

OPTIMAL REGULARITY OF MINIMAL GRAPHS IN THE HYPERBOLIC SPACE

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ABSTRACT. We discuss the global regularity of solutions f to the Dirichlet problem for minimal graphs in the hyperbolic space when the boundary of the domain $\Omega \subset \mathbb{R}^n$ has a nonnegative mean curvature and prove an optimal regularity $f \in C^{\frac{1}{n+1}}(\bar{\Omega})$. We can improve the Hölder exponent for f if certain combinations of principal curvatures of the boundary do not vanish, a phenomenon observed by F.-H. Lin.

1. INTRODUCTION

Anderson [1], [2] studied complete area-minimizing submanifolds and proved that, for any given closed embedded $(n - 1)$ -dimensional submanifold N at the infinity of \mathbb{H}^{n+1} , there exists a complete area minimizing integral n -current which is asymptotic to N at infinity. Hardt and Lin [5] discussed the C^1 -boundary regularity of such hypersurfaces. Subsequently, Lin [8] studied the higher order boundary regularity for solutions to the Dirichlet problem for minimal graphs in the hyperbolic space.

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain. Lin [8] studied the Dirichlet problem of the form

$$(1.1) \quad \begin{aligned} \Delta f - \frac{f_i f_j}{1 + |\nabla f|^2} f_{ij} + \frac{n}{f} &= 0 && \text{in } \Omega, \\ f &= 0 && \text{on } \partial\Omega, \\ f &> 0 && \text{in } \Omega. \end{aligned}$$

We note that the equation in (1.1) is a quasilinear non-uniformly elliptic equation. It becomes singular on $\partial\Omega$ since $f = 0$ there. Lin [8] proved that (1.1) admits a unique solution $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ if $\Omega \subset \mathbb{R}^n$ is a C^2 -domain with a nonnegative boundary mean curvature $H_{\partial\Omega} \geq 0$ with respect to the inward normal direction of $\partial\Omega$. Moreover, the graph of f is a complete minimal hypersurface in the hyperbolic space \mathbb{H}^{n+1} with the asymptotic boundary $\partial\Omega$. Concerning the higher global regularity, Lin proved $f \in C^{1/2}(\bar{\Omega})$ if $H_{\partial\Omega} > 0$. He also expected certain relations between the Hölder exponents for f and the vanishing order of $H_{\partial\Omega}$ at boundary points. (See Remark 3.7 [8].)

The primary goal of this paper is to discuss the global regularity of the solution f of (1.1). We first discuss the optimal regularity of f in the general case $H_{\partial\Omega} \geq 0$ and prove $f \in C^{\frac{1}{n+1}}(\bar{\Omega})$. We can improve the Hölder regularity of the solution if certain combinations of principal curvatures of the boundary do not vanish and thus establish a

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relation between the Hölder exponents for f and principal curvatures of the boundary. We will also discuss the global regularity of the solution f for certain domains with singularity.

The first main result is given by the following theorem.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded C^2 -domain with $H_{\partial\Omega} \geq 0$ and that $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ is the solution of (1.1). Then, $f \in C^{\frac{1}{n+1}}(\bar{\Omega})$. Moreover,*

$$[f]_{C^{\frac{1}{n+1}}(\bar{\Omega})} \leq [(n+1) \text{diam}(\Omega)^n]^{\frac{1}{n+1}},$$

where $\text{diam}(\Omega)$ is the diameter of Ω .

We point out that the Hölder exponent $\frac{1}{n+1}$ is optimal. By Remark 2.3 below, we cannot improve the regularity for f in domains with nonnegative mean curvature in general. We also note that $n+1$ is the power of the first global term in the expansions of minimal graphs in the hyperbolic space. See [4]. However, if certain combinations of principal curvatures of the boundary do not vanish, then we can improve the global Hölder regularity. We will prove in Theorem 3.5 that solutions can be $C^{1/i}$ up to the boundary under appropriate conditions, for an even integer $2 \leq i \leq n$. This confirms what Lin suggested in Remark 3.7 [8].

The proof of Theorem 1.1 is based on the maximum principle and the recent work of Han and Jiang [4] on the boundary expansions for minimal graphs in the hyperbolic space.

We note that the estimate in Theorem 1.1 does not depend on the regularity of the domain. This allows us to discuss (1.1) in domains with singularity. Along this direction, we prove the following result.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain which is the intersection of finitely many bounded convex C^2 -domains Ω_i with $H_{\partial\Omega_i} > 0$, and let $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ be the solution of (1.1). Then $f \in C^{1/2}(\bar{\Omega})$, and*

$$[f]_{C^{1/2}(\bar{\Omega})} \leq C,$$

where C is a positive constant depending only on n , $H_{\partial\Omega_i}$ and the diameter of Ω_i .

The equation in (1.1) for $n = 2$ also appears in the study of the Chaplygin gas. See [9] for details.

The paper is organized as follows. In Section 2, we discuss (1.1) in domains with nonnegative boundary mean curvature and prove Theorem 1.1. In Section 3, we discuss (1.1) in convex domains and prove Theorem 1.2. In Section 4, we discuss the regularity of solutions of an equivalent form of the equation in (1.1), which appears in the study of the Chaplygin gas.

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2. GENERAL MEAN CONVEX DOMAINS

We first note that, for $\Omega = B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$, the unique solution of (1.1) is given by

$$f_R(x) = (R^2 - |x - x_0|^2)^{\frac{1}{2}}.$$

Hence, for a domain Ω with $H_{\partial\Omega} \geq 0$, we have, by the maximum principle,

$$|f|_{L^\infty(\Omega)} \leq \text{diam}(\Omega).$$

We also note that the gradient of f blows up near $\partial\Omega$.

Now we prove two lemmas. Throughout the proof, we denote by $d(x)$ the distance from x to $\partial\Omega$ and by Λ the maximum of the absolute value of principle curvatures of $\partial\Omega$.

Lemma 2.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded C^2 -domain with $H_{\partial\Omega} \geq 0$ and that $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ is the solution of (1.1). Then,*

$$f \leq Cd^{\frac{1}{n+1}} \quad \text{in } \Omega,$$

where C is a positive constant depending only on n , Λ and the diameter of Ω .

Proof. Set $w = \psi(d)$, for some function ψ to be determined. We will require $\psi > 0$ and $\psi' > 0$ on $(0, \delta)$, for some $\delta > 0$. Then,

$$\begin{aligned} w_i &= \psi' d_i, \\ w_{ij} &= \psi' d_{ij} + \psi'' d_i d_j, \end{aligned}$$

and hence,

$$w_i w_j w_{ij} = \psi'^2 d_i d_j (\psi' d_{ij} + \psi'' d_i d_j) = \psi'^2 \psi'',$$

by $d_i d_{ij} = 0$. Therefore,

$$\begin{aligned} \Delta w - \frac{w_i w_j}{1 + |\nabla w|^2} w_{ij} + \frac{n}{w} &= \psi' \Delta d + \psi'' - \frac{1}{1 + \psi'^2} \psi'' \psi'^2 + \frac{n}{\psi} \\ &= \frac{1}{1 + \psi'^2} \psi'' + \frac{n}{\psi} + \psi' \Delta d \\ &\leq \frac{1}{1 + \psi'^2} \psi'' + \frac{n}{\psi}, \end{aligned}$$

where we used the assumption $H_{\partial\Omega} \geq 0$ and the expansion of Δd as in [3]. Set

$$m(\psi) = \frac{\psi'' \psi}{1 + \psi'^2} + n.$$

Then,

$$\Delta w - \frac{w_i w_j}{1 + |\nabla w|^2} w_{ij} + \frac{n}{w} \leq \frac{1}{\psi} m(\psi).$$

In the following, we set

$$\psi(d) = A[d^p - d^q],$$

for some positive constants A , p and q to be determined. Then,

$$\begin{aligned}\psi' &= A[pd^{p-1} - qd^{q-1}], \\ \psi'' &= A[p(p-1)d^{p-2} - q(q-1)d^{q-2}].\end{aligned}$$

Hence,

$$1 + \psi'^2 = 1 + A^2[p^2d^{2p-2} - 2pqd^{q+p-2} + q^2d^{2q-2}],$$

and

$$\begin{aligned}\psi''\psi + n(1 + \psi'^2) &= n + A^2[p(p-1 + pn)d^{2p-2} \\ &\quad - (p^2 + q^2 - p - q + 2pqn)d^{q+p-2} + q(q-1 + nq)d^{2q-2}].\end{aligned}$$

In the following, we set

$$(2.1) \quad p = \frac{1}{n+1}.$$

Then, $p-1 + pn = 0$ and hence

$$\psi''\psi + n(1 + \psi'^2) = n + A^2[-h(q)d^{q+p-2} + q(q-1 + nq)d^{2q-2}],$$

where

$$h(q) = p^2 + q^2 - p - q + 2pqn.$$

Then,

$$m(\psi) = \frac{n + A^2[-h(q)d^{q+p-2} + q(q-1 + nq)d^{2q-2}]}{1 + A^2[p^2d^{2p-2} - 2pqd^{q+p-2} + q^2d^{2q-2}]}.$$

Note that $h(q)$ is a quadratic polynomial of q . With the expression of p in (2.1), we have

$$\begin{aligned}h(q) &= q^2 + (2pn - 1)q + p^2 - p = q^2 + (np - p)q - np^2 \\ &= (q - p)(q + np).\end{aligned}$$

Then, $h(q) > 0$ for $q > p$. A simple rearrangement yields

$$m(\psi) = d^{q-p} \cdot \frac{-A^2(q-p)(q+np) + nd^{2-p-q} + A^2q(q-1+nq)d^{q-p}}{A^2p^2 + d^{2-2p} - 2A^2pqd^{q-p} + A^2q^2d^{2q-2p}}.$$

Now, we take q such that

$$p < q < 2 - p.$$

Then, we can take δ small depending only on n and Λ such that $m(\psi) \leq 0$ for $d \in (0, \delta)$. Next, we choose A large, depending only on n , Λ and the $L^\infty(\Omega)$ -norm of f , such that

$$|f|_{L^\infty(\Omega)} \leq A(\delta^p - \delta^q) = \psi(\delta).$$

Hence, by the maximum principle, we have $f \leq \psi$ for $0 < d < \delta$, and therefore

$$f \leq Ad^{\frac{1}{n+1}} \quad \text{in } \{x \in \Omega : d(x) \leq \delta\}.$$

This implies the desired result. \square

Next, we proceed as Lin [8]. Locally near each boundary point, the graph of f can be represented by a function over its vertical tangent plane. Specifically, we fix a boundary point of Ω , say the origin, and assume that the vector $e_n = (0, \dots, 0, 1)$ is the interior normal vector to $\partial\Omega$ at the origin. Then, with $x = (x', x_n)$, the x' -hyperplane is the tangent plane of $\partial\Omega$ at the origin, and the boundary $\partial\Omega$ can be expressed in a neighborhood of the origin as a graph of a smooth function over $\mathbb{R}^{n-1} \times \{0\}$, say

$$x_n = \varphi(x').$$

We now denote points in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ by (x', x_n, t) . The vertical hyperplane given by $x_n = 0$ is the tangent plane to the graph of f at the origin in \mathbb{R}^{n+1} , and we can represent the graph of f as a graph of a new function u defined in terms of $(x', 0, t)$ for small x' and t , with $t > 0$. In other words, we treat $\mathbb{R}^n = \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$ as our new base space and write $u = u(x', t)$. Then, for some $R > 0$, u satisfies

$$(2.2) \quad \Delta u - \frac{u_i u_j}{1 + |\nabla u|^2} u_{ij} - \frac{nu_t}{t} = 0 \quad \text{in } B_R^+,$$

and

$$(2.3) \quad u(\cdot, 0) = \varphi \quad \text{on } B'_R.$$

We note that u and f are related by

$$(2.4) \quad x_n = u(x', t),$$

and

$$(2.5) \quad t = f(x', x_n).$$

Set

$$(2.6) \quad u_{n+1}(x', t) = \varphi(x') + \sum_{i=2}^{n+1} c_i(x') t^i + c_{n+1,1}(x') t^{n+1} \log t.$$

In fact, $c_i = 0$ for odd i between 2 and n and $c_{n+1,1} = 0$ for even n . We have the following result.

Lemma 2.2. *For some constant $\alpha \in (0, 1)$, let $\varphi \in C^{n+1, \alpha}(B'_R)$ be a given function and $u \in C(\bar{B}_R^+) \cap C^\infty(B_R^+)$ be a solution of (2.2)-(2.3). Then, there exist functions $c_i \in C^{n+1-i, \epsilon}(B'_R)$, for $i = 0, 2, 4, \dots, n+1$, $c_{n+1,1} \in C^\epsilon(B'_R)$, and any $\epsilon \in (0, \alpha)$, such that, for u_{n+1} defined as in (2.6), for any $m = 0, 1, \dots, n+1$, any $\epsilon \in (0, \alpha)$, and any $r \in (0, R)$,*

$$(2.7) \quad \partial_t^m (u - u_{n+1}) \in C^\epsilon(\bar{B}_r^+),$$

and, for any $(x', t) \in B_{R/2}^+$,

$$(2.8) \quad |\partial_t^m (u - u_{n+1})(x', t)| \leq C t^{n+1-m+\alpha},$$

for some positive constant C depending only on n, α, R , the L^∞ -norm of u in B_R^+ and the $C^{n+1, \alpha}$ -norm of φ in B'_R .

Lemma 2.2 follows from Theorem 1.1 in [4] by taking $\ell = k = n + 1$. In fact, c_2, \dots, c_n and $c_{n+1,1}$ are coefficients for local terms and have explicit expressions in terms of φ . Meanwhile, c_{n+1} is the coefficient of the first nonlocal term.

Remark 2.3. The growth rate $d^{\frac{1}{n+1}}$ in Lemma 2.1 is optimal for f in general domains with nonnegative boundary mean curvature. This can be seen as follows. Consider a C^2 -domain Ω with $H_{\partial\Omega} \geq 0$ such that $B_r \cap \partial\Omega \subset \{x_n = 0\}$ and $B_r \cap \Omega = B_r^+$, for some $r > 0$. Then, $\varphi = 0$ on B_r' . Hence, in the expression of u_{n+1} , we have $c_i = 0$ for $i \leq n$ and $c_{n+1,1} = 0$. The estimate (2.8) with $m = 0$ implies $|u| \leq Ct^{n+1}$ near the origin. Moreover, $x_n = u \geq 0$ in $B_r \cap \Omega$. Hence, $C^{-1}d^{\frac{1}{n+1}} \leq f$ near the origin. By combining with Lemma 2.1, we obtain

$$C^{-1}d^{\frac{1}{n+1}} \leq f \leq Cd^{\frac{1}{n+1}} \quad \text{near the origin.}$$

Therefore, the growth rate of f is exactly $d^{\frac{1}{n+1}}$.

The next result plays a crucial role in the proof of Theorem 1.1.

Lemma 2.4. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded C^2 -domain with $H_{\partial\Omega} \geq 0$ and that $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ is the solution of (1.1). Then,*

$$f^n \sqrt{1 + |\nabla f|^2} \leq |f|_{L^\infty(\Omega)}^n \quad \text{in } \Omega.$$

Proof. We will prove, for any $\varepsilon > 0$,

$$(2.9) \quad f^{n+\varepsilon} \sqrt{1 + |\nabla f|^2} \leq |f|_{L^\infty(\Omega)}^{n+\varepsilon} \quad \text{in } \Omega.$$

By letting $\varepsilon \rightarrow 0$, we have the desired result.

We first consider the case that $\partial\Omega$ is smooth and set

$$F_\varepsilon = f^{n+\varepsilon} \sqrt{1 + |\nabla f|^2}.$$

The proof consists of two steps.

Step 1. We will prove $F_\varepsilon \rightarrow 0$ as x approaches $\partial\Omega$. Take any $x \in \Omega$. Without loss of generality, take a coordinate such that $x = (0, x_n)$, with $x_n = d(x)$, and the origin is the nearest point on $\partial\Omega$ to x . We can express x_n by a function $x_n = u(x', t)$ as in (2.4), which satisfies (2.2).

By Lemma 2.1, we have, for $x_n \in (0, \delta)$,

$$f(0, x_n) \leq Ax_n^{\frac{1}{n+1}}.$$

Hence,

$$(2.10) \quad u(0, t) \geq Ct^{n+1},$$

for some positive constant C . Therefore, x_n approaching 0 is equivalent to t approaching 0.

Next, we note $\varphi(0) = 0$ since $\{x_n = 0\}$ is tangent to $\partial\Omega$ at the origin. By (2.10) and (2.8) with $m = 0$, there is a nonzero term in the expression of $u_{n+1}(0, t)$ in (2.6). We

now write, for some $\alpha \in (0, 1)$,

$$u(0, t) = \sum_{i=2}^n c_i(0)t^i + c_{n+1,1}(0)t^{n+1} \log t + c_{n+1}(0)t^{n+1} + O(t^{n+1+\alpha}).$$

By $u(0, t) > 0$ for small $t > 0$, we note that either the first nonzero coefficient $c_i(0)$ is positive, for some $i = 0, 1, \dots, n+1$, or $c_{n+1,1}(0) < 0$ if $c_i(0) = 0$ for any $i = 2, \dots, n$. Next,

$$\begin{aligned} tu_t(0, t) &= \sum_{i=2}^n i c_i(0)t^i + (n+1)c_{n+1,1}(0)t^{n+1} \log t \\ &\quad + [c_{n+1,1}(0) + (n+1)c_{n+1}(0)]t^{n+1} + O(t^{n+1+\alpha}). \end{aligned}$$

Therefore, we have, for t small,

$$\frac{tu_t(0, t)}{u(0, t)} > 1.$$

By (2.4) and (2.5), we get

$$1 = u_t f_{x_n}, \quad 0 = u_{x'} + u_t f_{x'}.$$

Hence,

$$|u_t \nabla_x f|^2 = 1 + |\nabla_{x'} u|^2.$$

As a result, we obtain

$$F_\varepsilon \leq f^{n+\varepsilon} + f^{n+\varepsilon} \frac{\sqrt{n}}{u_t} \leq f^{n+\varepsilon} + C(n) f^\varepsilon,$$

where we used

$$f^{n+\varepsilon} \frac{1}{u_t} < \frac{f^{n+\varepsilon} t}{u} = \frac{f^{n+1}}{d} f^\varepsilon \leq C f^\varepsilon.$$

Hence, $F_\varepsilon \rightarrow 0$, as x_n or t approaches 0. We point out that it is important to have the extra power of ε .

Step 2. By Step 1, we note that F_ε attains its maximum at some x_0 in Ω . We will prove $\nabla f(x_0) = 0$ by contradiction. Without loss of generality, we assume $|\nabla f(x_0)| = f_1(x_0) \neq 0$. Set

$$g_\varepsilon = \log F_\varepsilon = \log(f^{n+\varepsilon} \sqrt{1 + |\nabla f|^2}).$$

Then, g_ε attains its maximum at x_0 . Hence, $g_{\varepsilon,i}(x_0) = 0$ and $(g_{\varepsilon,ij}(x_0)) \leq 0$. A simple calculation yields

$$g_{\varepsilon,i} = (n + \varepsilon) \frac{f_i}{f} + \frac{f_k f_{ki}}{1 + |\nabla f|^2},$$

and

$$g_{\varepsilon,ij} = (n + \varepsilon) \frac{f_{ij}}{f} - (n + \varepsilon) \frac{f_i f_j}{f^2} + \frac{f_{ki} f_{kj}}{1 + |\nabla f|^2} + \frac{f_{kij} f_k}{1 + |\nabla f|^2} - \frac{2f_k f_l f_{ki} f_{lj}}{(1 + |\nabla f|^2)^2}.$$

In the following, we calculate at the point x_0 . By $g_{\varepsilon,i} = 0$, $f_2 = \cdots = f_n = 0$ and $f_1 \neq 0$, we have

$$\begin{aligned} f_{11} &= -\frac{n+\varepsilon}{f}(1+f_1^2), \\ f_{1i} &= 0 \quad (i \neq 1). \end{aligned}$$

Set

$$a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}.$$

Then,

$$a_{ij}(\nabla f) f_{ij} = -\frac{n}{f}.$$

A simple differentiation yields

$$(2.11) \quad a_{ij} f_{1ij} + a_{ij,p_k} f_{ij} f_{1k} = \frac{n f_1}{f^2}.$$

By the assumption $|\nabla f| = f_1$ at x_0 , we have

$$a_{11} = \frac{1}{1+f_1^2}, \quad a_{ii} = 1 \quad (i \neq 1), \quad a_{ij} = 0 \quad (i \neq j).$$

Moreover, a straightforward calculation yields

$$\begin{aligned} a_{11,p_1} &= -\frac{2f_1}{(1+f_1^2)^2}, \quad a_{11,p_i} = 0 \quad (i \neq 1), \\ a_{1i,p_i} &= -\frac{f_1}{1+f_1^2} \quad (i \neq 1), \quad a_{1i,p_j} = 0 \quad (i \neq 1, i \neq j), \\ a_{ij,p_k} &= 0 \quad (i \neq 1, j \neq 1). \end{aligned}$$

Then, we can rewrite (2.11) as

$$a_{ii} f_{1ii} + a_{11,p_1} f_{11}^2 + 2 \sum_{i \geq 2} a_{1i,p_i} f_{1i}^2 = \frac{n f_1}{f^2},$$

or

$$a_{ii} f_{1ii} = \frac{2f_1}{1+f_1^2} \sum_{i \geq 1} a_{ii} f_{1i}^2 + \frac{n f_1}{f^2}.$$

If $f_1 \neq 0$, we have, by a simple substitution,

$$a_{ij} g_{\varepsilon,ij} = -\frac{n(n+\varepsilon)}{f^2} - (n+\varepsilon) \frac{a_{11} f_1^2}{f^2} + \frac{a_{ii} f_{ki}^2}{1+f_1^2} + \frac{a_{ii} f_{1ii} f_1}{1+f_1^2} - \frac{2a_{ii} f_1^2 f_{1i}^2}{(1+f_1^2)^2},$$

and, keeping only $k=1$ in the middle term,

$$\begin{aligned} a_{ij} g_{\varepsilon,ij} &\geq -\frac{n(n+\varepsilon)}{f^2} - (n+\varepsilon) \frac{a_{11} f_1^2}{f^2} + \frac{a_{11} f_{11}^2}{1+f_1^2} \\ &\quad + \frac{f_1}{1+f_1^2} \left(\frac{2f_1}{1+f_1^2} a_{11} f_{11}^2 + n \frac{f_1}{f^2} \right) - \frac{2a_{11} f_1^2 f_{11}^2}{(1+f_1^2)^2}. \end{aligned}$$

Hence,

$$a_{ij}g_{\varepsilon,ij} \geq \frac{\varepsilon}{f^2} \left(n + \varepsilon - \frac{f_1^2}{1 + f_1^2} \right) > 0.$$

On the other hand, since g_ε attains its maximum at x_0 , we have $a_{ij}g_{\varepsilon,ij} \leq 0$, which leads to a contradiction. Therefore, $\nabla f(x_0) = 0$, and by the definition of F_ε , we have

$$F_\varepsilon \leq |f|_{L^\infty(\Omega)}^{n+\varepsilon}.$$

This implies (2.9) in the case that $\partial\Omega$ is smooth.

We now consider the general case that $\partial\Omega$ is C^2 with $H_{\partial\Omega} \geq 0$. We can take a sequence of smooth domains $\{\Omega_k\}$ with $H_{\partial\Omega_k} \geq 0$ such that $\partial\Omega_k$ approaches $\partial\Omega$ in C^2 . Let $f_k \in C(\bar{\Omega}_k) \cap C^\infty(\Omega_k)$ be the solution of

$$\begin{aligned} \Delta f_k - \frac{f_{k,i}f_{k,j}}{1 + |\nabla f_k|^2} f_{k,ij} + \frac{n}{f_k} &= 0 && \text{in } \Omega_k, \\ f_k &= 0 && \text{on } \partial\Omega_k, \\ f_k &> 0 && \text{in } \Omega_k. \end{aligned}$$

By what we just proved, we have

$$f_k^{n+\varepsilon} \sqrt{1 + |\nabla f_k|^2} \leq |f_k|_{L^\infty(\Omega_k)}^{n+\varepsilon}.$$

By the interior estimate and Lemma 2.1, we have $f_k(x) \rightarrow f(x)$ and $\nabla f_k(x) \rightarrow \nabla f(x)$ for any $x \in \Omega$. Hence, by taking the limit, we obtain

$$f^{n+\varepsilon} \sqrt{1 + |\nabla f|^2} \leq |f|_{L^\infty(\Omega)}^{n+\varepsilon}.$$

This is (2.9) in the general case. \square

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.4, we have

$$f^n \sqrt{1 + |\nabla f|^2} \leq |f|_{L^\infty(\Omega)}^n.$$

Hence,

$$(2.12) \quad |\nabla f^{n+1}| < (n+1) \text{diam}(\Omega)^n.$$

By the mean value theorem, we obtain, for any $x \in \Omega$,

$$f^{n+1}(x) = |f^{n+1}(x) - 0| \leq |\nabla f^{n+1}| d(x),$$

and hence

$$(2.13) \quad f(x) \leq [(n+1) \text{diam}(\Omega)^n]^{\frac{1}{n+1}} d(x)^{\frac{1}{n+1}}.$$

We point out that (2.13) is sharper than Lemma 2.1.

Next, we note, for any $x_1, x_2 \in \Omega$,

$$(2.14) \quad |f(x_1) - f(x_2)|^{n+1} \leq |f(x_1)^{n+1} - f(x_2)^{n+1}|.$$

In fact, for $f(x_1), f(x_2) > 0$, we may assume $f(x_1) = \max\{f(x_1), f(x_2)\}$, and then employ $|1 - y|^{n+1} \leq |1 - y^{n+1}|$, with $y = \frac{f(x_2)}{f(x_1)}$, to derive (2.14).

Now, we claim, for any $x_1, x_2 \in \Omega$,

$$(2.15) \quad |f(x_1) - f(x_2)| \leq [(n+1) \text{diam}(\Omega)^n]^{\frac{1}{n+1}} |x_1 - x_2|^{\frac{1}{n+1}}.$$

With $d_i = \text{dist}(x_i, \partial\Omega)$ for $i = 1, 2$, we assume $d_1 \geq d_2$.

If $|x_1 - x_2| \geq d_1$, by (2.13), we have

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq |f(x_1)| \leq [(n+1) \text{diam}(\Omega)^n]^{\frac{1}{n+1}} d_1^{\frac{1}{n+1}} \\ &\leq [(n+1) \text{diam}(\Omega)^n]^{\frac{1}{n+1}} |x_1 - x_2|^{\frac{1}{n+1}}. \end{aligned}$$

If $|x_1 - x_2| < d_1$, by (2.14) and (2.12), we have,

$$\begin{aligned} |f(x_1) - f(x_2)|^{n+1} &\leq |f(x_1)^{n+1} - f(x_2)^{n+1}| \leq |\nabla f(\tilde{x})|^{n+1} |x_1 - x_2| \\ &\leq (n+1) \text{diam}(\Omega)^n |x_1 - x_2|, \end{aligned}$$

where \tilde{x} is some point in $B_{d_1}(x_1) \subset \Omega$. In summary, we have (2.15). \square

3. CONVEX DOMAINS

In this section, we discuss refined regularity for f in general convex domains. First, we prove that (1.1) admits a solution in convex domains. We point out that there is no higher regularity assumptions on the boundary of the domains.

Theorem 3.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain. Then, (1.1) admits a unique solution $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ and f is concave. Moreover, $f \in C^{\frac{1}{n+1}}(\bar{\Omega})$ and*

$$[f]_{C^{\frac{1}{n+1}}(\bar{\Omega})} \leq [(n+1) \text{diam}(\Omega)^n]^{\frac{1}{n+1}},$$

where $\text{diam}(\Omega)$ is the diameter of Ω .

Proof. We first prove the existence and note that the uniqueness is a simple consequence of the maximum principle.

We take a sequence of bounded smooth convex domains $\{\Omega_k\}$ such that $\partial\Omega_k$ approaches $\partial\Omega$ in the Hausdorff metric. Let $f_k \in C(\bar{\Omega}_k) \cap C^\infty(\Omega_k)$ be the solution of

$$\begin{aligned} \Delta f_k - \frac{f_{k,i} f_{k,j}}{1 + |\nabla f_k|^2} f_{k,ij} + \frac{n}{f_k} &= 0 \quad \text{in } \Omega_k, \\ f_k &= 0 \quad \text{on } \partial\Omega_k, \\ f_k &> 0 \quad \text{in } \Omega_k. \end{aligned}$$

By (2.13) and the interior estimate, we have, for any $m \geq 1$ and any $\Omega' \subset\subset \Omega$,

$$f_k \rightarrow f \quad \text{in } C^m(\Omega'),$$

for some function $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ with $f = 0$ on $\partial\Omega$. Therefore, f is the unique solution of (1.1). Next, we apply Theorems 3.1 and 3.2 [6] in Ω_k and conclude that f_k are concave. Hence, f is concave in Ω .

For the global regularity, we take any $x_1, x_2 \in \Omega$. Then, $x_1, x_2 \in \Omega_k$ for k large, and hence

$$|f_k(x_1) - f_k(x_2)| \leq [(n+1) \text{diam}(\Omega_k)^n]^{\frac{1}{n+1}} |x_1 - x_2|^{\frac{1}{n+1}}.$$

By letting $k \rightarrow \infty$, we get

$$|f(x_1) - f(x_2)| \leq [(n+1) \text{diam}(\Omega)^n]^{\frac{1}{n+1}} |x_1 - x_2|^{\frac{1}{n+1}}.$$

This implies the desired result on the Hölder semi-norm of f . \square

The Hölder exponent $\frac{1}{n+1}$ is optimal. By Remark 2.3, we cannot improve the regularity for f in general convex domains.

We next consider the local regularity for f . We write the equation (1.1) in its divergence form

$$\nabla \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} + \frac{n}{f \sqrt{1 + |\nabla f|^2}} = 0.$$

Then, we have, for any $\varphi \in C_0^\infty(\Omega)$,

$$(3.1) \quad \int_{\Omega} \frac{\nabla f \nabla \varphi}{\sqrt{1 + |\nabla f|^2}} - \int_{\Omega} \frac{n\varphi}{f \sqrt{1 + |\nabla f|^2}} = 0.$$

We now prove a local regularity for solutions of (1.1). We point out that there is no regularity assumption on the domain. Hence, it may be applied to domains with singularity.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ be a solution of (1.1). For some $x_0 \in \partial\Omega$ and $\alpha \in (0, 1)$, assume*

$$f \leq Md^\alpha, \quad \Delta f \leq 0 \quad \text{in } \Omega \cap B_r(x_0),$$

for some constants $M \geq 1$ and $r > 0$. Then, $f \in C^\alpha(\bar{\Omega} \cap B_{r/2}(x_0))$, and

$$(3.2) \quad [f]_{C^\alpha(\bar{\Omega} \cap B_{r/2}(x_0))} \leq CM,$$

where C is a positive constant depending only on n and α . If, in addition, f is concave in $\Omega \cap B_r(x_0)$ and $\alpha \in (0, 1/2]$, then

$$(3.3) \quad |\nabla f^\frac{1}{\alpha}|_{L^\infty(\Omega \cap B_{r/2}(x_0))} \leq CM^\frac{1}{\alpha},$$

where C depends on n and α .

Proof. Step 1. We will prove, for any $x \in \Omega \cap B_{r/2}(x_0)$,

$$(3.4) \quad [f]_{C^\alpha(B_{d_x/2}(x))} \leq CM,$$

where C is a positive constant depending only on n and α . Then, we have (3.2) by combining with $f \leq Md^\alpha$ in $\Omega \cap B_r(x_0)$. Here and hereafter, we write $d_x = d(x)$.

We now fix a point $x \in \Omega \cap B_{r/2}(x_0)$ and a cutoff function $\psi \in C_0^\infty(B_{d_x}(x))$, with $0 \leq \psi \leq 1$ and $|\nabla \psi| \leq Cd_x^{-1}$, for some positive constant C depending only on n . In addition, we assume $\psi = 1$ in $B_{d_x/2}(x)$.

We note $d \leq 2d_x$ in $B_{d_x}(x)$. To verify this, we take $\tilde{x} \in \partial B_{d_x}(x) \cap \partial\Omega$. Then, for any $y \in B_{d_x}(x)$, $d(y) \leq |y - \tilde{x}| \leq |y - x| + |x - \tilde{x}| \leq 2d_x$. This implies

$$f \leq 2Md_x^\alpha \quad \text{in } B_{d_x}(x).$$

Next, we note $f \geq d$. This follows from a simple comparison of f and the corresponding solution in $B_{d(y)}(y)$, for any $y \in \Omega$. Then, for any $y \in B_{d_x}(x)$, we have, for some $y_0 \in \partial B_{d_x}(x)$,

$$f(y) \geq d(y) \geq |y - y_0|,$$

and, for some \tilde{y} between y and y_0 ,

$$\psi(y) = \psi(y) - \psi(y_0) \leq |\nabla \psi(\tilde{y})| |y - y_0| \leq C d_x^{-1} |y - y_0|.$$

Hence,

$$(3.5) \quad \frac{\psi}{f} \leq C d_x^{-1} \quad \text{in } B_d(x).$$

Taking $\varphi = f\psi$ in (3.1), we have

$$\int \psi \sqrt{1 + |\nabla f|^2} = \int \frac{(n+1)\psi}{\sqrt{1 + |\nabla f|^2}} - \frac{f \nabla f \nabla \psi}{\sqrt{1 + |\nabla f|^2}} \leq (n+1) \int \psi + \int f |\nabla \psi|.$$

Then,

$$(3.6) \quad \int \psi \sqrt{1 + |\nabla f|^2} \leq C M d_x^{\alpha-1+n}.$$

For an integer $k \geq 2$, take $\varphi = \psi^k (1 + |\nabla f|^2)^{\frac{k-1}{2}}$ in (3.1). Then,

$$\begin{aligned} & \int k \psi^{k-1} (1 + |\nabla f|^2)^{\frac{k-2}{2}} \nabla f \nabla \psi - \int \frac{n}{f} \psi^k (1 + |\nabla f|^2)^{\frac{k-2}{2}} \\ & + (k-1) \int \psi^k (1 + |\nabla f|^2)^{\frac{k-2}{2}} \frac{f_i f_j}{1 + |\nabla f|^2} f_{ij} = 0. \end{aligned}$$

Note

$$\frac{f_i f_j}{1 + |\nabla f|^2} f_{ij} = \Delta f + \frac{n}{f}.$$

A simple substitution yields

$$\begin{aligned} & \int k \psi^{k-1} (1 + |\nabla f|^2)^{\frac{k-2}{2}} \nabla f \nabla \psi + n(k-2) \int \frac{1}{f} \psi^k (1 + |\nabla f|^2)^{\frac{k-2}{2}} \\ & + (k-1) \int \psi^k (1 + |\nabla f|^2)^{\frac{k-2}{2}} \Delta f = 0. \end{aligned}$$

By $\Delta f \leq 0$ and (3.5), we obtain

$$\begin{aligned} & \int \psi^k (1 + |\nabla f|^2)^{\frac{k-2}{2}} |\Delta f| \\ & \leq C d_x^{-1} \left\{ \psi^{k-1} (1 + |\nabla f|^2)^{\frac{k-1}{2}} + (k-2) \int \psi^{k-1} (1 + |\nabla f|^2)^{\frac{k-2}{2}} \right\}. \end{aligned}$$

We include the factor $k-2$ to emphasize that the corresponding term disappears if $k = 2$. Then,

$$(3.7) \quad \begin{aligned} & \int \psi^k f (1 + |\nabla f|^2)^{\frac{k-2}{2}} |\Delta f| \\ & \leq CM d_x^{\alpha-1} \left\{ \int \psi^{k-1} (1 + |\nabla f|^2)^{\frac{k-1}{2}} + (k-2) \int \psi^{k-2} (1 + |\nabla f|^2)^{\frac{k-2}{2}} \right\}. \end{aligned}$$

Next, take $\varphi = \psi^k f (1 + |\nabla f|^2)^{\frac{k-1}{2}}$ in (3.1). A similar calculation yields

$$\begin{aligned} & \int k \psi^{k-1} f (1 + |\nabla f|^2)^{\frac{k-2}{2}} \nabla f \nabla \psi + n(k-2) \int \psi^k (1 + |\nabla f|^2)^{\frac{k-2}{2}} \\ & + \int \psi^k (1 + |\nabla f|^2)^{\frac{k-2}{2}} |\nabla f|^2 + (k-1) \int \psi^k f (1 + |\nabla f|^2)^{\frac{k-2}{2}} \Delta f = 0. \end{aligned}$$

Combining with (3.7), we obtain

$$\begin{aligned} & \int \psi^k (1 + |\nabla f|^2)^{\frac{k-2}{2}} |\nabla f|^2 \\ & \leq CM d_x^{\alpha-1} \left\{ \int \psi^{k-1} (1 + |\nabla f|^2)^{\frac{k-1}{2}} + (k-2) \int \psi^{k-2} (1 + |\nabla f|^2)^{\frac{k-2}{2}} \right\}, \end{aligned}$$

and hence

$$\begin{aligned} & \int \psi^k (1 + |\nabla f|^2)^{\frac{k}{2}} \\ & \leq CM d_x^{\alpha-1} \left\{ \int \psi^{k-1} (1 + |\nabla f|^2)^{\frac{k-1}{2}} + (k-2) \int \psi^{k-2} (1 + |\nabla f|^2)^{\frac{k-2}{2}} \right\}. \end{aligned}$$

With the help of (3.6), a simple iteration yields

$$\int \psi^k (1 + |\nabla f|^2)^{\frac{k}{2}} \leq CM^k d_x^{k\alpha-k+n},$$

and hence

$$\int \psi^k |\nabla f|^k \leq CM^k d_x^{k\alpha-k+n}.$$

Therefore, for any integer $k \geq 1$,

$$\int_{B_{d_x}(x)} |\nabla(\psi f)|^k \leq CM^k d_x^{k\alpha-k+n},$$

where C is a positive constant depending only on n , k and α .

Next, take any $p \geq 1$. If p is not an integer, by fixing some integer $k > p$, we have

$$\begin{aligned} \int_{B_{d_x}(x)} |\nabla(\psi f)|^p & \leq \left(\int_{B_{d_x}(x)} 1 \right)^{1-\frac{p}{k}} \left(\int_{B_{d_x}(x)} (|\nabla(\psi f)|^p)^{\frac{k}{p}} \right)^{\frac{p}{k}} \\ & \leq CM^p d_x^{n(1-\frac{p}{k}) + [k\alpha-k+n]\frac{p}{k}} = CM^p d_x^{p\alpha-p+n}. \end{aligned}$$

Therefore, for any $p \geq 1$,

$$(3.8) \quad \|\nabla(\psi f)\|_{L^p(B_{d_x}(x))} \leq CM d_x^{\alpha-1+\frac{n}{p}},$$

where C is a positive constant depending only on n , p and α .

If $\alpha \in (0, 1)$, we take $p = \frac{n}{1-\alpha}$ so that $\alpha = 1 - \frac{n}{p}$. Then,

$$\|\nabla(\psi f)\|_{L^p(B_{d_x}(x))} \leq CM.$$

Hence, (3.4) follows from the Sobolev embedding.

Step 2. We now prove (3.3). We first claim

$$(3.9) \quad |\Delta f| \leq \frac{n}{f}(1 + |\nabla f|^2) \quad \text{in } \Omega \cap B_r(x_0).$$

We fix an $x \in \Omega \cap B_r(x_0)$. Since Δf and $|\nabla f|$ are invariant under orthogonal transformations, we assume $|\nabla f| = f_1$ at x by a rotation. Then the equation in (1.1) reduces to

$$\Delta f - \frac{f_1^2}{1 + f_1^2} f_{11} + \frac{n}{f} = 0.$$

Therefore, at x ,

$$\frac{1}{1 + f_1^2} f_{11} + f_{22} + \cdots + f_{nn} + \frac{n}{f} = 0.$$

Since $f_{ii} \leq 0$, for $i = 1, \dots, n$, we have

$$\frac{1}{1 + f_1^2} \Delta f + \frac{n}{f} \geq 0.$$

This implies (3.9) by $\Delta f \leq 0$.

Next,

$$\Delta(f^{\frac{1}{\alpha}}) = \frac{1}{\alpha} f^{\frac{1}{\alpha}-1} \Delta f + \frac{1}{\alpha} \left(\frac{1}{\alpha} - 1 \right) f^{\frac{1}{\alpha}-2} |\nabla f|^2.$$

By $f \leq Md^\alpha$, we have

$$|\Delta(f^{\frac{1}{\alpha}})| \leq \frac{n}{\alpha^2} f^{\frac{1}{\alpha}-2} (1 + |\nabla f|^2) \quad \text{in } \Omega \cap B_r(x_0).$$

Fix an $x \in \Omega \cap B_{r/2}(x_0)$ and a $p > n$. By applying the $W^{2,p}$ -estimate in $B_{d_x/2}(x)$ and (3.8), we get

$$\begin{aligned} d_x |\nabla(f^{\frac{1}{\alpha}})(x)| &\leq C \left\{ \frac{n}{\alpha^2} d_x^{2-\frac{n}{p}} \|f^{\frac{1}{\alpha}-2} (1 + |\nabla f|^2)\|_{L^p(B_{d_x/2}(x))} + \|f^{\frac{1}{\alpha}}\|_{L^\infty(B_{d_x/2}(x))} \right\} \\ &\leq CM^{\frac{1}{\alpha}} \left\{ d_x^{2-\frac{n}{p}} d_x^{\alpha(\frac{1}{\alpha}-2)} d_x^{2\alpha+\frac{n}{p}-2} + d_x \right\} \leq CM^{\frac{1}{\alpha}} d_x, \end{aligned}$$

where we used the fact $\alpha \leq 1/2$. Hence, for any $x \in \Omega \cap B_{r/2}(x_0)$,

$$|\nabla(f^{\frac{1}{\alpha}})(x)| \leq CM^{\frac{1}{\alpha}}.$$

This is (3.3). □

We now prove a result concerning the local growth.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ be a solution of (1.1). Suppose $\partial\Omega$ is C^2 and $H_{\partial\Omega} > 0$ near $x_0 \in \partial\Omega$. Then,*

$$f \leq Cd^{\frac{1}{2}} \quad \text{in } \Omega \cap B_r(x_0),$$

where r and C are positive constants depending only on n , the geometry of $\partial\Omega$ near x_0 and the diameter of Ω .

Proof. Without loss of generality, we assume x_0 is the origin and the x_n -direction is the interior normal to $\partial\Omega$. Furthermore, we assume $\Delta d \leq -c_0$ in $\Omega \cap B_R$, for some positive c_0 and R . Set, for some $r < R^2/4$,

$$G_r = \{(x', x_n) \in \Omega : x' \in B'_{\sqrt{r}}, 0 < d < r\},$$

and, for some $\alpha \in (0, 1)$,

$$w = Ad^\alpha + B|x'|^2,$$

where A and B are constants such that

$$(3.10) \quad A \geq \frac{\tau}{r^\alpha} |f|_{L^\infty}, \quad B = \frac{1}{r} |f|_{L^\infty},$$

for some large constant $\tau \geq 1$ to be determined. Then $f \leq w$ on ∂G_r .

A straightforward calculation yields

$$\begin{aligned} w_a &= A\alpha d^{\alpha-1} d_a + 2Bx_a, \\ w_n &= A\alpha d^{\alpha-1} d_n, \end{aligned}$$

and

$$\begin{aligned} w_{ab} &= A\alpha(\alpha-1)d^{\alpha-2} d_a d_b + A\alpha d^{\alpha-1} d_{ab} + 2B\delta_{ab}, \\ w_{an} &= A\alpha(\alpha-1)d^{\alpha-2} d_n d_a + A\alpha d^{\alpha-1} d_{na}, \\ w_{nn} &= A\alpha(\alpha-1)d^{\alpha-2} d_n^2 + A\alpha d^{\alpha-1} d_{nn}. \end{aligned}$$

Then,

$$\begin{aligned} |\nabla w|^2 &= A^2 \alpha^2 d^{2\alpha-2} + 4AB\alpha d^{\alpha-1} x' \cdot \nabla_{x'} d + 4B^2 |x'|^2, \\ \Delta w &= A\alpha(\alpha-1)d^{\alpha-2} + A\alpha d^{\alpha-1} \Delta d + 2(n-1)B, \end{aligned}$$

and

$$\begin{aligned} w_i w_j w_{ij} &= A^3 \alpha^3 (\alpha-1) d^{3\alpha-4} + 4A^2 B \alpha^2 (\alpha-1) d^{2\alpha-3} x' \cdot \nabla_{x'} d \\ &\quad + 2A^2 B \alpha^2 d^{2\alpha-2} |\nabla_{x'} d|^2 + 4AB^2 \alpha (\alpha-1) d^{\alpha-2} (x' \cdot \nabla_{x'} d)^2 \\ &\quad + 4AB^2 \alpha d^{\alpha-1} [2x' \cdot \nabla_{x'} d + x_a x_b d_{ab}] + 8B^3 |x'|^2. \end{aligned}$$

By taking $\alpha \in (0, 1)$ and a straightforward calculation, we obtain

$$\begin{aligned} &A\alpha(\alpha-1)d^{\alpha-2} - \frac{w_i w_j}{1 + |\nabla w|^2} w_{ij} \\ &\leq \frac{4AB^2 \alpha d^{\alpha-1} [2|x' \cdot \nabla_{x'} d| + |x_a x_b d_{ab}|]}{1 + A^2 \alpha^2 d^{2\alpha-2} + 4AB\alpha d^{\alpha-1} x' \cdot \nabla_{x'} d + 4B^2 |x'|^2}. \end{aligned}$$

In fact, the numerator in the left-hand side is given by

$$\begin{aligned} & A\alpha(\alpha-1)d^{\alpha-2} + 4AB^2\alpha(\alpha-1)d^{\alpha-2}[|x'|^2 - (x' \cdot \nabla_{x'}d)^2] \\ & - 2A^2B\alpha^2d^{2\alpha-2}|\nabla_{x'}d|^2 - 8B^3|x'|^2 \\ & - 4AB^2\alpha d^{\alpha-1}[2x' \cdot \nabla_{x'}d + x_ax_b d_{ab}]. \end{aligned}$$

The first four terms are nonpositive. By dropping some positive terms in the denominator and some rearrangements, we get

$$\begin{aligned} \Delta w - \frac{w_i w_j}{1 + |\nabla w|^2} w_{ij} + \frac{n}{w} &\leq Ad^{\alpha-1} \left\{ \alpha \Delta d + 2(n-1)A^{-1}Bd^{1-\alpha} + \frac{n}{A^2}d^{1-2\alpha} \right. \\ &\quad \left. + \frac{4(A^{-1}Bd^{1-\alpha})^2[2|x' \cdot \nabla_{x'}d| + |x_ax_b d_{ab}|]}{\alpha - 4A^{-1}Bd^{1-\alpha}|x' \cdot \nabla_{x'}d|} \right\}. \end{aligned}$$

By the choice of A and B in (3.10) and $d < r$, we have

$$\begin{aligned} \Delta w - \frac{w_i w_j}{1 + |\nabla w|^2} w_{ij} + \frac{n}{w} &\leq Ad^{\alpha-1} \left\{ \alpha \Delta d + \frac{2(n-1)}{\tau} + \frac{n}{\tau^2} r^{2\alpha} |f|_{L^\infty}^{-2} d^{1-2\alpha} \right. \\ &\quad \left. + \frac{4[2|x' \cdot \nabla_{x'}d| + |x_ax_b d_{ab}|]}{\tau(\alpha\tau - 4|x' \cdot \nabla_{x'}d|)} \right\}. \end{aligned}$$

Note $\Delta d \leq -c_0$ in G_r . We take $\alpha = 1/2$. By taking τ large, we obtain

$$\Delta w - \frac{w_i w_j}{1 + |\nabla w|^2} w_{ij} + \frac{n}{w} \leq 0 \quad \text{in } G_r.$$

We can apply the maximum principle and obtain

$$f \leq Ad^\alpha + B|x'|^2 \quad \text{in } G_r.$$

By taking $x' = 0$, we conclude $f(0, x_n) \leq Ad^{1/2}$ for any $x_n \in (0, d)$.

In general, we consider

$$w = Ad^\alpha + B|x' - x'_0|^2,$$

and conclude $f(x'_0, x_n) \leq Ad^{1/2}$ for any $x_n \in (0, d)$. \square

We now prove a regularity result in convex domains with singularity.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain such that, in a neighborhood of $x_0 \in \partial\Omega$, $\partial\Omega$ consists of finitely many C^2 -hypersurfaces S_i intersecting at x_0 with $H_{S_i} > 0$, and let $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ be the solution of (1.1). Then, $f \in C^{1/2}(\bar{\Omega} \cap B_r(x_0))$ and*

$$[f]_{C^{1/2}(\bar{\Omega} \cap B_r(x_0))} \leq C,$$

where r and C are positive constants depending only on n , H_{S_i} and the geometry of Ω .

Proof. We extend each S_i to form a bounded convex C^2 -domain Ω_i with $\Omega \subset \Omega_i$ such that $(\cap_i \Omega_i) \cap B_R(x_0) = \Omega \cap B_R(x_0)$ and $H_{\partial\Omega_i} > 0$ on $\partial\Omega_i \cap B_R(x_0)$. Let f_i be the solution of (1.1) in Ω_i . By the maximum principle, we have $f \leq f_i$ in Ω . By applying Lemma 3.3 to f_i in Ω_i near x_0 and then restricting to Ω , we have

$$f \leq Cd^{1/2} \quad \text{in } \Omega \cap B_r(x_0),$$

where C and r are positive constants depending only on n , $H_{\partial\Omega_i \cap B_R}$ and the diameter of Ω_i . By Theorem 3.1, f is concave in Ω . Then, we can apply Theorem 3.2 and get the desired result. \square

Theorem 1.2 follows easily from Theorem 3.4.

To end this section, we discuss another application of Theorem 3.2, which demonstrates that the Hölder exponent for the regularity can be taken as $1/(n+1)$ and $1/i$, for any even integer i between 2 and n .

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{n+1,\alpha}$ -domain with $H_{\partial\Omega} \geq 0$, for some $\alpha \in (0, 1)$, and $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ be the solution of (1.1). Assume $c_i(x')$ is the first nonzero term in the expansion of u near 0, for some even i between 2 and n , or $i = n+1$. Then,*

$$[f]_{C^{1/i}(\bar{\Omega} \cap B_R(0))} \leq C,$$

where C and R are positive constants depending only on $c_i(0)$, n and the $C^{n+1,\alpha}$ -norm of $\partial\Omega$ near 0.

Here, for $i = n+1$, $c_{n+1} = c_{n+1,1}$ if $c_{n+1,1}(0)$ does not vanish and $c_{n+1} = c_{n+1,0}$ if $c_{n+1,1}$ vanishes near 0.

Proof. Let $x_n = \varphi(x')$ be a $C^{n+1,\alpha}$ -function representing the boundary $\partial\Omega$ near the origin, with $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. By (2.4) and (2.5), we get

$$1 = u_t f_{x_n}, \quad 0 = u_{x'} + u_t f_{x'}.$$

and

$$\begin{aligned} 0 &= u_{tt} f_{x_n}^2 + u_t f_{x_n x_n}, \\ 0 &= u_{x_\alpha x_\alpha} + 2u_{x_\alpha t} f_{x_\alpha} + u_{tt} f_{x_\alpha}^2 + u_t f_{x_\alpha x_\alpha}. \end{aligned}$$

Hence,

$$u_t^2 \Delta f + u_t u_{tt} |\nabla f|^2 - 2\nabla_{x'} u_t \nabla_{x'} u + u_t \Delta_{x'} u = 0.$$

Since $c_i(x')$ is the the first nonzero term in the expansion of u , by taking r small depending only on $c_i(0)$, n and the $C^{n+1,\alpha}$ -norm of φ near 0, we have $\Delta f \leq 0$, for $(x', t) \in B'_r(0) \times (0, r)$.

Next, we verify $f \leq Cd^{1/i}$ in a neighborhood of 0. By taking r small, we have

$$|u(x', t) - \varphi(x')| \geq \frac{1}{2} c_i(0) t^i \quad \text{for any } (x', t) \in B'_r(0) \times (0, r).$$

Note $x_n = u$ and $t = f$. It is easy to verify that $|x_n - \varphi(x')| \leq 2d(x)$ for x sufficiently small. Hence, by taking r sufficiently small and $R = c_i(0)r^i/2$, we obtain

$$f \leq \left(\frac{4}{c_i(0)} d \right)^{\frac{1}{i}} \quad \text{in } \Omega \cap B_R.$$

We then have the desired estimate by Theorem 3.2. \square

4. AN EQUIVALENT FORM OF THE MINIMAL SURFACE EQUATION

Let Ω be a bounded domain and $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ be the solution of (1.1). Set

$$w = \frac{1}{4}f^2.$$

Then, u satisfies

$$(4.1) \quad \begin{aligned} \Delta w - \frac{w_i w_j}{w + |Dw|^2} u_{ij} + \frac{w}{2w + 2|Dw|^2} + \frac{n-1}{2} &= 0 && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega, \\ w &> 0 && \text{in } \Omega. \end{aligned}$$

The equation in (4.1) for $n = 2$ appears in the study of Chaplygin gas. See the equation (22) in [9]. (The equation in (1.1) for $n = 2$ is the equation (24) in [9].)

Concerning (4.1), we have the following global regularity for its solutions. Compare with Theorem 6.1 in [9].

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain which is the intersection of finitely many bounded convex C^2 -domains Ω_i with $H_{\partial\Omega_i} > 0$, and let $w \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ be the solution of (4.1). Then $w \in C^{0,1}(\bar{\Omega})$, and*

$$|w|_{C^{0,1}(\bar{\Omega})} \leq C,$$

where C is a positive constant depending only on n , $H_{\partial\Omega_i}$ and the diameter of Ω_i .

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{n+1,\alpha}$ -domain with $H_{\partial\Omega} > 0$, for some $\alpha \in (0, 1)$, and let $w \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ be the solution of (4.1). Then, $w \in C^{(n+1)/2}(\bar{\Omega})$ if n is even, and $w \in C^{(n+1-\varepsilon)/2}(\bar{\Omega})$ for any $\varepsilon \in (0, 1)$ if n is odd. In particular, if $n = 2$, $w \in C^{1,1/2}(\bar{\Omega})$.*

In fact, local versions as Theorem 3.4 hold for both Theorem 4.1 and Theorem 4.2. Moreover, the regularity in both Theorem 4.1 and Theorem 4.2 is optimal in general. Even if the domain Ω is smooth in Theorem 4.2, the regularity of w cannot be improved.

The proof of Theorem 4.1 follows from that of Theorem 3.4 by employing Theorem 3.2 for $\alpha = 1/2$, with (3.3) replacing (3.2).

We need to point out that the global Lipschitz property of solutions u of (4.1) established in Theorem 4.1 is optimal if domains Ω admit singularity, in which case the solutions u cannot be C^1 up to the boundary. We demonstrate this by considering Ω in \mathbb{R}^2 . Assume that a part of $\partial\Omega$ near $0 \in \partial\Omega$ consists of curves c_1 and c_2 and that the tangent lines l_1 and l_2 of c_1 and c_2 intersect at 0 with an angle $\alpha\pi$, with $0 < \alpha < 1$. If w is C^1 up to the boundary near 0 , then $\nabla w(0) = 0$ by the linear independence of l_1 and l_2 . However by the local expansion of w (see below), we have

$$\frac{\partial w}{\partial \nu} = \frac{1}{2H_i} \quad \text{on } c_i \setminus \{0\},$$

where H_i is the curvature of c_i . This leads to a contradiction.

We now discuss briefly the global regularity of solutions of (4.1) if the boundary $\partial\Omega$ is smooth. First, we recall a result established in [4].

Set, for n even,

$$(4.2) \quad f_n = a_1\sqrt{d} + a_3(\sqrt{d})^3 + \cdots + a_{n-1}(\sqrt{d})^{n-1} + a_n(\sqrt{d})^n,$$

and, for n odd,

$$(4.3) \quad f_n = a_1\sqrt{d} + a_3(\sqrt{d})^3 + \cdots + a_{n-2}(\sqrt{d})^{n-2} + a_{n,1}(\sqrt{d})^n \log \sqrt{d} + a_{n,0}(\sqrt{d})^n,$$

where a_i and $a_{i,j}$ are functions on $\partial\Omega$. For example,

$$a_1 = \sqrt{\frac{2}{H}}.$$

The following result is a special case in Theorem 7.1 in [4] by taking $k = n$.

Theorem 4.3. *Let Ω be a bounded $C^{n+1,\alpha}$ -domain in \mathbb{R}^n with $H_{\partial\Omega} > 0$, for some $\alpha \in (0, 1)$, and (y', d) be the principal coordinates near $\partial\Omega$. Suppose that $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ is a solution of (4.1). Then, there exist functions $a_i, a_{i,j} \in C^{n-i,\epsilon}(\partial\Omega)$, for $i = 1, 3, \dots, n$ and any $\epsilon \in (0, \alpha)$, and a positive constant d_0 such that, for f_k defined as in (4.2) or (4.3), for any $m = 0, 1, \dots, n$, and any $\epsilon \in (0, \alpha)$,*

$$\partial_{\sqrt{d}}^m(f - f_n) \text{ is } C^\epsilon \text{ in } (y', \sqrt{d}) \in \partial\Omega \times [0, d_0],$$

and, for any $0 < d < d_0$,

$$(4.4) \quad |\partial_{\sqrt{d}}^m(f - f_n)| \leq C(\sqrt{d})^{n-m+\alpha},$$

where C is a positive constant depending only on n, α and the $C^{n+1,\alpha}$ -norm of $\partial\Omega$.

In fact, the remainder $f - f_n$ can be characterized by a multiple integral of multiplicity n . Then, with such a characterization and the explicit expression of f_n in (4.2) and (4.3), Theorem 4.2 follows easily.

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