\zeta\cdot\Psi dx &\le \frac{1}{2}\int_\Omega u^2\Delta(\zeta^{2m-4}\nabla\zeta\cdot\Psi)dx + \int_\Omega |u||\Delta u|\zeta^{2m-4}\nabla\zeta\cdot\Psi dx\\ &\le \int_\Omega u^2\Big[|\Delta(\zeta^{2m-4}\nabla\zeta\cdot\Psi)| + C_\epsilon|\Psi|^2\zeta^{2m-6}\Big]dx + \epsilon\int_\Omega (\Delta u)^2|\nabla\zeta|^2\zeta^{2m-2}dx\\ &\le C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx + \epsilon\int_\Omega (\Delta u)^2|\nabla\zeta^m|^2dx.\end{aligned} \tag{3.5}$$

Combining (3.1) and (3.4)-(3.5), there holds

$$\int_\Omega |\nabla^2 u|^2|\nabla\zeta^m|^2dx \le \epsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.$$

Furthermore, integrating by parts,

$$\begin{aligned}\int_\Omega |\nabla u|^2(\Delta\zeta^m)^2dx &= -2\int_\Omega \nabla^2 u(\nabla u,\nabla\zeta^m)\Delta\zeta^m dx - \int_\Omega |\nabla u|^2\nabla(\Delta\zeta^m)\nabla\zeta^m dx\\ &\le \frac{1}{2}\int_\Omega |\nabla u|^2(\Delta\zeta^m)^2dx + C\int_\Omega |\nabla^2 u|^2|\nabla\zeta^m|^2dx\\ &\quad + \frac{1}{2}\int_\Omega u^2\Delta\Big[\nabla(\Delta\zeta^m)\nabla\zeta^m\Big]dx + \int_\Omega |u||\Delta u|\nabla(\Delta\zeta^m)\nabla\zeta^m dx.\end{aligned}$$

We deduce that

$$\int_\Omega |\nabla u|^2(\Delta\zeta^m)^2dx \le C\int_\Omega \Big[|\nabla^2 u|^2|\nabla\zeta^m|^2 + |\Delta u|^2|\nabla\zeta^m|^2\Big]dx + C\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx,$$

so using the previous estimates, we are done. □

Let $R>0$, $y\in\Omega_{1,R}\cup\Gamma(R)$, $0<a<b$. Denote $A := A_a^b$ and $A_\rho := A_{a+\rho}^{b-\rho}$, similar to Lemma 2.3, we have

Lemma 3.3. *There exists a constant $C>0$ depending only on N such that for any $u\in H_0^3(\Omega)$ and $0<\rho<\min(1,\frac{b-a}{4})$, we have*

$$\|\Delta u\|^2_{L^2(A_\rho\cap\Omega)} \le C\left(\frac{1}{\rho^4}\|u\|^2_{L^2(A\cap\Omega)} + \|\nabla(\Delta u)\|^2_{L^2(A\cap\Omega)}\right).$$

3.2. Explicit estimate via Morse index

Lemma 3.4. *Let f satisfies (H_1) and u be a solution to (E_3) with finite Morse index $i(u)$. Then for any $y \in \Gamma(R) \cup \Omega_{1,R}$ with $R > 0$, there exists $j_0 \in \{1, 2, ..., 1 + i(u)\}$ such that*

$$\int_{A_{j_0}\cap\Omega} |\nabla(\Delta u)|^2 dx + \int_{A_{j_0}\cap\Omega} f(x,u)u dx \le C\left(\frac{1+i(u)}{R}\right)^{\frac{6\mu+12}{\mu}}.$$

Proof. Take $\eta \in C^6(\overline{\Omega})$. By direct calculations, we get, as $u \in H_0^3(\Omega)$,

$$\begin{aligned}\int_\Omega [\nabla(\Delta(u\eta))]^2 dx &= \int_\Omega \left(\nabla(\Delta u)\eta + \Delta u\nabla\eta + 2\nabla^2 u\nabla\eta + \nabla u\Delta\eta + 2\nabla u\nabla^2\eta + u\nabla(\Delta\eta)\right)^2 \\ &\le (1+\epsilon)\int_\Omega |\nabla(\Delta u)|^2\eta^2 dx \\ &\quad + C_\epsilon \int_\Omega \Big[|\Delta u|^2|\nabla\eta|^2 + |\nabla^2 u|^2|\nabla\eta|^2 + |\nabla u|^2\left(|\nabla^2\eta|^2 + |\Delta\eta|^2\right) + u^2|\nabla(\Delta\eta)|^2\Big] dx.\end{aligned}$$

Using Lemmas 3.1-3.2, let $\eta = \zeta^m$ with $m = 3 + \frac{6}{\mu} > 3$, we derive that

$$\int_\Omega |\nabla(\Delta(u\zeta^m))|^2 dx \le (1+\epsilon)\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m} dx + C_\epsilon \int_\Omega u^2[\zeta]_6\zeta^{2m-6} dx.$$

As in section 2, we can easily check that $\{u\phi_j^m\}_{1\le j\le i(u)+1}$ are linearly independent, so there exists $j_0 \in \{1, 2, ..., 1 + i(u)\}$ such that $\Lambda_u(u\phi_{j_0}^m) \ge 0$. The above estimate with $\zeta = \phi_{j_0}$ implies then

$$\int_\Omega f'(x,u)u^2\phi_{j_0}^{2m} dx - (1+\epsilon)\int_\Omega |\nabla(\Delta u)|^2\phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^6}(1+i(u))^6 \int_\Omega u^2\phi_{j_0}^{2m-6} dx. \tag{3.6}$$

Now, take $u\phi_{j_0}^{2m}$ as the test function for (E_3), the integration by parts yields that

$$\int_\Omega |\nabla(\Delta u)|^2\phi_{j_0}^{2m} dx - \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx = \int_\Omega \nabla(\Delta u)\cdot\Big[\nabla\left(\Delta(u\phi_{j_0}^{2m})\right) - \nabla(\Delta u)\phi_{j_0}^{2m}\Big] dx.$$

Developing the right hand side, applying again Lemmas 3.1-3.2, we can conclude: For any $\epsilon > 0$, there exists C_ϵ such that

$$(1-\epsilon)\int_\Omega |\nabla(\Delta u)|^2\phi_{j_0}^{2m} dx - \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^6}(1+i(u))^6 \int_\Omega u^2\phi_{j_0}^{2m-6} dx. \tag{3.7}$$

Multiplying (3.7) by $\frac{1+2\epsilon}{1-\epsilon}$ adding it with (3.6), we obtain from (H_1) that

$$\epsilon\int_\Omega |\nabla(\Delta u)|^2\phi_{j_0}^{2m} dx + \left(\mu - \frac{3\epsilon}{1-\epsilon}\right)\int_\Omega f(x,u)u\phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^6}(1+i(u))^6 \int_\Omega u^2\phi_{j_0}^{2m-6} dx + C.$$

Fix $0 < \epsilon < \frac{\mu}{3+\mu}$, we get

$$\int_\Omega |\nabla(\Delta u)|^2\phi_{j_0}^{2m} dx + \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx \le \frac{C}{R^6}(1+i(u))^6 \int_\Omega u^2\phi_{j_0}^{2m-6} dx + C.$$

By Young's inequality, for any $\epsilon' > 0$, there holds

$$\begin{aligned}\int_\Omega |\nabla(\Delta u)|^2\phi_{j_0}^{2m} dx + \int_\Omega uf(x,u)\phi_{j_0}^{2m} dx &\le C_{\epsilon'}\left(\frac{1+i(u)}{R}\right)^{\frac{6\mu+12}{\mu}} + \epsilon'\int_\Omega |u|^{\mu+2}\phi_{j_0}^{(m-3)(\mu+2)} dx \\ &\le C_{\epsilon'}\left(\frac{1+i(u)}{R}\right)^{\frac{6\mu+12}{\mu}} + C\epsilon'\int_\Omega f(x,u)u\phi_{j_0}^{2m} dx.\end{aligned}$$

We used (1.5) and $(m-3)(2+\mu) = 2m$ for the last line. Take ϵ' small enough, the claim follows. □

3.3. Proof of Theorem 1.1 for $k = 3$

We show firstly the Pohozaev identity for (E_3).

Lemma 3.5. *Let u be solution to (E_3). Let $\psi \in C_c^4(B_R(y))$. Then*

$$\begin{aligned} & N\int_\Omega F(x,u)\psi dx + \int_\Omega \nabla_x F(x,u)\cdot n\psi dx - \frac{N-6}{2}\int_\Omega |\nabla(\Delta u)|^2\psi dx \\ &= \frac{1}{2}\int_\Omega |\nabla(\Delta u)|^2(\nabla\psi\cdot n)dx - \int_\Omega F(x,u)\nabla\psi\cdot n dx \\ &\quad - \int_\Omega \Delta\psi\nabla(\Delta u)\nabla(n\cdot\nabla u)dx - 2\int_\Omega \nabla(\Delta u)\nabla\Big[\nabla^2 u(n,\nabla\psi) + \nabla u\nabla\psi\Big]dx \\ &\quad + \int_{\partial\Omega_R(y)} \frac{\partial\Delta u}{\partial\nu}\left(\nabla(\Delta u)\cdot n\right)\psi d\sigma - \frac{1}{2}\int_{\partial\Omega_R(y)} |\nabla(\Delta u)|^2(\nu\cdot n)\psi d\sigma. \end{aligned}$$

For the boundary terms, we have

Lemma 3.6. *There exists $R_1 > 0$ depending only on Ω such that for any u smooth function in $H_0^3(\Omega)$, any $0 < R < R_1$, $y \in \Gamma(R)$ and any nonnegative function ψ, there holds*

$$\int_{\partial\Omega_R(y)} \frac{\partial\Delta u}{\partial\nu}(\nabla(\Delta u)\cdot n)\psi d\sigma - \frac{1}{2}\int_{\partial\Omega_R(y)} |\nabla(\Delta u)|^2\nu\cdot n\psi d\sigma \le 0.$$

Proof. Take $R_1 > 0$ such that $v\cdot n \le 0$ on $\partial\Omega_R(y)$ for any $0 < R \le R_1$ and $y \in \Gamma(R)$. As $u \in H_0^3(\Omega)$, we know that $\nabla(\Delta u)$ is parallel to ν on $\partial\Omega$, in other words $\nabla(\Delta u)(x) = \lambda(x)\nu(x)$ on $\partial\Omega$. Therefore

$$\frac{\partial\Delta u}{\partial\nu}(\nabla(\Delta u)\cdot n) - \frac{1}{2}(\nu\cdot n)|\nabla(\Delta u)|^2 = \frac{\lambda^2}{2}(\nu\cdot n) \le 0, \quad \forall\, x \in \partial\Omega_R(y).$$

So we are done. □

Similar to Proposition 2.1, we can claim

Proposition 3.1. *There exists $R_0 > 0$ small who satisfies the following property: Let u be a classical solution of (E_3) with f verifying (H_1)-(H_3). Then for any $0 < R \le R_0$, $y \in \Gamma(R)$ and $\zeta \in C_c^6(B_R(y))$ verifying $0 \le \zeta \le 1$ and $\psi = \zeta^{2m}$ with $m \ge 3$, there holds*

$$\begin{aligned} & \int_\Omega f(x,u)u\psi dx + \int_\Omega |\nabla(\Delta u)|^2\psi dx \\ &\le CR\|\nabla\zeta\|_\infty \int_{A_{R,\psi}(y)} f(x,u)udx + C\Big(1 + R\|\nabla\zeta\|_\infty + R^2|\zeta|_{2,\infty}\Big)\|\nabla(\Delta u)\|^2_{L^2(A_{R,\psi}(y))} \\ &\quad + CR^2|\zeta|_{6,\infty}\|\nabla u\|^2_{L^2(A_{R,\psi}(y))} + C\Big(|\zeta|_{6,\infty} + R^2|\zeta|_{8,\infty}\Big)\|u\|^2_{L^2(A_{R,\psi}(y))}. \end{aligned} \tag{3.8}$$

Proof. Using Lemmas 3.5- 3.6, (H_1)–(H_3) and by (1.4), we obtain

$$\begin{aligned} & \frac{N-6}{2}\left[(1+\theta)\int_\Omega f(x,u)u\psi dx - \int_\Omega |\nabla(\Delta u)|^2\psi dx\right] \\ &\le CR\|\nabla\psi\|_\infty \int_{A_{R,\psi}(y)} |\nabla(\Delta u)|^2 dx + CR\int_\Omega f(x,u)u\psi dx + CR\|\nabla\psi\|_\infty \int_{A_{R,\psi}(y)} f(x,u)udx \\ &\quad + \int_{A_{R,\psi}(y)} \Big|\nabla(\Delta u)\nabla\Big[\nabla^2 u(n,\nabla\psi) + \nabla u\nabla\psi\Big]\Big|dx + \int_\Omega \Big|\Delta\psi\nabla(\Delta u)\nabla(n\cdot\nabla u)\Big|dx + CR^N. \end{aligned} \tag{3.9}$$

We will use also the following lemma.

Lemma 3.7. *For any $R < 1$, $\psi = \zeta^{2m}$ with $\zeta \in C_c^6(B_R(y))$ in Proposition 3.1, there exists a positive constant C such that*

$$\begin{aligned} & \int_{A_{R,\psi}(y)} \Big|\nabla(\Delta u)\nabla\Big[\nabla^2 u(n,\nabla\psi) + \nabla u\nabla\psi\Big]\Big|dx + \int_\Omega \Big|\Delta\psi\nabla(\Delta u)\nabla(n\cdot\nabla u)\Big|dx \\ &\le C\int_{A_{R,\psi}(y)} |\nabla(\Delta u)|^2 dx + CR^2\int_{A_{R,\psi}(y)} |\nabla(\Delta u)|^2[\zeta]_2 dx \\ &\quad + CR^2\int_{A_{R,\psi}(y)} |\nabla u|^2[\zeta]_6 dx + \int_{A_{R,\psi}(y)} u^2\left([\zeta]_6 + R^2[\zeta]_8\right)dx. \end{aligned} \tag{3.10}$$

Proof. Indeed, in $B_R(y)\cap\Omega$,

$$\Big|\nabla(\Delta u)\nabla\Big[\nabla^2 u(n,\nabla\psi)+\nabla u\nabla\psi\Big]\Big| \le CR|\nabla(\Delta u)|\Big(|\nabla^3(u\nabla\psi)|+|\nabla^2 u||\nabla^2\psi|+|\nabla u||\nabla^3\psi|+|u||\nabla^4\psi|\Big)$$
$$+\,C|\nabla(\Delta u)|\Big(|\nabla^2 u||\nabla\psi|+|\nabla u||\nabla^2\psi|\Big).$$

We get then

$$\begin{aligned}&\int_{A_{R,\psi}(y)}\Big|\nabla(\Delta u)\nabla\Big[\nabla^2 u(n,\nabla\psi)+\nabla u\nabla\psi\Big]\Big|dx\\ &\le C\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2dx+CR^2\int_{A_{R,\psi}(y)}|\nabla^3(u\nabla\psi)|^2dx+CR^2\int_{A_{R,\psi}(y)}|\nabla^2 u|^2|\nabla^2\psi|^2dx\\ &\quad+CR^2\int_{A_{R,\psi}(y)}|\nabla u|^2|\nabla^3\psi|^2dx+CR^2\int_{A_{R,\psi}(y)}u^2|\nabla^4\psi|^2dx\\ &\quad+C\int_{A_{R,\psi}(y)}|\nabla^2 u|^2|\nabla\psi|^2dx+C\int_{A_{R,\psi}(y)}|\nabla u|^2|\nabla^2\psi|^2dx.\end{aligned}$$

First, using Lemmas 3.1-3.2 on $A_{R,\psi}(y)\cap\Omega$, the last two terms can be upper bounded by

$$C\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2dx+C\int_{A_{R,\psi}(y)}u^2[\zeta]_6dx.$$

Moreover, as $u\nabla\psi\in H_0^3(\Omega)$, there exists $C>0$ depending only on Ω such that

$$\int_{A_{R,\psi}(y)}|\nabla^3(u\nabla\psi)|^2dx=\int_\Omega|\nabla^3(u\nabla\psi)|^2dx\le C\int_\Omega|\nabla\Delta(u\nabla\psi)|^2dx=C\int_{A_{R,\psi}(y)}|\nabla\Delta(u\nabla\psi)|^2dx.$$

Remark that (as $\psi=\zeta^{2m}$)

$$\begin{aligned}|\nabla\Delta(u\nabla\psi)|^2&\le C\Big(|\nabla(\Delta u)|^2|\nabla\psi)|^2+|\nabla^2 u|^2|\nabla^2\psi|^2+|\nabla u|^2|\nabla^3\psi|^2+u^2|\nabla^4\psi|^2\Big)\\ &\le C\Big(|\nabla(\Delta u)|^2[\zeta]_2+|\nabla u|^2[\zeta]_6+u^2[\zeta]_8\Big)+C|\nabla^2 u|^2|\nabla^2\psi|^2.\end{aligned}$$

Using the equality $2|\nabla^2 u|^2=\Delta(|\nabla u|^2)-2\nabla u\cdot\nabla(\Delta u)$, we obtain

$$\begin{aligned}\int_{A_{R,\psi}(y)}|\nabla^2 u|^2|\nabla^2\psi|^2dx&\le\frac12\int_{A_{R,\psi}(y)}|\nabla u|^2\Delta(|\nabla^2\psi|^2)dx+\int_{A_{R,\psi}(y)}|\nabla u\cdot\nabla(\Delta u)||\nabla^2\psi|^2dx\\ &\le\frac12\int_{A_{R,\psi}(y)}|\nabla u|^2|\Delta(|\nabla^2\psi|^2)|dx+C\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2|\nabla^2\psi|dx\\ &\quad+C\int_{A_{R,\psi}(y)}|\nabla u|^2|\nabla^2\psi|^3dx\\ &\le\int_{A_{R,\psi}(y)}|\nabla u|^2[\zeta]_6dx+C\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2[\zeta]_2dx.\end{aligned}\tag{3.11}$$

Combining all these inequalities, we obtain the estimate for the first left term in (3.10).

On the other hand,

$$\begin{aligned}&\int_{A_{R,\psi}(y)}\Big|\Delta\psi\nabla(\Delta u)\nabla(n\cdot\nabla u)\Big|dx\\ &\le\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|\Big[R|\nabla^2 u||\Delta\psi|+|\nabla u||\Delta\psi|\Big]dx\\ &\le\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2dx+C\int_{A_{R,\psi}(y)}\Big[R^2|\nabla^2 u|^2|\nabla^2\psi|^2+|\nabla u|^2(\Delta\psi)^2\Big]dx.\end{aligned}$$

Applying (3.11) and Lemma 3.2, the proof is completed. □

Coming back to the proof of (3.8) . Take $u\zeta^{2m}$ as the test function for (E_3), using Lemmas 3.1-3.2, for any $\epsilon > 0$ there exists C_ϵ such that

$$\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx - \int_\Omega f(x,u)u\zeta^{2m}dx \le \epsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx. \tag{3.12}$$

Remark that

$$\frac{\theta}{2}\int_\Omega |\nabla(\Delta u)|^2\psi dx + \frac{\theta}{2}\int_\Omega f(x,u)u\psi dx = (1+\theta)\int_\Omega f(x,u)u\psi dx - \int_\Omega |\nabla(\Delta u)|^2\psi dx + \left(1+\frac{\theta}{2}\right)\left[\int_\Omega |\nabla(\Delta u)|^2\psi dx - \int_\Omega f(x,u)u\psi dx\right].$$

Combining (3.9)-(3.10) and (3.12), for $\epsilon, R > 0$ small enough, we have (3.8). □

Proof of Theorem 1.1 for $k = 3$ completed.

Now, we are in position to prove Theorem 1.1 for $k = 3$. Fix

$$R = R_0, \quad m = 3 + \frac{6}{\mu}, \quad \rho := \frac{R}{10(i(u)+1)}, \quad A_{j_0,\rho} := A^{b_{j_0}-\rho}_{a_{j_0}+\rho} \subset A_{j_0} \text{ be as in } (*).$$

Using Remark 2.2 and lemma 3.4, there holds

$$\|u\|^2_{L^2(A_{j_0}\cap\Omega)} \le C\left(\int_{A_{j_0}\cap\Omega} f(x,u)u\right)^{\frac{2}{2+\mu}} + C \le C(1+i(u))^{\frac{12}{\mu}}. \tag{3.13}$$

According to Lemmas 2.3, 3.3, 3.4 and (3.13), there exists a positive constant C independent of $y \in \Gamma(R) \cup \Omega_{1,R}$ such that

$$\|\nabla(\Delta u)\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} + \|\nabla u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} \le C(1+i(u))^{\frac{6\mu+12}{\mu}}. \tag{3.14}$$

Combining (3.8), (3.13) and (3.14), one obtains

$$\int_\Omega f(x,u)u\xi_{j_0}dx + \int_\Omega (\Delta u)^2\xi_{j_0}dx \le C(1+i(u))^{\frac{12\mu+12}{\mu}}.$$

As $\frac{R}{2} < a_{j_0}$ and $R = R_0$, we get then for any $y \in \Gamma(R) \cup \Omega_{1,R}$,

$$\int_{B_{\frac{R_0}{2}}(y)\cap\Omega} \left[|\Delta u|^2 + f(x,u)u\right]dx \le C(1+i(u))^{\frac{12\mu+12}{\mu}}.$$

The proof is completed by the covering argument. □

Acknowledgment: A.H. and F.M. would like to express their deepest gratitude to our Research Laboratory *LR11ES53 Algebra, Geometry and Spectral Theory (AGST) Sfax University*, for providing us with an excellent atmosphere for doing this work.

References

[1] A. Bahri and P.L. Lions: Solutions of superlinear elliptic equations and their Morse indices, Comm. Pure Appl. Math. **XLV**, 1205-1215 (1992).

[2] K.C. Chang: Ininite-dimensional Morse theory and multiple solution problems, Birkhäuser, Boston, 1993.

[3] H. Hajlaoui, A. Harrabi and F. Mtiri : Morse indices of solutions for super-linear elliptic PDEs, Nonlinear Analysis **116**, 180-192 (2015).

[4] M. Schecher and W. Zou: Critical point theory and its applications, Springer, New York, 2006.

[5] X. Yang, *Nodal Sets and Morse Indices of Solutions of Super-linear Elliptic PDEs*, J. Funct. Anal. **160**, 223-253 (1998).