

EXTREMAL METRICS FOR THE Q' -CURVATURE IN THREE DIMENSIONS

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ABSTRACT. We construct contact forms with constant Q' -curvature on compact three-dimensional CR manifolds which admit a pseudo-Einstein contact form and satisfy some natural positivity conditions. These contact forms are obtained by minimizing the CR analogue of the II -functional from conformal geometry. Two crucial steps are to show that the P' -operator can be regarded as an elliptic pseudodifferential operator and to compute the leading order terms of the asymptotic expansion of the Green's function for $\sqrt{P'}$.

1. INTRODUCTION

On an even-dimensional manifold (M^{2n}, g) , the pair (P, Q) of the (critical) GJMS operator P and the (critical) Q -curvature Q possesses many of the same properties of the pair $(-\Delta, K)$ on surfaces, where K is the Gauss curvature. For example, P is a conformally covariant formally self-adjoint operator with leading order term $(-\Delta)^{n/2}$ which annihilates constants [13, 15] and Q is a Riemannian invariant with leading order term $c_n(-\Delta)^{\frac{n-2}{2}}R$, where R is the scalar curvature, which transforms in a particularly simple way within a conformal class [4]: if $\hat{g} = e^{2u}g$, then

$$e^{n\sigma}\widehat{Q} = Q + Pu.$$

In particular, $\int Q$ is conformally invariant on closed even-dimensional manifolds; indeed, it computes the Euler characteristic modulo integrals of pointwise conformal invariants [1]. It also follows that metrics of constant Q -curvature within a conformal class are in one-to-one correspondence with critical points of the functional

$$II[u] = \int_M uPu + 2 \int_M Qu - \frac{2}{n} \left(\int_M Q \right) \log \left(\frac{1}{\text{Vol}(M)} \int_M e^{nu} \right).$$

This functional can always be minimized on the two-sphere [21] and on four-manifolds with positive Yamabe constant and nonnegative Paneitz operator [2, 9, 16], with important applications to logarithmic functional determinants [5, 21] and sharp Onofri-type inequalities [2]. Due to the parallels between conformal and CR geometry, it is interesting to determine whether a similar pair exists in the latter setting.

Work of Graham and Lee [14] and Hirachi [17] identified CR analogues of the Paneitz operator and Q -curvature in dimension three. However, the kernel of the Paneitz operator contains the (generally infinite-dimensional) space \mathcal{P} of CR pluriharmonic functions and the total Q -curvature is always zero. In particular,

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an Onofri-type inequality involving the CR Paneitz operator cannot be satisfied. Branson, Fontana and Morpurgo overcame this latter issue on the CR spheres by introducing a formally self-adjoint operator P' which is CR covariant on CR pluriharmonic functions and in terms of which one has the sharp Onofri-type inequality

$$\int_{S^{2n+1}} u P' u + 2 \int_M Q' u - \frac{2}{n} \int \left(\int_M Q' \right) \log \left(\frac{1}{\text{Vol}(M)} \int_M e^{nu} \right)$$

for all $u \in W^{n+1,2} \cap \mathcal{P}$, where Q' is an explicit dimensional constant [6]. The construction of P' is analogous to the construction of the Q -curvature from the GJMS operators by analytic continuation in the dimension.

It was observed by the first- and third-named authors in dimension three [8] and by Hirachi in general dimension [18] that one can define the P' -operator on general pseudohermitian manifolds $(M^{2n+1}, T^{1,0}, \theta)$. Roughly speaking, if P_{2n+2}^N is the CR GJMS operator of order $2n+2$ on a $(2N+1)$ -dimensional manifold, one defines P' as the limit of $\frac{2}{(N-n)} P_{2n+2}^N|_{\mathcal{P}}$ as $N \rightarrow n$. This is made rigorous by explicit computation in dimension three [8] and via the ambient metric in general dimension [18]. Regarded as a map from \mathcal{P} to $C^\infty(M)/\mathcal{P}^\perp$, the P' -operator is CR covariant: if $\widehat{\theta} = e^\sigma \theta$, then $e^{(n+1)\sigma} \widehat{P}' = P'$.

If θ is a pseudo-Einstein contact form (cf. [8, 18, 20]), then the P' -operator is formally self-adjoint and annihilates constants. Note that if M^{2n+1} is the boundary of a domain in \mathbb{C}^{n+1} , then the defining functions constructed by Fefferman [11] induce pseudo-Einstein contact forms on M . One can construct a pseudohermitian invariant Q' on pseudo-Einstein manifolds by formally considering the limit $\left(\frac{2}{N-n}\right)^2 P_{2n+2}^N(1)$ as $N \rightarrow n$; this can be made rigorous by direct computation in dimension three [8] and via the ambient metric in general dimension [18]. Regarded as $C^\infty(M)/\mathcal{P}^\perp$ -valued, the Q' -curvature transforms linearly with a change of contact form: if $\widehat{\theta} = e^\sigma \theta$ is also pseudo-Einstein, then

$$(1.1) \quad e^{2(n+1)\sigma} \widehat{Q}' = Q' + P'(\sigma).$$

Since $\widehat{\theta}$ is pseudo-Einstein if and only if $\sigma \in \mathcal{P}$ [17, 20], this makes sense. It follows from the properties of P' that $\int Q'$ is independent of the choice of pseudo-Einstein contact form. Direct computation on S^{2n+1} shows that it is a nontrivial invariant; indeed, in dimension three it is a nonzero multiple of the Burns–Epstein invariant [8]. In particular, the pair (P', Q') on pseudo-Einstein manifolds has the same properties as the pair (P, Q) on Riemannian manifolds.

If $(M^{2n+1}, T^{1,0}, \theta)$ is a compact pseudo-Einstein manifold, the self-adjointness of P' and (1.1) imply that critical points of the functional $II: \mathcal{P} \rightarrow \mathbb{R}$ defined by

$$(1.2) \quad II[u] = \int_M u P' u + 2 \int_M Q' u - \frac{2}{n+1} \left(\int_M Q' \right) \log \left(\frac{1}{\text{Vol}(M)} \int_M e^{(n+1)u} \right)$$

are in one-to-one correspondence with pseudo-Einstein contact forms with constant Q' -curvature (still regarded as $C^\infty(M)/\mathcal{P}^\perp$ -valued). The existence and classification of minimizers of the II -functional on the standard CR spheres was given by Branson, Fontana and Morpurgo [6]. In this note, we discuss the main ideas used by the authors to give criteria which guarantee that minimizers exist for the II -functional on a given pseudo-Einstein three-manifold [7].

Theorem 1.1. *Let $(M^3, T^{1,0}, \theta)$ be a compact, embeddable pseudo-Einstein three-manifold such that $P' \geq 0$ and $\ker P' = \mathbb{R}$. Suppose additionally that*

$$(1.3) \quad \int_M Q' \theta \wedge d\theta < 16\pi^2.$$

Then there exists a function $w \in \mathcal{P}$ which minimizes the II-functional. Moreover, the contact form $\widehat{\theta} := e^w \theta$ is such that \widehat{Q}' is constant.

The assumptions on P' mean that the pairing $(u, v) := \int u P' v$ defines a positive definite quadratic form on \mathcal{P} . It is important to emphasize that the conclusion is that \widehat{Q}'_4 is constant as a $C^\infty(M)/\mathcal{P}^\perp$ -valued invariant: a local formula for the Q' -curvature was given by the first- and third-named authors [8], while we observe that, on $S^1 \times S^2$ with any of its locally spherical contact structures, there is no pseudo-Einstein contact form with Q' pointwise zero; see [7, Section 5].

As in the study of Riemannian four-manifolds (cf. [9, 16]), the hypotheses of Theorem 1.1 can be replaced by the nonnegativity of the pseudohermitian scalar curvature and of the CR Paneitz operator. Indeed, Chanillo, Chiu and the third-named author proved that these assumptions imply that $(M^3, T^{1,0})$ is embeddable [10]; the first- and third-named authors proved that these assumptions imply both that $P' \geq 0$ with $\ker P' = \mathbb{R}$ and that $\int Q' \leq 16\pi^2$ with equality if and only if $(M^3, T^{1,0})$ is CR equivalent to the standard CR three-sphere [8]; and Branson, Fontana and Morpurgo showed that minimizers of the II-functional exist on the standard CR three-sphere [6].

Corollary 1.2. *Let $(M^3, T^{1,0}, \theta)$ be a compact pseudo-Einstein manifold with non-negative scalar curvature and nonnegative CR Paneitz operator. Then there exists a function $w \in \mathcal{P}$ which minimizes the II-functional. Moreover, the contact form $\widehat{\theta} := e^w \theta$ is such that \widehat{Q}' is constant.*

2. SKETCH OF THE PROOF OF THEOREM 1.1

The proof of Theorem 1.1 proceeds analogously to the proof of the corresponding result on four-dimensional Riemannian manifolds [9] with one important difference: P' is defined as a $C^\infty(M)/\mathcal{P}^\perp$ -valued operator; in particular, it is a nonlocal operator. Let $\tau: C^\infty(M) \rightarrow \mathcal{P}$ be the orthogonal projection with respect to the standard L^2 -inner product. A key observation is that the operator $\overline{P}' := \tau P': \mathcal{P} \rightarrow \mathcal{P}$ is a self-adjoint elliptic pseudodifferential operator of order -2 ; see [7, Theorem 9.1]. This follows from the observation that, while the sublaplacian Δ_b is subelliptic, the Toeplitz operator $\tau \Delta_b \tau$ is a classical elliptic pseudodifferential operator of order -1 . This is achieved by writing $\Delta_b = 2\Box_b + iT$, relating τ to the Szegő projector S , and using well-known properties of the latter operator (cf. [3, 19]).

Since $\int u P' v = \int u \overline{P}' v$ for all $u, v \in \mathcal{P}$, it follows that \overline{P}' is a nonnegative operator with $\ker \overline{P}' = \mathbb{R}$. In particular, the positive square root $(\overline{P}')^{1/2}$ of \overline{P}' is well-defined and such that $\ker(\overline{P}')^{1/2} = \mathbb{R}$. Using the pseudodifferential calculus and the fact that, as a local operator, P' equals Δ_b^2 plus lower order terms [8], we then observe that the Green's function of $(\overline{P}')^{1/2}$ is of the form $c\rho^{-2} + O(\rho^{-1-\varepsilon})$ for $\rho^4(z, t) = |z|^4 + t^2$ the Heisenberg pseudo-distance, $\varepsilon \in (0, 1)$, and c the same constant as the computation on the three-sphere [6]; for a more precise statement, see [7, Theorem 1.3].

From this point, the remaining argument is fairly standard. The above fact about the Green's function of $(\overline{P'})^{1/2}$ allows us to apply the Adams-type theorem of Fontana and Morpurgo [12] to conclude that the former operator satisfies an Adams-type inequality with the same constant as on the standard CR three-sphere. This has two important effects. First, it implies that the II -functional is coercive under the additional assumption $\int Q' < 16\pi^2$; see [7, Theorem 4.1]. Second, it implies that if $w \in W^{2,2} \cap \mathcal{P}$ satisfies

$$\tau(P'w + Q' - \lambda e^{2w}) = 0,$$

then $w \in C^\infty(M)$; see [7, Theorem 4.2]. The former assumption allows us to minimize II within $W^{2,2} \cap \mathcal{P}$ and the latter assumption yields the regularity of the minimizers. The final conclusion follows from the transformation formula (1.1) for the Q' -curvature.

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