TOPOLOGY OF CLOSED HYPERSURFACES OF SMALL ENTROPY

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ABSTRACT. We use a weak mean curvature flow together with a surgery procedure to show that all closed hypersurfaces in $\mathbb{R}^4$ with entropy less than or equal to that of $\mathbb{S}^2 \times \mathbb{R}$, the round cylinder in $\mathbb{R}^4$, are diffeomorphic to $\mathbb{S}^3$.

1. INTRODUCTION

If Σ is a hypersurface, that is, a smooth properly embedded codimension-one submanifold of $\mathbb{R}^{n+1}$, then the *Gaussian surface area* of Σ is

$$F[\Sigma] = \int_\Sigma \Phi \, d\mathcal{H}^n = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|\mathbf{x}|^2}{4}} d\mathcal{H}^n, \tag{1.1}$$

where $\mathcal{H}^n$ is n-dimensional Hausdorff measure. Following Colding-Minicozzi [11], define the *entropy* of Σ to be

$$\lambda[\Sigma] = \sup_{(\mathbf{y},\rho)\in\mathbb{R}^{n+1}\times\mathbb{R}^+} F[\rho\Sigma + \mathbf{y}].$$

That is, the entropy of Σ is the supremum of the Gaussian surface area over all translations and dilations of Σ. Observe that the entropy of a hyperplane is one. In [3], we show that, for $2 \le n \le 6$, the entropy of a closed (i.e. compact and without boundary) hypersurface in $\mathbb{R}^{n+1}$ is uniquely (modulo translations and dilations) minimized by $\mathbb{S}^n$, the unit round sphere. This verifies a conjecture of Colding-Ilmanen-Minicozzi-White [10, Conjecture 0.9] (cf. [27]). We further show, in [4, Corollary 1.3], that surfaces in $\mathbb{R}^3$ of small entropy are topologically rigid. That is, if Σ is a closed surface in $\mathbb{R}^3$ and $\lambda[\Sigma] \le \lambda[\mathbb{S}^1 \times \mathbb{R}]$, then Σ is diffeomorphic to $\mathbb{S}^2$.

In this article, we use a weak mean curvature flow (see [13–16] and [8]) to obtain a new topological rigidity of closed hypersurfaces in $\mathbb{R}^4$ of small entropy. This generalizes a result of Colding-Ilmanen-Minicozzi-White [10] for closed self-shrinkers to arbitrary closed hypersurfaces and contrasts with the methods of both [10] and [4, Corollary 1.3], which both use only the classical mean curvature flow.

Theorem 1.1. *If $\Sigma \subset \mathbb{R}^4$ is a closed hypersurface with $\lambda[\Sigma] \le \lambda[\mathbb{S}^2 \times \mathbb{R}]$, then Σ is diffeomorphic to $\mathbb{S}^3$.*

One of the key ingredients in the proof of Theorem 1.1 is a refinement of [4, Theorem 0.1] about the topology of asymptotically conical self-shrinkers of small entropy. Recall, a hypersurface Σ is said to be *asymptotically conical* if it is smoothly asymptotic to a regular cone; i.e., $\lim_{\rho\to 0} \rho\Sigma = \mathcal{C}(\Sigma)$ in $C^\infty_{loc}(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})$ for $\mathcal{C}(\Sigma)$ a regular cone. A *self-shrinker*, Σ, is a hypersurface that satisfies

$$\mathbf{H}_\Sigma + \frac{\mathbf{x}^\perp}{2} = \mathbf{0}, \tag{1.2}$$

The first author was partially supported by the NSF Grant DMS-1307953. The second author was partially supported by the Chapman Fellowship of Imperial College and NSF Grant DMS-1406240.

where $\mathbf{H}_\Sigma = -H_\Sigma \mathbf{n}_\Sigma = \Delta_\Sigma \mathbf{x}$ is the mean curvature vector of Σ and $\mathbf{x}^\perp$ is the normal component of the position vector. Let us denote the set of self-shrinkers in $\mathbb{R}^{n+1}$ by $\mathcal{S}_n$ and the set of asymptotically conical self-shrinkers by $\mathcal{ACS}_n$. Self-shrinkers generate solutions to the mean curvature flow that move self-similarly by scaling. That is, if $\Sigma \in \mathcal{S}_n$, then

$$\{\Sigma_t\}_{t\in(-\infty,0)} = \left\{\sqrt{-t}\,\Sigma\right\}_{t\in(-\infty,0)}$$

moves by mean curvature vectors. Important examples are the maximally symmetric self-shrinking cylinders with k-dimensional spine,

$$\mathbb{S}^{n-k}_* \times \mathbb{R}^k = \left\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{n-k+1}\times\mathbb{R}^k = \mathbb{R}^{n+1} : |\mathbf{x}|^2 = 2(n-k)\right\},$$

where $0 \leq k \leq n$. As $\mathbb{S}^{n-k}_* \times \mathbb{R}^k$ are self-shrinkers, their Gaussian surface area and entropy agree (cf. [11, Lemma 7.20]).. That is,

$$\lambda_n = \lambda[\mathbb{S}^n] = F[\mathbb{S}^n_*] = F[\mathbb{S}^n_* \times \mathbb{R}^l] = \lambda[\mathbb{S}^n \times \mathbb{R}^l].$$

Hence, a computation of Stone [34], gives that

$$2 > \lambda_1 > \frac{3}{2} > \lambda_2 > \ldots > \lambda_n > \ldots \to \sqrt{2}.$$

Theorem 1.2. *Let $\Sigma \in \mathcal{ACS}_n$ for $n \geq 2$. If $\lambda[\Sigma] \leq \lambda_{n-1}$, then Σ is contractible and $\mathcal{L}(\Sigma)$, the link of the asymptotic cone $\mathcal{C}(\Sigma)$, is a homology $(n-1)$-sphere.*

For $n = 3$, the classification of surfaces and Alexander's theorem [1] gives

Corollary 1.3. *Let $\Sigma \in \mathcal{ACS}_3$. If $\lambda[\Sigma] \leq \lambda_2$, then Σ is diffeomorphic to $\mathbb{R}^3$.*

Theorem 1.1 follows from Corollary 1.3 together with a topological surgery procedure applied to the weak mean curvature flow associated to Σ. This same surgery procedure may also be used with Theorem 1.2 to show a (strictly weaker) extension of Theorem 1.1 valid in any dimension where the two hypotheses below are satisfied. These hypotheses are needed in order to ensure that if the entropy of an initial hypersurface is small enough, then tangent flows at all singularities are modeled by self-shrinkers that are either closed or asymptotically conical.

In order to state these hypotheses, first let $\mathcal{S}^*_n$ denote the set of non-flat elements of $\mathcal{S}_n$ and, for any $\Lambda > 0$, let

$$\mathcal{S}_n(\Lambda) = \{\Sigma \in \mathcal{S}_n : \lambda[\Sigma] < \Lambda\} \text{ and } \mathcal{S}^*_n(\Lambda) = \mathcal{S}^*_n \cap \mathcal{S}_n(\Lambda).$$

Next, let $\mathcal{RMC}_n$ denote the space of *regular minimal cones* in $\mathbb{R}^{n+1}$, that is $\mathcal{C} \in \mathcal{RMC}_n$ if and only if it is a proper subset of $\mathbb{R}^{n+1}$ and $\mathcal{C}\setminus\{\mathbf{0}\}$ is a hypersurface in $\mathbb{R}^{n+1}\setminus\{\mathbf{0}\}$ that is invariant under dilation about $\mathbf{0}$ and with vanishing mean curvature. Let $\mathcal{RMC}^*_n$ denote the set of non-flat elements of $\mathcal{RMC}_n$. For any $\Lambda > 0$, let

$$\mathcal{RMC}_n(\Lambda) = \{\mathcal{C} \in \mathcal{RMC}_n : \lambda[\mathcal{C}] < \Lambda\} \text{ and } \mathcal{RMC}^*_n(\Lambda) = \mathcal{RMC}^*_n \cap \mathcal{RMC}_n(\Lambda).$$

Let us now fix a dimension $n \geq 3$ and a value $\Lambda > 1$. The first hypothesis is

$$(\star_{n,\Lambda}) \qquad \text{For all } 3 \leq k \leq n,\ \mathcal{RMC}^*_k(\Lambda) = \emptyset.$$

Observe that all regular minimal cones in $\mathbb{R}^2$ consist of unions of rays and so $\mathcal{RMC}^*_1 = \emptyset$. Likewise, as great circles are the only geodesics in $\mathbb{S}^2$, $\mathcal{RMC}^*_2 = \emptyset$. The second hypothesis is

$$(\star\star_{n,\Lambda}) \qquad \mathcal{S}^*_{n-1}(\Lambda) = \emptyset.$$

We then show the following conditional result:

Theorem 1.4. *Fix $n \geq 3$ and $\Lambda \in (\lambda_n, \lambda_{n-1}]$. If ($\star_{n,\Lambda}$) and ($\star\star_{n,\Lambda}$) both hold and Σ is a closed hypersurface in $\mathbb{R}^{n+1}$ with $\lambda[\Sigma] \leq \Lambda$, then Σ is a homology n-sphere.*

For general n, neither the validity of ($\star_{n,\Lambda}$) nor that of ($\star\star_{n,\Lambda}$) is known. However, both can be established for $n = 3$ and $\Lambda = \lambda_2$. First, as part of their proof of the Willmore conjecture, Marques-Neves gave a lower bound on the density of non-trivial regular minimal cones in $\mathbb{R}^4$. In particular, it follows from [28, Theorem B] that if $\mathcal{C} \in \mathcal{RMC}_3^*$, then $\lambda[\mathcal{C}] > \lambda_2$ and so $(\star_{3,\lambda_2})$ holds. Furthermore, it follows from [4, Corollary 1.2] that $\mathcal{S}_2^*(\lambda_2) = \emptyset$ and so $(\star\star_{3,\lambda_2})$ holds.

For $n \geq 4$, some partial results suggest that ($\star_{n,\Lambda}$) and ($\star\star_{n,\Lambda}$) hold for $\Lambda = \lambda_{n-1}$. For instance, Ding [12, Theorem 9.5], building on work of Ilmanen-White [26, Theorem 2], has shown that if $\mathcal{C} \in \mathcal{RMC}_n^*$ and is topologically non-trivial, then $\lambda[\mathcal{C}] \geq \lambda_n$. Additionally, [10, Theorem 0.1] says that the self-shrinking sphere has the lowest entropy among all compact self-shrinkers and [10, Conjecture 0.10] posits that $(\star\star_{n,\lambda_{n-1}})$ holds for $n \leq 7$. An important caveat is that there exist many topologically trivial elements of $\mathcal{RMC}_n^*$. Indeed, the work of Hsiang [18, 19] and Hsiang-Sterling [20], shows that there exist topologically trivial elements of $\mathcal{RMC}_n^*$ for $n = 5, 7$ and for all even $n \geq 4$.

The paper is organized as follows. In Section 2, we introduce notation and recall basic facts about the mean curvature flow that we use. In Section 3, we show regularity of self-shrinking measures of low entropy. In Section 4, we study the structure of the singular set for weak mean curvature flows of small entropy. Importantly, we give a topological decomposition of the regular part of the flow which is the basis of the surgery procedure. In Section 5, we prove Theorem 1.2 and Corollary 1.3. Finally, in Section 6, we prove Theorems 1.1 and 1.4.

2. Notation and Background

In this section, we fix notation for the rest of the paper and recall some background on mean curvature flow.

2.1. **Singular hypersurfaces.** We will make heavy use of the results of [23] on weak mean curvature flows. For this reason, we follow the notation of [23] as closely as possible.

Denote by

- $\mathcal{M}(\mathbb{R}^{n+1}) = \{\mu : \mu \text{ is a Radon measure on } \mathbb{R}^{n+1}\}$ (see [32, Section 4]);
- $\mathcal{IM}_k(\mathbb{R}^{n+1}) = \{\mu : \mu \text{ is an integer } k\text{-rectifiable Radon measure on } \mathbb{R}^{n+1}\}$ (see [23, Section 1]);
- $\mathbf{IV}_k(\mathbb{R}^{n+1}) = \{V : V \text{ is an integer rectifiable } k\text{-varifold on } \mathbb{R}^{n+1}\}$ (see [23, Section 1] or [32, Chapter 8]).

The space $\mathcal{M}(\mathbb{R}^{n+1})$ is given the weak* topology. That is,

$$\mu_i \to \mu \iff \int f \, d\mu_i \to \int f \, d\mu \text{ for all } f \in C_c^0(\mathbb{R}^{n+1}).$$

And the topology on $\mathcal{IM}_k(\mathbb{R}^{n+1})$ is the subspace topology induced by the natural inclusion into $\mathcal{M}(\mathbb{R}^{n+1})$. For the details of the topologies considered on $\mathbf{IV}_k(\mathbb{R}^{n+1})$, we refer to [23, Section 1] or [32, Chapter 8]. There are natural bijective maps

$$V : \mathcal{IM}_k(\mathbb{R}^{n+1}) \to \mathbf{IV}_k(\mathbb{R}^{n+1}) \text{ and } \mu : \mathbf{IV}_k(\mathbb{R}^{n+1}) \to \mathcal{IM}_k(\mathbb{R}^{n+1}).$$

The second map is continuous, but the first is not. Henceforth, write $V(\mu) = V_\mu$ and $\mu(V) = \mu_V$.

If $\Sigma \subset \mathbb{R}^{n+1}$ is a k-dimensional smooth properly embedded submanifold, we denote by $\mu_\Sigma = \mathcal{H}^k \lfloor \Sigma \in \mathcal{IM}_k(\mathbb{R}^{n+1})$. Given $(\mathbf{y}, \rho) \in \mathbb{R}^{n+1} \times \mathbb{R}^+$ and $\mu \in \mathcal{IM}_k(\mathbb{R}^{n+1})$, we define the rescaled measure $\mu^{\mathbf{y},\rho} \in \mathcal{IM}_k(\mathbb{R}^{n+1})$ by

$$\mu^{\mathbf{y},\rho}(\Omega) = \rho^k \mu\left(\rho^{-1}\Omega + \mathbf{y}\right).$$

This is defined so that if Σ a k-dimensional smooth properly embedded submanifold,

$$\mu_\Sigma^{\mathbf{y},\rho} = \mu_{\rho(\Sigma - \mathbf{y})}.$$

One of the defining properties of $\mu \in \mathcal{IM}_k(\mathbb{R}^{n+1})$ is that for μ-a.e. $\mathbf{x} \in \mathbb{R}^{n+1}$, there is an integer value $\theta_\mu(\mathbf{x})$ so that

$$\lim_{\rho \to \infty} \mu^{\mathbf{x},\rho} = \theta_\mu(\mathbf{x}) \mu_P,$$

where P is a k-dimensional plane through the origin. When such P exists, we denote it by $T_{\mathbf{x}}\mu$ the *approximate tangent plane at* $\mathbf{x}$. The value $\theta_\mu(\mathbf{x})$ is the *multiplicity of* μ *at* $\mathbf{x}$ and by definition, $\theta_\mu(\mathbf{x}) \in \mathbb{N}$ for μ-a.e. $\mathbf{x}$. Notice that if $\mu = \mu_\Sigma$, then $T_{\mathbf{x}}\mu = T_{\mathbf{x}}\Sigma$ and $\theta_\mu(\mathbf{x}) = 1$. Given a $\mu \in \mathcal{IM}_n(\mathbb{R}^{n+1})$, set

$$\mathrm{reg}(\mathrm{spt}(\mu)) = \left\{\mathbf{x} \in \mathrm{spt}(\mu) : \exists \rho > 0 \text{ s.t. } B_\rho(\mathbf{x}) \cap \mathrm{spt}(\mu) \text{ is a hypersurface}\right\},$$

and $\mathrm{sing}(\mathrm{spt}(\mu)) = \mathrm{spt}(\mu) \setminus \mathrm{reg}(\mathrm{spt}(\mu))$. Here $B_\rho(\mathbf{x})$ is the open ball in $\mathbb{R}^{n+1}$ centered at $\mathbf{x}$ with radius ρ. Likewise,

$$\mathrm{reg}(\mu) = \{\mathbf{x} \in \mathrm{reg}(\mathrm{spt}(\mu)) : \theta_\mu(\mathbf{x}) = 1\} \text{ and } \mathrm{sing}(\mu) = \mathrm{spt}(\mu) \setminus \mathrm{reg}(\mu).$$

For $\mu \in \mathcal{IM}_n(\mathbb{R}^{n+1})$, we extend the definitions of F and λ in the obvious manner, namely,

$$F[\mu] = F[V_\mu] = \int \Phi \, d\mu \text{ and } \lambda[\mu] = \lambda[V_\mu] = \sup_{(\mathbf{y},\rho) \in \mathbb{R}^{n+1} \times \mathbb{R}^+} F[\mu^{\mathbf{y},\rho}].$$

2.2. **Brakke flow.** Historically, the first weak mean curvature flow was the measure-theoretic flow introduced by Brakke [5]. This flow is called a *Brakke flow*. Brakke's original definition considered the flow of varifolds. Here we use the (slightly stronger) notion introduced by Ilmanen [23]. For our purposes, the Brakke flow has two important roles. The first is the fact that Huisken's monotonicity formula [21] holds also for Brakke flows; see [24]. The second is the powerful regularity theory of Brakke [5] for such flows.

Let $\mu \in \mathcal{M}(\mathbb{R}^{n+1})$ and $\phi \in C^2_c(\mathbb{R}^{n+1}, \mathbb{R}^{\geq 0})$. Following [24], a family $\mathcal{K} = \{\mu_t\}_{t \in I}$ with $\mu_t \in \mathcal{M}(\mathbb{R}^{n+1})$ is a *codimension-one Brakke flow in* $\mathbb{R}^{n+1}$, if for all $t_1, t_2 \in I$ with $t_1 \leq t_2$ and $\phi \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{\geq 0})$,

$$\int \phi \, d\mu_{t_2} \leq \int \phi \, d\mu_{t_1} + \int_{t_1}^{t_2} \int -\phi H^2 + D\phi \cdot S^\perp \cdot \mathbf{H} \, d\mu \, dt. \tag{2.1}$$

Here $S = S(\mathbf{x}) = T_{\mathbf{x}}\mu$ for $\mathcal{H}^n$-a.e. $\mathbf{x} \in \{\phi > 0\}$, $S^\perp \cdot \mathbf{y}$ means to project the vector $\mathbf{y}$ onto the line $S^\perp$ perpendicular to S, and $\mathbf{H}$ is generalized mean curvature vector.

Given $U \subset \mathbb{R}^{n+1}$ a non-empty open subset and a subinterval $I' \subset I$, we could restrict $\mathcal{K}$ to $U \times I'$ by

$$\mathcal{K} \lfloor U \times I' = \{\mu_t \lfloor U\}_{t \in I'},$$

which clearly satisfies the inequality (2.1) for all $\phi \in C^2_c(U, \mathbb{R}^{\geq 0})$. When the meaning is clear from context, we will suppress mention of the codimension and ambient domain and speak of a *Brakke flow*. A Brakke flow, $\mathcal{K} = \{\mu_t\}_{t \in I}$ is *integral* if $\mu_t \in \mathcal{IM}_n(\mathbb{R}^{n+1})$ for a.e. $t \in I$.

An important consequence of Huisken's monotonicity formula is that if a Brakke flow $\mathcal{K}=\{\mu_t\}_{t\geq t_0}$ has bounded area ratios, then $\mathcal{K}$ has a well-defined *Gaussian density* at every point $(\mathbf{y},s)\in\mathbb{R}^{n+1}\times(t_0,\infty)$ given by

$$\Theta_{(\mathbf{y},s)}(\mathcal{K})=\lim_{t\to s^-}\int\Phi_{(\mathbf{y},s)}(\mathbf{x},t)\,d\mu_t(\mathbf{x}),$$

where

$$\Phi_{(\mathbf{y},s)}(\mathbf{x},t)=(4\pi)^{-\frac{n}{2}}e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-s)}}.$$

It is easy to see from the monotonicity formula that $\Theta_{(\mathbf{y},s)}(\mathcal{K})\geq\theta_{\mu_s}(\mathbf{y})$. Moreover, the Gaussian density is upper semi-continuous.

Combining the compactness of Brakke flows with the monotonicity formula, one establishes the existence of tangent flows. For a Brakke flow $\mathcal{K}=\{\mu_t\}_{t\geq t_0}$ and a point $(\mathbf{y},s)\in\mathbb{R}^{n+1}\times(t_0,\infty)$, define a new Brakke flow

$$\mathcal{K}^{(\mathbf{y},s),\rho}=\left\{\mu_t^{(\mathbf{y},s),\rho}\right\}_{t\geq\rho^2(t_0-s)},$$

where

$$\mu_t^{(\mathbf{y},s),\rho}=\mu^{\mathbf{y},\rho}_{s+\rho^{-2}t}.$$

Definition 2.1. Let $\mathcal{K}=\{\mu_t\}_{t\geq t_0}$ be an integral Brakke flow with bounded area ratios. A non-trivial Brakke flow $\mathcal{T}=\{\nu_t\}_{t\in\mathbb{R}}$ is a *tangent flow* to $\mathcal{K}$ at $(\mathbf{y},s)\in\mathbb{R}^{n+1}\times(t_0,\infty)$, if there is a sequence $\rho_i\to\infty$ so that $\mathcal{K}^{(\mathbf{y},s),\rho_i}\to\mathcal{T}$. Denote by $\mathrm{Tan}_{(\mathbf{y},s)}\mathcal{K}$ the set of tangent flows to $\mathcal{K}$ at $(\mathbf{y},s)$.

The monotonicity formula implies that any tangent flow is backwardly self-similar.

Theorem 2.2 ([24, Lemma 8])**.** *Given an integral Brakke flow $\mathcal{K}=\{\mu_t\}_{t\geq t_0}$ with bounded area ratios, a point $(\mathbf{y},s)\in\mathbb{R}^{n+1}\times(t_0,\infty)$ with $\Theta_{(\mathbf{y},s)}(\mathcal{K})\geq 1$, and a sequence $\rho_i\to\infty$, there exists a subsequence ρ_{i_j} and a $\mathcal{T}\in\mathrm{Tan}_{(\mathbf{y},s)}\mathcal{K}$ so that $\mathcal{K}^{(\mathbf{y},s),\rho_{i_j}}\to\mathcal{T}$.*

Furthermore, $\mathcal{T}=\{\nu_t\}_{t\in\mathbb{R}}$ is backwardly self-similar with respect to parabolic rescaling about $(\mathbf{0},0)$. That is, for all $t<0$ and $\rho>0$,

$$\nu_t=\nu_t^{(\mathbf{0},0),\rho}.$$

Moreover, $V_{\nu_{-1}}$ is a stationary point of the F functional and

$$\Theta_{(\mathbf{y},s)}(\mathcal{K})=F[\nu_{-1}].$$

2.3. **Level-set flow.** We will also need a set-theoretic weak mean curvature flow called the level-set flow. This flow was first studied in the context of numerical analysis by Osher-Sethian [30]. The mathematical theory was developed by Evans-Spruck [13–16] and Chen-Giga-Goto [8]. For our purposes, it has the important advantages of being uniquely defined and satisfying a nice maximum principle.

We will follow the formulation of the level-set flow of Evans-Spruck [13]. Let Γ be a compact non-empty subset of $\mathbb{R}^{n+1}$. Select a continuous function u_0 so that $\Gamma=\{\mathbf{x}:u_0(\mathbf{x})=0\}$ and there are constants $C,R>0$ so that

$$u_0=-C\quad\text{on }\left\{\mathbf{x}\in\mathbb{R}^{n+1}:|\mathbf{x}|\geq R\right\}$$

for some sufficiently large R. In particular, $\{u_0\geq a>-C\}$ is compact. In [13], Evans-Spruck established the existence and uniqueness of viscosity weak solutions to the initial value problem:

$$\left\{\begin{array}{ll} u_t=\sum_{i,j=1}^{n+1}\left(\delta_{ij}-u_{x_i}u_{x_j}|Du|^{-2}\right)u_{x_ix_j} & \text{on }\mathbb{R}^{n+1}\times(0,\infty)\\ u=u_0 & \text{on }\mathbb{R}^{n+1}\times\{0\}.\end{array}\right. \tag{2.2}$$

Setting $\Gamma_t = \{\mathbf{x} : u(\mathbf{x}, t) = 0\}$, define $L(\Gamma) = \{\Gamma_t\}_{t\geq 0}$ to be the *level-set* flow of Γ. It was shown in [13, Theorem 5.3] that $L(\Gamma)$ is independent of the choice of u_0.

A technical feature of the level-set flow is that the Γ_t of $L(\Gamma)$ may develop non-empty interiors for positive times. This phenomena is called fattening and is unavoidable for certain initial sets Γ. It is closely related to non-uniqueness phenomena of weak solutions of the flow. A level-set flow $L(\Gamma) = \{\Gamma_t\}_{t\geq 0}$ is *non-fattening*, if each Γ_t has no interior.

2.4. **Boundary motion.** In [23], Ilmanen synthesized both notions of weak flow. In particular, he showed that for a large class of initial sets, there is a canonical way to associate a Brakke flow to the level-set flow, and observed that this allows, among other things, for the application of Brakke's partial regularity theorem. For our purposes, it is important that the Brakke flow constructed does not vanish gratuitously. A similar synthesis may be found in [16]. The result we need is the following:

Theorem 2.3 ([23, Theorem 11.4])**.** *If Σ_0 is a closed hypersurface in $\mathbb{R}^{n+1}$ and the level-set flow $L(\Sigma_0)$ is non-fattening, then there is a set $E \subset \mathbb{R}^{n+1} \times \mathbb{R}$ and a Brakke flow $\mathcal{K} = \{\mu_t\}_{t\geq 0}$ so that:*

(1) $E = \{(\mathbf{x}, t) : u(\mathbf{x}, t) > 0\}$, where u solves (2.2) *with $E_0 = \{\mathbf{x} : u_0(\mathbf{x}) > 0\}$ so that $\partial E_0 = \{\mathbf{x} : u_0(\mathbf{x}) = 0\} = \Sigma_0$;*
(2) each $E_t = \{\mathbf{x} : (\mathbf{x}, t) \in E\}$ is of finite perimeter and $\mu_t = \mathcal{H}^n \lfloor \partial^* E_t$, where $\partial^* E_t$ is the reduced boundary of E_t.

It is relatively straightforward to see that the non-fattening condition is generic; see for instance [23, Theorem 11.3].

3. Regularity of Self-Shrinking Measures of Small Entropy

We establish some regularity properties of self-shrinking measures of small entropy when $n \geq 3$. We restrict to $n \geq 3$ in order to avoid certain technical complications coming from the fact that $\lambda_1 > \frac{3}{2}$.

3.1. **Self-shrinking measures.** We will need a singular analog of $\mathcal{S}_n$. To that end, we define the set of self-shrinking measures on $\mathbb{R}^{n+1}$ by

$$\mathcal{SM}_n = \left\{\mu \in \mathcal{IM}_n(\mathbb{R}^{n+1}) : V_\mu \text{ is stationary for the } F \text{ functional}, \operatorname{spt}(\mu) \neq \emptyset\right\}.$$

Clearly, if $\Sigma \in \mathcal{S}_n$, then $\mu_\Sigma \in \mathcal{SM}_n$. There are many examples of singular self-shrinkers. For instance, for any element of $C \in \mathcal{RMC}_n$ one verifies that $\mu_C = \mathcal{H}^n \lfloor C \in \mathcal{SM}_n$. For $\mu \in \mathcal{SM}_n$, we define the *associated Brakke flow* $\mathcal{K} = \{\mu_t\}_{t\in\mathbb{R}}$ by

$$\mu_t = \begin{cases} 0 & t \geq 0 \\ \mu^{\mathbf{0},\sqrt{-t}} & t < 0. \end{cases}$$

One can verify that this is a Brakke flow. Given $\Lambda > 0$, set

$$\mathcal{SM}_n(\Lambda) = \{\mu \in \mathcal{SM}_n : \lambda[\mu] < \Lambda\} \text{ and } \mathcal{SM}_n[\Lambda] = \{\mu \in \mathcal{SM}_n : \lambda[\mu] \leq \Lambda\}. \tag{3.1}$$

3.2. **Regularity and asymptotic properties of self-shrinking measures of small entropy.** A $\mu \in \mathcal{IM}_n(\mathbb{R}^{n+1})$ is a *cone*, if $\mu^{\mathbf{0},\rho} = \mu$. Likewise, $\mu \in \mathcal{IM}_n(\mathbb{R}^{n+1})$ *splits off a line*, if, up to an ambient rotation of $\mathbb{R}^{n+1}$, $\mu = \hat{\mu} \times \mu_{\mathbb{R}}$ for $\hat{\mu} \in \mathcal{IM}_{n-1}(\mathbb{R}^n)$. Observe that if $\mu \in \mathcal{SM}_n$ is a cone, then V_μ is stationary (for area). Similarly, if $\mu \in \mathcal{SM}_n$ splits off a line, then $\hat{\mu} \in \mathcal{SM}_{n-1}$ and $\lambda[\mu] = \lambda[\hat{\mu}]$.

Standard dimension reduction arguments give the following:

Lemma 3.1. *Fix $n \geq 3$ and $\Lambda \leq 3/2$ and suppose that $(\star_{n,\Lambda})$ holds. If $\mu \in \mathcal{SM}_n(\Lambda)$ is a cone, then $\mu = \mu_P$ for some hyperplane P.*

Proof. We will prove this by showing that if $(\star_{n,\Lambda})$ holds, then for all $3 \leq m \leq n$, if $\mu \in \mathcal{SM}_m(\Lambda)$ is a cone, then $\mu = \mu_P$ for a hyperplane P in $\mathbb{R}^{m+1}$.

We proceed by induction on m. When $m = 3$, note that $\Lambda \leq \frac{3}{2}$ and so we have that $\mu = \mu_{\mathcal{C}}$ for some $\mathcal{C} \in \mathcal{RMC}_3$ by [3, Proposition 4.3]. Hence, by the assumption that $\mathcal{RMC}^*_3(\Lambda) = \emptyset$, we must have that $\mathcal{C}$ is a hyperplane through the origin. To complete the induction argument, we observe that it suffices to show that if $\mu \in \mathcal{SM}_m(\Lambda)$ is a cone, then $\mu = \mu_{\mathcal{C}}$ for some $\mathcal{C} \in \mathcal{RMC}_m(\Lambda)$. Indeed, such a $\mathcal{C}$ must be a hyperplane because $(\star_{n,\Lambda})$ holds and so, by definition, $\mathcal{RMC}^*_m(\Lambda) = \emptyset$ for $3 \leq m \leq n$.

To complete the proof, we argue by contradiction. Suppose that $\mathrm{spt}(\mu)$ is not a regular cone. Then there is a point $\mathbf{y} \in \mathrm{sing}(\mu) \setminus \{\mathbf{0}\}$. As V_μ is stationary, and $\mu \in \mathcal{IM}_m$ with $\lambda[\mu] < \Lambda$, we may apply Allard's integral compactness theorem (see [32, Theorem 42.7 and Remark 42.8]) to conclude that there exists a sequence $\rho_i \to \infty$ so that $\mu^{\mathbf{y},\rho_i} \to \nu$ and V_ν is a stationary integral varifold. Moreover, it follows from the monotonicity formula [32, Theorem 17.6] that ν is a cone; see also [32, Theorem 19.3].

As μ is a cone, ν splits off a line.That is, $\nu = \hat{\nu} \times \mu_{\mathbb{R}}$, where $\hat{\nu} \in \mathcal{IM}_{m-1}$ and $V_{\hat{\nu}}$ is a self-shrinking cone. Moreover, by the lower semi-continuity of entropy,

$$\lambda[\hat{\nu}] = \lambda[\hat{\nu} \times \mu_{\mathbb{R}}] \leq \lambda[\mu] < \Lambda.$$

Thus, it follows from the induction hypotheses that $\hat{\nu} = \mu_{\hat{P}}$, where $\hat{P}$ is a hyperplane in $\mathbb{R}^m$ and so V_ν is a multiplicity-one hyperplane. Hence, by Allard's regularity theorem (see [32, Theorem 24.2]), $\mathbf{y} \in \mathrm{reg}(\mu)$, giving a contradiction. Therefore, $\mu = \mu_{\mathcal{C}}$ for a $\mathcal{C} \in \mathcal{RMC}_m(\Lambda)$. □

As a consequence, we obtain regularity for elements of $\mathcal{SM}_n(\Lambda)$ under the hypotheses that $(\star_{n,\Lambda})$ holds.

Proposition 3.2. *Fix $n \geq 3$ and $\Lambda \leq 3/2$ and suppose that $(\star_{n,\Lambda})$ holds. If $\mu \in \mathcal{SM}_n(\Lambda)$, then $\mu = \mu_\Sigma$ for some $\Sigma \in \mathcal{S}_n(\Lambda)$.*

Proof. Observe that for $\mu \in \mathcal{SM}_n(\Lambda)$, the mean curvature of V_μ is locally bounded by (1.2). Following the same reasoning in the proof of Lemma 3.1, given $\mathbf{y} \in \mathrm{sing}(\mu)$, there exists a sequence $\rho_i \to \infty$ so that $\mu^{\mathbf{y},\rho_i} \to \nu$ and V_ν is an integral stationary cone. By the lower semi-continuity of entropy, $\lambda[\nu] \leq \lambda[\mu] < \Lambda$. Hence, together with Lemma 3.1, it follows that $\mathrm{sing}(\mu) = \emptyset$. That is $\mathrm{spt}(\mu)$ is a smooth submanifold of $\mathbb{R}^{n+1}$ that, moreover, satisfies (1.2). Finally, the entropy bound on μ implies that $\mu(B_R) \leq CR^n$ for some $C > 0$ and so, by [9, Theorem 1.3], $\mathrm{spt}(\mu)$ is proper. That is, $\mu = \mu_\Sigma$ for some $\Sigma \in \mathcal{S}_n$. □

If, in addition, $(\star\star_{n,\Lambda})$ is assumed, one can prove:

Proposition 3.3. *Fix an $n \geq 3$ and $\Lambda \in (\lambda_n, \lambda_{n-1}]$ and suppose that both $(\star_{n,\Lambda})$ and $(\star\star_{n,\Lambda})$ hold. If $\mu \in \mathcal{SM}_n(\Lambda)$, then $\mu = \mu_\Sigma$ for some $\Sigma \in \mathcal{S}_n(\Lambda)$, and either Σ is diffeomorphic to $\mathbb{S}^n$ or $\Sigma \in \mathcal{ACS}_n$.*

Proof. First observe that, by Proposition 3.2, $\mu = \mu_\Sigma$ for some $\Sigma \in \mathcal{S}_n(\Lambda)$. If Σ is closed, then it follows from [10, Theorem 0.7] that Σ is diffeomorphic to $\mathbb{S}^n$. On the other hand, if Σ is not closed, then it is non-compact.

Let $\mathcal{K} = \{\mu_t\}_{t\in\mathbb{R}}$ be the Brakke flow associated to μ. Note that $\mu_t = \mu_{\sqrt{-t}\Sigma}$ for $t < 0$. Let $\mathcal{X} = \left\{\mathbf{y} : \mathbf{y} \neq \mathbf{0}, \Theta_{(\mathbf{y},0)}(\mathcal{K}) \geq 1\right\} \subset \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$. As Σ is non-compact, $\mathcal{X}$

is non-empty. Indeed, pick any sequence of points $\mathbf{y}_i \in \Sigma$ with $|\mathbf{y}_i| \to \infty$. The points $\hat{\mathbf{y}}_i = |\mathbf{y}_i|^{-1}\mathbf{y}_i \in |\mathbf{y}_i|^{-1}\Sigma$. Hence, $\Theta_{(\hat{\mathbf{y}}_i, -|\mathbf{y}_i|^{-2})}(\mathcal{K}) \geq 1$. As the $\hat{\mathbf{y}}_i$ are in a compact subset, up to passing to a subsequence and relabeling, $\hat{\mathbf{y}}_i \to \hat{\mathbf{y}}$, and so the upper semi-continuity of Gaussian density implies that $\Theta_{(\hat{\mathbf{y}},0)}(\mathcal{K}) \geq 1$.

We next show that $\mathcal{X}$ is a regular cone. The fact that $\mathcal{X}$ is a cone readily follows from the fact that $\mathcal{K}$ is invariant under parabolic scalings. To see that $\mathrm{sing}(\mathcal{X}) \subset \{\mathbf{0}\}$, we note that, by [3, Lemma 4.6], for any $\mathbf{y} \in \mathcal{X}$ and $\mathcal{T} \in \mathrm{Tan}_{(\mathbf{y},0)}\mathcal{K}$, $\mathcal{T} = \{\nu_t\}_{t\in\mathbb{R}}$ splits off a line. That is, up to an ambient rotation, $\nu_t = \hat{\nu}_t \times \mu_{\mathbb{R}}$ with $\{\hat{\nu}_t\}_{t\in\mathbb{R}}$ the Brakke flow associated to $\hat{\nu}_{-1} \in \mathcal{SM}_{n-1}(\Lambda)$. Here we use the lower semi-continuity of entropy. Note that $\Lambda \leq \lambda_{n-1} < 3/2$. Thus, by Proposition 3.2 and the hypothesis that $(\star_{n,\Lambda})$ holds, $\hat{\nu}_{-1} = \mu_\Gamma$ for $\Gamma \in \mathcal{S}_{n-1}(\Lambda)$. Hence, as we assume that $(\star\star_{n,\Lambda})$ holds, Γ is a hyperplane through the origin. Therefore, it follows from Brakke's regularity theorem that, for $t<0$ close to 0, $\mathrm{spt}(\mu_t)$ has uniformly bounded curvature near $\mathbf{y}$ and so $\sqrt{-t}\,\Sigma \to \mathcal{X}$ in $C^\infty_{loc}\left(\mathbb{R}^{n+1}\setminus\{0\}\right)$, concluding the proof. □

As a consequence, we establish the following compactness theorem for asymptotically conical self-shrinkers of small entropy.

Corollary 3.4. *Fix $n \geq 3$, $\Lambda \in (\lambda_n, \lambda_{n-1}]$, and $\epsilon_0 > 0$. If both $(\star_{n,\Lambda})$ and $(\star\star_{n,\Lambda})$ hold, then the set*

$$\mathcal{ACS}_n[\Lambda - \epsilon_0] = \{\Sigma : \Sigma \in \mathcal{ACS}_n \text{ and } \lambda[\Sigma] \leq \Lambda - \epsilon_0\}$$

is compact in the $C^\infty_{loc}(\mathbb{R}^{n+1})$ topology.

Proof. Consider a sequence $\Sigma_i \in \mathcal{ACS}_n[\Lambda-\epsilon_0]$ and let $\mu_i = \mu_{\Sigma_i} \in \mathcal{SM}_n[\Lambda-\epsilon_0]$. By the integral compactness theorem for F-stationary varifolds, up to passing to a subsequence, $\mu_i \to \mu$ in the sense of Radon measures. Moreover, by the lower semi-continuity of the entropy, $\mu \in \mathcal{SM}_n[\Lambda - \epsilon_0]$. Hence, by Proposition 3.2, $\mu = \mu_\Sigma$ for $\Sigma \in \mathcal{S}_n[\Lambda - \epsilon_0]$ and so by Allard's regularity theorem, $\Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^{n+1})$. Finally, since each Σ_i is non-compact, so is Σ and so, by Proposition 3.3, $\Sigma \in \mathcal{ACS}_n[\Lambda - \epsilon_0]$, proving the claim. □

Recall that $\mathcal{C}(\Sigma)$ denotes the asymptotic cone of any $\Sigma \in \mathcal{ACS}_n$. Denote the link of the asymptotic cone by $\mathcal{L}(\Sigma) = \mathcal{C}(\Sigma) \cap \mathbb{S}^n$.

Proposition 3.5. *Fix $n \geq 3$, $\Lambda \in (\lambda_n, \lambda_{n-1}]$, and $\epsilon_0 > 0$. If both $(\star_{n,\Lambda})$ and $(\star\star_{n,\Lambda})$ hold, then the set*

$$\mathcal{L}_n[\Lambda - \epsilon_0] = \{\mathcal{L}(\Sigma) : \Sigma \in \mathcal{ACS}_n[\Lambda - \epsilon_0]\}$$

is compact in the $C^\infty(\mathbb{S}^n)$ topology.

Proof. Consider a sequence $L_i \in \mathcal{L}_n[\Lambda - \epsilon_0]$ and let $\Sigma_i \in \mathcal{ACS}_n[\Lambda - \epsilon_0]$ be chosen so that $\mathcal{L}(\Sigma_i) = L_i$ (observe that the Σ_i are uniquely determined by [35, Theorem 1.3]). By Corollary 3.4, up to passing to a subsequence, the $\Sigma_i \to \Sigma \in \mathcal{ACS}_n[\Lambda - \epsilon_0]$. We claim that $L_i \to L = \mathcal{L}(\Sigma)$ in $C^\infty(\mathbb{S}^n)$.

To see this, let $\mu_i = \mu_{\Sigma_i}$ and $\mu = \mu_\Sigma$ be the corresponding elements of $\mathcal{SM}_n[\Lambda - \epsilon_0]$ and let $\mathcal{K}_i$ and $\mathcal{K}$ be the associated Brakke flows. Clearly, $\mu_i \to \mu$ in the sense of measures. Hence, by construction, the $\mathcal{K}_i$ converge in the sense of Brakke flows to $\mathcal{K}$. Since

$$\mathcal{C}(\Sigma) = \left\{\mathbf{x} \in \mathbb{R}^{n+1} : \Theta_{(\mathbf{x},0)}(\mathcal{K}) \geq 1\right\}$$

and likewise for $\mathcal{C}(\Sigma_i)$, we have by Brakke's regularity theorem that $\mathcal{C}(\Sigma_i) \to \mathcal{C}(\Sigma)$ in $C^\infty_{loc}(\mathbb{R}^{n+1}\setminus\{\mathbf{0}\})$, that is $\mathcal{L}(\Sigma_i) \to \mathcal{L}(\Sigma)$ in $C^\infty(\mathbb{S}^n)$ as claimed. □

Combining Corollary 3.4 and Proposition 3.5 gives that

Corollary 3.6. *Fix $n \geq 3$, $\Lambda \in (\lambda_n, \lambda_{n-1}]$, and $\epsilon_0 > 0$. Suppose that ($\star_{n,\Lambda}$) and ($\star\star_{n,\Lambda}$) hold. There is an $R_0 = R_0(n, \Lambda, \epsilon_0)$ and $C_0 = C_0(n, \Lambda, \epsilon_0)$ so that if $\Sigma \in \mathcal{ACS}_n[\Lambda - \epsilon_0]$, then*

(1) $\Sigma \setminus \bar{B}_{R_0}$ is given by the normal graph of a smooth function u over $\mathcal{C}(\Sigma) \setminus \Omega$, where Ω is a compact set, satisfying that for $p \in \mathcal{C}(\Sigma) \setminus \Omega$,

$$|\mathbf{x}(p)| \, |u(p)| + |\mathbf{x}(p)|^2 \left|\nabla_{\mathcal{C}(\Sigma)} u(p)\right| + |\mathbf{x}(p)|^3 \left|\nabla^2_{\mathcal{C}(\Sigma)} u(p)\right| \leq C_0;$$

(2) given $\delta > 0$, there is a $\kappa \in (0,1)$ and $\mathcal{R} > 1$ depending only on n, Λ, ϵ_0 and δ so that if $p \in \Sigma \setminus B_{\mathcal{R}}$ and $r = \kappa |\mathbf{x}(p)|$, then $\Sigma \cap B_r(p)$ can be written as a connected graph of a function v over $T_p\Sigma$ with $|Dv| \leq \delta$.

Proof. For any sequence $\Sigma_i \in \mathcal{ACS}_n[\Lambda - \epsilon_0]$, by Corollary 3.4 and Proposition 3.5, up to passing to a subsequence, $\Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^{n+1})$ for some $\Sigma \in \mathcal{ACS}_n[\Lambda - \epsilon_0]$, and $\mathcal{L}(\Sigma_i) \to \mathcal{L}(\Sigma)$ in $C^\infty(\mathbb{S}^n)$. Let $\mathcal{K}_i$ and $\mathcal{K}$ be the associated Brakke flows to Σ_i and Σ, respectively. As $\Sigma \in \mathcal{ACS}_n$, $\mathcal{K} \lfloor (B_2 \setminus \bar{B}_1) \times [-1, 0]$ is a smooth mean curvature flow. Furthermore, since $\mathcal{K}_i \to \mathcal{K}$, it follows from Brakke's local regularity theorem that Σ_i have uniform curvature decay, more precisely, there exist $R, C > 0$ so that for all i and $p \in \Sigma_i \setminus B_R$,

$$\sum_{k=0}^{2} |\mathbf{x}(p)|^{k+1} \left|\nabla^k_{\Sigma_i} A_{\Sigma_i}(p)\right| \leq C,$$

where A_{Σ_i} is the second fundamental form of Σ_i. As the $\mathcal{C}(\Sigma_i) \to \mathcal{C}(\Sigma)$, by [35, Lemma 2.2] and [4, Proposition 4.2], there exist $R', C' > 0$ so that Items (1) and (2) in the statement hold for all Σ_i. This establishes the corollary by the arbitrariness of the Σ_i. □

4. Singularities of Flows with Small Entropy

Given a Brakke flow $\mathcal{K} = \{\mu_t\}_{t \in I}$ and a point $(\mathbf{x}_0, t_0) \in \mathrm{sing}(\mathcal{K})$ with $t_0 \in \mathring{I}$, a tangent flow $\mathcal{T} \in \mathrm{Tan}_{(\mathbf{x}_0, t_0)}\mathcal{K}$ is of *compact type* if $\mathcal{T} = \{\nu_t\}_{t \in (-\infty, \infty)}$ and $\mathrm{spt}(\nu_{-1})$ is compact. Otherwise, the tangent flow is of *non-compact type*. If every element of $\mathrm{Tan}_{(\mathbf{x}_0,t_0)}\mathcal{K}$ is of compact type, then $(\mathbf{x}_0, t_0)$ is a *compact singularity*. Likewise, if every element of $\mathrm{Tan}_{(\mathbf{x}_0,t_0)}\mathcal{K}$ is of non-compact type, then $(\mathbf{x}_0, t_0)$ is a *non-compact singularity*.

For the remainder of this section, we fix a dimension $n \geq 3$, constants $\Lambda \in (\lambda_n, \lambda_{n-1}]$ and $\epsilon_0 > 0$ and suppose that both ($\star_{n,\Lambda}$) and ($\star\star_{n,\Lambda}$) hold. We further assume that $\Sigma_0 \subset \mathbb{R}^{n+1}$ is a closed connected hypersurface with $\lambda[\Sigma_0] \leq \Lambda - \epsilon_0$ and with the property that the level set flow $L(\Sigma_0)$ is non-fattening and that $(E, \mathcal{K})$ is the pair given by Theorem 2.3.

Proposition 4.1. *Let $(\mathbf{x}_0, t_0) \in \mathrm{sing}(\mathcal{K})$ and $\mathcal{T} \in \mathrm{Tan}_{(\mathbf{x}_0,t_0)}\mathcal{K}$. If $\mathcal{T} = \{\nu_t\}_{t \in (-\infty,\infty)}$ is of non-compact type, then $\nu_{-1} = \mu_\Sigma$ for some $\Sigma \in \mathcal{ACS}_n$ and $\nu_0 = \mu_{\mathcal{C}(\Sigma)}$ for $\mathcal{C}(\Sigma)$ the asymptotic cone of Σ. Moreover, there is a constant $R_1 = R_1(n, \Lambda, \epsilon_0)$ so that*

$$\mathcal{T} \lfloor \left(B_{8R_1} \setminus \bar{B}_{R_1}\right) \times (-1, 1)$$

is a smooth mean curvature flow.

Proof. First, invoking Theorem 2.2 and the monotonicity formula, $\mathcal{T}$ is backwardly self-similar with respect to parabolic scalings about $(\mathbf{0}, 0)$ and $\nu_{-1} \in \mathcal{SM}_n[\Lambda - \epsilon_0]$. Furthermore, by Proposition 3.3, we have $\nu_{-1} = \mu_\Sigma$ for some $\Sigma \in \mathcal{ACS}_n[\Lambda - \epsilon_0]$ and so $\nu_0 = \mu_{\mathcal{C}(\Sigma)}$ for $\mathcal{C}(\Sigma)$ a regular cone. Finally, by Corollary 3.6, the pseudo-locality property of mean curvature flow [25, Theorem 1.5] and Brakke's local regularity theorem, there is an $R > 0$ depending only on n, Λ, ϵ_0 so that

$$\mathcal{T} \lfloor \left(B_{8R} \setminus \bar{B}_R\right) \times (-1, 1)$$

is a smooth mean curvature flow. □

Next we show that

Lemma 4.2. *Each* $(\mathbf{x}_0, t_0) \in \mathrm{sing}(\mathcal{K})$ *is either of compact type or of non-compact type.*

Proof. Suppose that $(\mathbf{x}_0, t_0)$ is not of non-compact type. Then there is a $\mathcal{T} = \{\nu_t\}_{t\in\mathbb{R}} \in \mathrm{Tan}_{(\mathbf{x}_0,t_0)}\mathcal{K}$ of compact type. By the monotonicity formula and Theorem 2.2, $\nu_{-1} \in \mathcal{SM}_n[\Lambda-\epsilon_0]$. Thus it follows from Proposition 3.3 that $\nu_{-1} = \mu_\Sigma$ for some $\Sigma \in \mathcal{S}_n[\Lambda-\epsilon_0]$ and Σ is diffeomorphic to $\mathbb{S}^n$. Hence, by [31, Corollary 1.2], $\mathcal{T}$ is the only element of $\mathrm{Tan}_{(\mathbf{x}_0,t_0)}\mathcal{K}$ and so the singularity $(\mathbf{x}_0, t_0)$ is of compact type, proving the claim. □

We further prove that

Theorem 4.3. *Given* $(\mathbf{x}_0, t_0) \in \mathrm{sing}(\mathcal{K})$*, there exist* $\rho_0 = \rho_0(\mathbf{x}_0, t_0, \mathcal{K}) > 0$ *and* $\alpha = \alpha(n, \Lambda, \epsilon_0) > 1$ *so that:*

(1) If $(\mathbf{x}_0, t_0)$ *is a singularity of non-compact type and* $\rho < \rho_0$*, then*

$$\mathcal{K}\lfloor\big(B_\rho(\mathbf{x}_0) \times (t_0 - \rho^2, t_0] \setminus \{(\mathbf{x}_0, t_0)\}\big)$$

and

$$\mathcal{K}\lfloor\big(B_{2\alpha\rho}(\mathbf{x}_0)\setminus\bar{B}_{\alpha\rho}(\mathbf{x}_0)\big) \times (t_0 - \rho^2, t_0 + \rho^2)$$

are both smooth mean curvature flows.

(2) If $(\mathbf{x}_0, t_0)$ *is a singularity of compact type and* $\rho < \rho_0$*, then*

$$\mathcal{K}\lfloor\big(B_\rho(\mathbf{x}_0) \times (t_0 - \rho^2, t_0 + \rho^2)\setminus \{(\mathbf{x}_0, t_0)\}\big)$$

is a smooth mean curvature flow.

Proof. Without loss of generality, we may assume $(\mathbf{x}_0, t_0) = (\mathbf{0}, 0)$. Suppose that there is a sequence $(\mathbf{x}_i, t_i)$ of points of $\mathrm{sing}(\mathcal{K})$ so that $(\mathbf{x}_i, t_i) \to (\mathbf{0}, 0)$. If $(\mathbf{0}, 0)$ is of non-compact type, we further assume $t_i \leq 0$. Let $r_i^2 = |\mathbf{x}_i|^2 + |t_i|$. Then, up to passing to a subsequence, it follows from Theorem 2.2 that $\mathcal{K}^{(\mathbf{0},0),r_i} \to \mathcal{T}$ in the sense of Brakke flows and $\mathcal{T} = \{\nu_t\}_{t\in\mathbb{R}} \in \mathrm{Tan}_{(\mathbf{0},0)}\mathcal{K}$. Let $\tilde{\mathbf{x}}_i = r_i^{-1}\mathbf{x}_i$ and $\tilde{t}_i = r_i^{-2}t_i$. Then $|\tilde{\mathbf{x}}_i|^2 + |\tilde{t}_i| = 1$, that is, $(\tilde{\mathbf{x}}_i, \tilde{t}_i)$ lies on the unit parabolic sphere in space-time. Thus, up to passing to a subsequence, $(\tilde{\mathbf{x}}_i, \tilde{t}_i) \to (\tilde{\mathbf{x}}_0, \tilde{t}_0)$, where $|\tilde{\mathbf{x}}_0|^2 + |\tilde{t}_0| = 1$. Moreover, the upper semi-continuity of Gaussian density implies $\Theta_{(\tilde{\mathbf{x}}_0,\tilde{t}_0)}(\mathcal{T}) \geq 1$. If $(\mathbf{0}, 0)$ is of non-compact type, then so is $\mathcal{T}$ and $\tilde{t}_0 \leq 0$. It follows from Proposition 4.1 that $(\tilde{\mathbf{x}}_0, \tilde{t}_0)$ is a regular point of $\mathcal{T}$. If $(\mathbf{0}, 0)$ is of compact type, then so is $\mathcal{T}$ and $\nu_{-1} = \mu_\Gamma$ for some $\Gamma \in \mathcal{S}_n(\Lambda)$ by Proposition 3.3. This implies that $\mathcal{T}$ is extinct at time 0 and $\mathrm{sing}(\mathcal{T}) = \{(\mathbf{0}, 0)\}$, again implying that $(\tilde{\mathbf{x}}_0, \tilde{t}_0)$ is a regular point of $\mathcal{T}$. Hence it follows from Brakke's local regularity theorem that for all i sufficiently large, $(\tilde{\mathbf{x}}_i, \tilde{t}_i) \notin \mathrm{sing}(\mathcal{K}^{(\mathbf{0},0),r_i})$, or equivalently, $(\mathbf{x}_i, t_i) \notin \mathrm{sing}(\mathcal{K})$. This is a contradiction. Therefore, for $\rho > 0$ sufficiently small, if $(\mathbf{0}, 0)$ is a non-compact singularity, then

$$\mathcal{K}\lfloor\big(B_\rho \times (-\rho^2, 0] \setminus \{(\mathbf{0}, 0)\}\big)$$

is a smooth mean curvature flow; if $(\mathbf{0}, 0)$ is a compact singularity, then

$$\mathcal{K}\lfloor\big(B_\rho \times (-\rho^2, \rho^2) \setminus \{(\mathbf{0}, 0)\}\big)$$

is a smooth mean curvature flow.

Next we choose α to be $2R_1$ given by Proposition 4.1. Given a sequence ρ_i of positive numbers so that $\rho_i \to 0$, up to passing to a subsequence, $\mathcal{K}^{(\mathbf{0},0),\rho_i}$ converges, in the sense of Brakke flows, to some $\mathcal{T} \in \mathrm{Tan}_{(\mathbf{0},0)}\mathcal{K}$. Again invoking Proposition 4.1,

$$\mathcal{T}\lfloor\big(B_{8R_1} \setminus \bar{B}_{R_1}\big) \times (-1, 1)$$

is a smooth mean curvature flow. Thus, by Brakke's local regularity theorem, for all i sufficiently large,

$$\mathcal{K}^{(\mathbf{0},0),\rho_i}\lfloor\left(B_{4R_1}\setminus\bar{B}_{2R_1}\right)\times(-1,1)$$

is a smooth mean curvature flow, and so is

$$\mathcal{K}\lfloor\left(B_{4R_1\rho_i}\setminus\bar{B}_{2R_1\rho_i}\right)\times(-\rho_i^2,\rho_i^2).$$

Because of the arbitrariness of the choice of the sequence ρ_i, Item (1) follows. □

We obtain a direct consequence of Theorem 4.3.

Corollary 4.4. *The set* $\mathrm{sing}(\mathcal{K})$ *has the following structure:*

(1) For each $t_0>0$, $\mathrm{sing}_{t_0}(\mathcal{K})=\{(\mathbf{x},t_0):(\mathbf{x},t_0)\in\mathrm{sing}(\mathcal{K})\}$ *is finite;*
(2) For each $t_0>0$*, there is a* $\tau=\tau(t_0)>0$ *so that if* $t\in(t_0-\tau,t_0)$*, then* $\mathrm{sing}_t(\mathcal{K})=\emptyset$.

Given a manifold M we say a subset $U\subset M$ is a *smooth domain* if U is open and ∂U is a smooth submanifold.

Theorem 4.5. *There is an* $N=N(\Sigma_0)\in\mathbb{N}$ *and a sequence of closed connected hypersurfaces* $\Sigma^1,\dots,\Sigma^N$ *so that:*

(1) $\Sigma^1=\Sigma_0$*;*
(2) Σ^N *is diffeomorphic to* $\mathbb{S}^n$*;*
(3) For each i *with* $1\le i\le N-1$*, there is an* $m=m(i)\in\mathbb{N}$ *and open connected pair-wise disjoint smooth domains* $U_1^i,\dots,U_{m(i)}^i\subset\Sigma^i$ *and* $V_1^i,\dots,V_{m(i)}^i\subset\Sigma^{i+1}$ *so that:*
  - *There are orientation preserving diffeomorphisms*
$$\bar{\Phi}^i:\Sigma^{i+1}\setminus\cup_{j=1}^{m(i)}V_j^i\to\Sigma^i\setminus\cup_{j=1}^{m(i)}U_j^i;$$
  - *Each* $\bar{U}_j^i$ *is diffeomorphic (as manifolds with boundary) to* $\bar{B}_{R_j^i}\cap\Gamma_j^i$ *where* $\Gamma_j^i\in\mathcal{ACS}_n^*(\Lambda)$ *and* R_j^i *is chosen large enough so that* $\partial B_{R_j^i}\cap\Gamma_j^i$ *is diffeomorphic to* $\mathcal{L}(\Gamma_j^i)$.

Proof. Let us denote the set of compact singularities of $\mathcal{K}$ by $\mathrm{sing}^C(\mathcal{K})$ and the set of non-compact singularities by $\mathrm{sing}^{NC}(\mathcal{K})$. By Lemma 4.2, $\mathrm{sing}(\mathcal{K})=\mathrm{sing}^{NC}(\mathcal{K})\cup\mathrm{sing}^C(\mathcal{K})$. We note that if $X\in\mathrm{sing}^{NC}(\mathcal{K})$, then, by Proposition 3.3, every element of $\mathrm{Tan}_X\mathcal{K}$ is the flow of an element of $\mathcal{ACS}_n$ and so the tangent flows are non-collapsed at time 0 in the sense of [3, Definition 4.11]. Hence, by [3, Propositions 5.2], $\mathrm{sing}^C(\mathcal{K})\neq\emptyset$. In fact, if we define the extinction time of $\mathcal{K}$ to be

$$T(\mathcal{K})=\sup\left\{t:\mathrm{spt}(\mu_t)\neq\emptyset\right\},$$

then

$$\emptyset\neq\left\{\mathbf{x}\in\mathbb{R}^{n+1}:\Theta_{(\mathbf{x}_0,T(\mathcal{K}))}(\mathcal{K})\ge1\right\}=\left\{\mathbf{x}\in\mathbb{R}^{n+1}:(\mathbf{x},T(\mathcal{K}))\in\mathrm{sing}^{\,C}(\mathcal{K})\right\}.$$

It follows from Theorem 4.3 that $\mathrm{sing}^C(\mathcal{K})$ consists of at most a finite number of points.

Observe that if $\mathrm{sing}(\mathcal{K})$ consists of exactly one point X_0, then we can take $N=1$. Indeed, by the above discussion, this singularity must be compact and hence, by Proposition 3.3, there is a $\Gamma\in\mathcal{S}_n(\Lambda)$ diffeomorphic to $\mathbb{S}^n$ so that one of the tangent flows at X_0 is the flow associated to μ_Γ. Let $\left\{\Sigma_t^1\right\}_{t\in[0,T(\mathcal{K}))}$ be the flow with initial hypersurface $\Sigma_0^1=\Sigma^1$. By Brakke's regularity theorem, there is a t near $T(\mathcal{K})$ so that Σ_t is a small normal graph over Γ and hence Σ^1 is diffeomorphic to Γ, verifying the claim.

Now let $\mathrm{ST}(\mathcal{K}) = \{t \in \mathbb{R} : (\mathbf{x}, t) \in \mathrm{sing}(\mathcal{K})\}$ be the set of singular times and define $\mathrm{ST}^C(\mathcal{K})$ and $\mathrm{ST}^{NC}(\mathcal{K})$ in the analogous way. Notice that by Corollary 4.4 there are at most a finite number of singular points associated to each singular time. We observe that as Σ^1 is smooth, there is a $\delta > 0$ so that $\mathrm{ST}(\mathcal{K}) \subset [\delta, T(\mathcal{K})]$. Furthermore, as $\mathrm{sing}(\mathcal{K})$ is a closed set, so is $\mathrm{ST}(\mathcal{K})$.

For each $t \in \mathrm{ST}(\mathcal{K})$, let

$$\rho(t) = \frac{1}{2\alpha} \min \{\rho_0(\mathbf{x}, t, \mathcal{K}) : \mathbf{x} \in \mathrm{sing}_t(\mathcal{K})\} > 0,$$

where $\rho_0(\mathbf{x}, t, \mathcal{K})$ is the constant given by Theorem 4.3. This minimum exists as $\mathrm{sing}_t(\mathcal{K})$ is a finite set by Corollary 4.4. Observe that by Theorem 4.3,

$$B_{\alpha\rho(t)}(\mathbf{x}) \cap B_{\alpha\rho(t)}(\mathbf{x}') = \emptyset \tag{4.1}$$

when $\mathbf{x}, \mathbf{x}'$ are distinct elements of $\mathrm{sing}_t(\mathcal{K})$. Next, choose $\tau(t) \in (0, \rho^2(t))$ so that

$$\mathcal{K}\lfloor \left(\mathbb{R}^{n+1} \setminus \bigcup_{\mathbf{x} \in \mathrm{sing}_t(\mathcal{K})} \bar{B}_{\alpha\rho(t)}(\mathbf{x}) \right) \times (t - \tau(t), t + \tau(t))$$

is a smooth mean curvature flow. Such a τ exists as $\mathrm{sing}(\mathcal{K})$ is a closed set.

As $\mathrm{ST}(\mathcal{K})$ is a closed subset of $[0, T(\mathcal{K})]$, it is a compact set and so the open cover

$$\{(t - \tau(t), t + \tau(t)) : t \in \mathrm{ST}(\mathcal{K})\}$$

of $\mathrm{ST}(\mathcal{K})$ has a finite subcover. That is, there are a finite number of times $t_1, \ldots, t_N \in \mathrm{ST}(\mathcal{K})$, labeled so that $t_i < t_{i+1}$ and chosen so that

$$\mathrm{ST}(\mathcal{K}) \subset \bigcup_{i=1}^{N} (t_i - \tau(t_i), t_i + \tau(t_i)).$$

Furthermore, we can assume that for each i:

(1) For all $j > i$, $t_i - \tau(t_i) < t_j - \tau(t_j)$,
(2) For all $j < i$, $t_i + \tau(t_i) < t_j + \tau(t_j)$, and
(3) For all $j < i < j'$, $t_j + \tau(t_j) < t_{j'} - \tau(t_{j'})$.

As otherwise, we could delete $(t_i - \tau(t_i), t_i + \tau(t_i))$ and still have an open cover. Note that, by the definition of $\tau(t)$, one must have $t_N = T(\mathcal{K})$.

By Theorem 4.3, $\mathrm{ST}(\mathcal{K})$ has empty interior and so we may choose a sequence of points $s_1^\pm, \ldots, s_N^\pm$ with $t_i \in (s_i^-, s_i^+)$, $|s_i^\pm - t_i| < \tau(t_i)$, $s_i^+ \leq s_{i+1}^-$ and so that

$$\left([0, s_1^-] \cup \bigcup_{i=1}^{N-1} [s_i^+, s_{i+1}^-] \right) \cap \mathrm{ST}(\mathcal{K}) = \emptyset.$$

For $1 \leq i \leq N$ set $\Sigma_\pm^i = \mathrm{spt}(\mu_{s_i^\pm})$. By the choice of $s_i^\pm$, each $\Sigma_\pm^i$ is a closed hypersurface and as there are no singular times between s_i^+ and s_{i+1}^- we have for $1 \leq i \leq N-1$ diffeomorphisms $\Phi^i : \Sigma_+^i \to \Sigma_{i+1}^-$ and likewise a diffeomorphism $\Phi^0 : \Sigma^1 \to \Sigma_-^1$. Observe that, *a priori*, the $\Sigma_\pm^i$ need not consist of one component (indeed, Σ_+^N is empty). By Corollary 4.4, $\mathrm{sing}_{t_i}(\mathcal{K})$ is finite for each $1 \leq i \leq N$ write

$$\left\{ \mathbf{x}_i^1, \ldots, \mathbf{x}_i^{M(i)} \right\} = \mathrm{sing}_{t_i}(\mathcal{K})$$

i.e. $\mathbf{x}_i^j$ are the singular points of the flow at time t_i. Up to relabeling, there is an $0 \leq m(i) \leq M(i)$ so that for $1 \leq j \leq m(i)$, $(\mathbf{x}_i^j, t_i) \in \mathrm{sing}^{NC}(\mathcal{K})$ while for $m(i) < j \leq M(i)$, $\mathbf{x}_i^j \in \mathrm{sing}^C(\mathcal{K})$. Set $R^i = \alpha\rho(t_i)$ and, for each $\mathbf{x}_i^j$, let $U_{j,\pm}^i \subset \Sigma_\pm^i$ be the sets

$B_{R^i}(\mathbf{x}_i^j) \cap \Sigma_\pm^i$. By (4.1) for fixed j, these are pairwise disjoint sets and, by Theorem 4.3, these intersections are transverse and so the $\sigma_{j,\pm}^i = \partial U_{j,\pm}^i$ are submanifolds of $\Sigma_\pm^i$. Hence, the $U_{j,\pm}^i$ are smooth pair-wise disjoint domains.

Furthermore, we have that $\bar{U}_{j,-}^i$ are diffeomorphic to $\bar{B}_{R^i} \cap \Gamma_j^i$ for some $\Gamma_j^i \in \mathcal{S}_n$. In particular, for $j > m(i)$ we have that $\bar{U}_{j,-}^i$ is a closed connected hypersurface, while for $1 \leq j \leq m(i)$, $\partial \bar{U}_{j,-}^i$ is non-empty and connected. Hence, for $j > m(i)$, $\bar{U}_{j,+}^i = \emptyset$, while for $1 \leq j \leq m(i)$, $\partial \bar{U}_{j,+}^i$ is non-empty and connected. Furthermore, Theorem 4.3 implies that there are diffeomorphisms (see Appendix A)

$$\Psi^i : \Sigma_-^i \setminus \bigcup_{j=1}^{M(i)} U_{j,-}^i \to \Sigma_+^i \setminus \bigcup_{j=1}^{M(i)} U_{j,+}^i.$$

As Σ^1 is connected and $\Phi^0(\Sigma^1) = \Sigma_-^1$, Σ_-^1 is also connected. As each $\sigma_{j,-}^1$ is connected, we obtain that $\hat{\Sigma}_-^1 = \Sigma_-^1 \setminus \bigcup_{j=1}^{M(1)} U_{j,-}^1$ is connected. Let $\tilde{\Sigma}_+^1$ be the connected component of Σ_+^1 that contains $\Psi^1(\hat{\Sigma}_-^1)$. Inductively, let $\tilde{\Sigma}_+^{i+1} = \Phi^i(\tilde{\Sigma}_-^i)$ and $\hat{\Sigma}_-^{i+1} = \tilde{\Sigma}_-^{i+1} \setminus \bigcup_{j=1}^{M(i+1)} U_{j,-}^{i+1}$ and define $\tilde{\Sigma}_+^{i+1}$ to be the connected component of Σ_+^{i+1} that contains $\Psi^{i+1}(\hat{\Sigma}_-^{i+1})$. It follows inductively that each $\tilde{\Sigma}_\pm^i$ is connected. Let $\tilde{\Phi}^i : \tilde{\Sigma}_+^i \to \tilde{\Sigma}_i^{i+1}$ be the diffeomorphisms given by restricting the Φ^i. To be consistent we also set $\tilde{\Sigma}_-^1 = \Sigma_-^1$ and $\tilde{\Phi}^0 = \Phi^0$.

As all the singularities at time $T(\mathcal{K}) = t_N$ are compact, it follows from [10, Theorem 0.7] that $\tilde{\Sigma}_-^N$ is diffeomorphic to $\mathbb{S}^n$. The theorem now follows by relabelling everything correctly. ☐

5. A Sharpening of [4]

In order to prove Theorem 1.2, we begin with an elementary lemma.

Lemma 5.1. *If* $\mathbf{x}_1, \ldots, \mathbf{x}_{m+1} \in \mathbb{R}^{n+1}$ *is a sequence of points so that*

$$|\mathbf{x}_i - \mathbf{x}_{i+1}| \leq \hat{K}(1 + |\mathbf{x}_i|)^{-1} \tag{5.1}$$

for $1 \leq i \leq m$ *and some* $\hat{K} \geq 0$, *then*

$$|\mathbf{x}_1 - \mathbf{x}_{m+1}| \leq K(m)(1 + |\mathbf{x}_1|)^{-1} \tag{5.2}$$

where $K(m) = (\hat{K} + 1)^m - 1$.

Proof. We proceed by induction on m. The lemma is obviously true when $m = 1$. Suppose (5.2) holds for $m = m'$. Using this induction hypothesis with (5.1) implies that

$$|\mathbf{x}_1 - \mathbf{x}_{m'+2}| \leq |\mathbf{x}_1 - \mathbf{x}_{m'+1}| + |\mathbf{x}_{m'+1} - \mathbf{x}_{m'+2}| \leq K(m')(1 + |\mathbf{x}_1|)^{-1} + \hat{K}(1 + |\mathbf{x}_{m'+1}|)^{-1}.$$

Furthermore, by the induction hypothesis and triangle inequality

$$|\mathbf{x}_1| \leq K(m')(1 + |\mathbf{x}_1|)^{-1} + |\mathbf{x}_{m'+1}|.$$

As $K(m') \geq 0$ and $(1 + |\mathbf{x}_1|)^{-1} \leq 1$, this implies that

$$1 + |\mathbf{x}_1| \leq 1 + K(m') + |\mathbf{x}_{m'+1}| \leq (1 + K(m'))(1 + |\mathbf{x}_{m'+1}|).$$

That is,

$$(1 + |\mathbf{x}_{m'+1}|)^{-1} \leq (1 + K(m'))(1 + |\mathbf{x}_1|)^{-1}.$$

Hence,

$$|\mathbf{x}_1 - \mathbf{x}_{m'+2}| \leq (K(m') + \hat{K}(1 + K(m')))(1 + |\mathbf{x}_1|)^{-1}$$

and, by the induction hypothesis, $K(m') = (\hat{K}+1)^{m'} - 1$ and so setting

$$K(m'+1) = (K(m') + \hat{K}(1 + K(m'))) = (\hat{K}+1)^{m'+1} - 1$$

verifies that (5.2) holds for $m = m'+1$ and finishes the proof. □

We next observe that the proof of the main result of [4, Theorem 0.1] actually allows us to make the following more refined conclusion.

Proposition 5.2. *Fix $n \geq 2$, if $\Sigma \in \mathcal{ACS}_n[\lambda_{n-1}]$, then there is a homeomorphic involution $\phi : \mathbb{S}^n \to \mathbb{S}^n$ which fixes $\mathcal{L}(\Sigma)$, the link of the asymptotic cone, $\mathcal{C}(\Sigma)$, of Σ, and swaps the two components of $\mathbb{S}^n \backslash \mathcal{L}(\Sigma)$.*

Proof. By [4, Theorem 0.1], the link $\mathcal{L}(\Sigma)$ is connected and separates $\mathbb{S}^n$ into two components Ω_+ and Ω_-. In particular, $\mathcal{L}(\Sigma) = \partial\bar{\Omega}_+ = \partial\bar{\Omega}_-$. In order to construct ϕ, it is enough to show the existence of a homeomorphism $\psi : \bar{\Omega}_+ \to \bar{\Omega}_-$ so that

(1) $\psi|_{\mathcal{L}(\Sigma)} : \mathcal{L}(\Sigma) \to \mathcal{L}(\Sigma)$ is the identity map; and;
(2) $\psi|_{\Omega_+} : \Omega_+ \to \Omega_-$ is a homeomorphism.

Indeed, if such a ψ exists, one defines ϕ by

$$\phi(p) = \begin{cases} \psi(p) & p \in \bar{\Omega}_+ \\ \psi^{-1}(p) & p \in \Omega_- \end{cases}$$

To explain the construction of ψ let us first summarize the main objects used in the proof of [4, Theorem 0.1]. First, recall that it is shown there that associated to Σ are two smooth mean curvature flows $\left\{\Gamma_t^{\pm}\right\}_{t\in[-1,0]}$ with Γ_{-1}^+ the normal exponential graph over Σ of a small positive multiple of the lowest eigenfunction of the self-shrinker stability operator of Σ (normalized to be positive) and Γ_{-1}^- to be a small negative multiple of this function. In particular, by choosing the multiple small enough, one can ensure both that Γ_{-1}^+ is the exponential normal graph of some function on Γ_{-1}^- and that Γ_{-1}^- is the exponential normal graph of some function on Γ_{-1}^+. Furthermore, up to relabeling, each $\Gamma^{\pm} = \Gamma_0^{\pm}$ is diffeomorphic to $\Omega^{\pm}$ the components of $\mathbb{S}^n \backslash \mathcal{L}(\Sigma)$. Moreover, these diffeomorphisms, which we denote by $\Pi^{\pm}$, are given by restricting the map

$$\Pi(p) = \frac{\mathbf{x}(p)}{|\mathbf{x}(p)|}$$

to $\Gamma^{\pm}$.

We next use the flow $\left\{\Gamma_t^{\pm}\right\}_{t\in[-1,0]}$ to construct a natural diffeomorphism $\Psi : \Gamma^+ \to \Gamma^-$ which has the property that there is a constant $K > 0$ so that

$$|\mathbf{x}(p) - \mathbf{x}(\Psi(p))| \leq \frac{K}{1 + |\mathbf{x}(p)|}. \tag{5.3}$$

We do so iteratively. Specifically, by [4, Items (1) and (2) of Proposition 4.4 and Proposition 4.5] there is a constant $C_0 > 0$ so that

$$\sup_{t\in[-1,0]} \sup_{\Gamma_t^{\pm}} \left(|A_{\Gamma_t^{\pm}}| + |\nabla_{\Gamma_t^{\pm}} A_{\Gamma_t^{\pm}}| \right) < C_0. \tag{5.4}$$

This, together with [4, Item (3) of Proposition 4.4], implies that there is a $\rho > 0$ so that for each $t \in [-1, 0]$, $\mathcal{T}_\rho(\Gamma_t^{\pm})$ is a regular tubular neighborhood of $\Gamma_t^{\pm}$. It follows from this and (5.4) that there is a $\delta > 0$ so that if $t_1, t_2 \in [-1, 0]$ and $|t_1 - t_2| < \delta$, then $\Gamma_{t_1}^{\pm}$ is a normal exponential graph over $\Gamma^{\pm}_{t_2}$ and vice versa. As such, for all $t_1, t_2 \in [-1, 0]$ with $|t_1 - t_2| < \delta$, there is a diffeomorphism

$$\Psi^{\pm}_{t_2,t_1} : \Gamma^{\pm}_{t_1} \to \Gamma^{\pm}_{t_2}$$

defined by nearest point projection from $\Gamma^{\pm}_{t_1}$ to $\Gamma^{\pm}_{t_2}$. Pick $N \in \mathbb{N}$ so $N\delta > 1$ and choose $0 = s_0 > s_1 > \ldots > s_N = -1$ so that $|s_i - s_{i+1}| < \delta$ and define a diffeomorphism $\Psi^- : \Gamma^-_{-1} \to \Gamma^-$ by

$$\Psi^- = \Psi^-_{s_0,s_1} \circ \Psi^-_{s_1,s_2} \circ \cdots \circ \Psi^-_{s_{N-1},s_N}.$$

Likewise, define a diffeomorphism $\Psi^+ : \Gamma^- \to \Gamma^-_{-1}$ by

$$\Psi^+ = \Psi^+_{s_N,s_{N-1}} \circ \Psi^+_{s_{N-1},s_{N-2}} \circ \cdots \circ \Psi^+_{s_1,s_0}$$

and let $\Psi^{+,-} : \Gamma^+_{-1} \to \Gamma^-_{-1}$ be given by nearest point projection. By construction, this is also a diffeomorphism and so the map

$$\Psi = \Psi^- \circ \Psi^{+,-} \circ \Psi^+$$

is a diffeomorphism $\Psi : \Gamma^+ \to \Gamma^-$.

By construction, if $t_1, t_2 \in [-1, 0]$ and $|t_1 - t_2| < \delta$, then for all $p \in \Gamma^{\pm}_{t_1}$

$$|\mathbf{x}(p) - \mathbf{x}(\Psi^{\pm}_{t_2,t_1}(p))| < \rho. \tag{5.5}$$

Furthermore, [4, Item (1) of Proposition 4.4] implies that for $t \in [-1, 0]$ each $\Gamma^{\pm}_t$ is smoothly asymptotic to $\mathcal{C}(\Sigma)$. In particular, there is a $R > 0$ and functions $u^{\pm}_t : \mathcal{C}(\Sigma)\backslash B_R \to \mathbb{R}$ whose normal exponential graph over $\mathcal{C}(\Sigma)$ sits inside of $\Gamma^{\pm}_t$ and contains $\Gamma^{\pm}_t\backslash B_{2R}$. Moreover, by [4, Item (2) of Proposition 4.2] and [4, Lemma 4.3] there is a constant $K'' > 0$ so that for $p \in \mathcal{C}(\Sigma)\backslash B_R$

$$|u^{\pm}_t(p)| \leq K''(1 + |\mathbf{x}(p)|)^{-1}.$$

Hence, for any $t_1, t_2 \in [-1, 0]$, if $p \in \Gamma^{\pm}_{t_1}\backslash B_{2R}$, then there is a point $p' \in \mathcal{C}(\Sigma)\backslash B_R$ so that

$$|\mathbf{x}(p) - \mathbf{x}(p')| \leq K''(1 + |\mathbf{x}(p')|)^{-1} \tag{5.6}$$

and also a point $p'' \in \Gamma^{\pm}_{t_2}$ so that

$$|\mathbf{x}(p') - \mathbf{x}(p'')| \leq K''(1 + |\mathbf{x}(p')|)^{-1}. \tag{5.7}$$

Hence, if $|t_1 - t_2| < \delta$, then as $\Psi^{\pm}_{t_2,t_1}$ is given by nearest point projection,

$$\begin{aligned} |\mathbf{x}(p) - \mathbf{x}(\Psi^{\pm}_{t_2,t_1}(p))| &\leq |\mathbf{x}(p) - \mathbf{x}(p'')| \\ &\leq |\mathbf{x}(p) - \mathbf{x}(p')| + |\mathbf{x}(p') - \mathbf{x}(p'')| \\ &\leq 2K''(1 + |\mathbf{x}(p')|)^{-1}. \end{aligned}$$

As $K'' > 0$ and $1 + |\mathbf{x}(p')| \geq 1$, (5.6) implies that

$$(1 + |\mathbf{x}(p')|)^{-1} \leq (1 + K'')(1 + |\mathbf{x}(p)|)^{-1}.$$

and so

$$|\mathbf{x}(p) - \mathbf{x}(\Psi^{\pm}_{t_2,t_1}(p))| \leq 2K''(1 + K'')(1 + |\mathbf{x}(p)|)^{-1}.$$

Combining this with (5.5) one obtains that for all $p \in \Gamma^{\pm}_{t_1}$

$$|\mathbf{x}(p) - \mathbf{x}(\Psi^{\pm}_{t_1,t_2}(p))| \leq \hat{K}(1 + |\mathbf{x}(p)|)^{-1}$$

where $\hat{K} = 2K''(1 + K'') + \rho(1 + R)$. By the same arguments, for all $p \in \Gamma^+_{-1}$

$$|\mathbf{x}(p) - \mathbf{x}(\Psi^{+,-}(p))| \leq \hat{K}(1 + |\mathbf{x}(p)|)^{-1}.$$

Hence, it follows from Lemma 5.1, that

$$|\mathbf{x}(p) - \mathbf{x}(\Psi(p))| \leq K(1 + |\mathbf{x}(p)|)^{-1}$$

where $K = ((1+\hat{K})^{2N+2} - 1)$.

To complete the proof set

$$\psi(p) = \begin{cases} \Pi^-(\Psi((\Pi^+)^{-1}(p))) & p \in \Omega_+ \\ p & p \in \partial\Omega_+. \end{cases}$$

We claim that ψ is a homeomorphism. First, note that, by [4, Item (3) of Proposition 4.4], there is an $R > 1$ and $\tilde{C}_1 > 1$ so that if $p \in \Gamma^\pm \setminus B_R$, then

$$\tilde{C}_1^{-1} |\mathbf{x}(p)|^{2\mu} < \mathrm{dist}_{\mathbb{R}^{n+1}}(p, \mathcal{C}(\Sigma)) < \tilde{C}_1 |\mathbf{x}(p)|^{-1}$$

where here $\mu < -1$. Hence,

$$C^{-1}|\mathbf{x}(p)|^{2\mu-1} < \mathrm{dist}_{\mathbb{S}^n}(\Pi^\pm(p), \mathcal{L}(\Sigma)) < C|\mathbf{x}(p)|^{-2} \tag{5.8}$$

where here $C \geq \tilde{C}_1$. Hence, for $q \in \Omega^+$, with $\mathrm{dist}_{\mathbb{S}^n}(q, \mathcal{L}(\Sigma))$ sufficiently small, if we set $q' = (\Pi^+)^{-1}(q) \in \Gamma^+$

$$|\mathbf{x}(q')| \geq C^{\frac{1}{2\mu-1}} \mathrm{dist}_{\mathbb{S}^n}(q, \mathcal{L}(\Sigma))^{\frac{1}{2\mu-1}}.$$

By (5.3),

$$\begin{aligned} ||\mathbf{x}(\Psi(q'))| - |\mathbf{x}(q')|| &\leq |\mathbf{x}(\Psi(q')) - \mathbf{x}(q')| \\ &\leq KC^{-\frac{1}{2\mu-1}} \mathrm{dist}_{\mathbb{S}^n}(q, \mathcal{L}(\Sigma))^{-\frac{1}{2\mu-1}}. \end{aligned}$$

Hence, for $\mathrm{dist}_{\mathbb{S}^n}(q, \mathcal{L}(\Sigma))$ sufficiently small,

$$\mathrm{dist}_{\mathbb{S}^n}(q, \psi(q)) \leq 4KC^{-\frac{1}{2\mu-1}} \mathrm{dist}_{\mathbb{S}^n}(q, \mathcal{L}(\Sigma))^{-\frac{1}{2\mu-1}} |\mathbf{x}(q')|^{-1}.$$

Using (5.8), again gives

$$\mathrm{dist}_{\mathbb{S}^n}(q, \psi(q)) \leq 4KC^{-\frac{2}{2\mu-1}} \mathrm{dist}_{\mathbb{S}^n}(q, \mathcal{L}(\Sigma))^{-\frac{2}{2\mu-1}}.$$

As $\mu < -1$, for any $q_0 \in \mathcal{L}(\Sigma)$, the right hand side goes to 0 as $q \to q_0$ and so ψ is continuous. Finally, as $\bar{\Omega}_+$ is compact and $\bar{\Omega}_-$ is Hausdorff, ψ is a closed map and hence, as ψ is a bijection, it is a homeomorphism. □

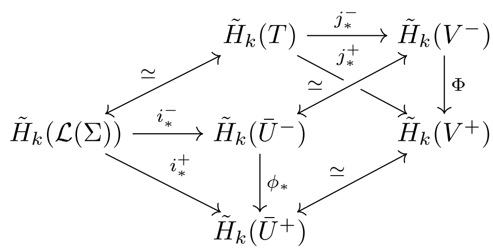
Theorem 1.2 is a standard topological consequence of Proposition 5.2.

Proof. (of Theorem 1.2)

Observe that as $\mathcal{L}(\Sigma)$ is connected, by [4, Theorem 0.1], there are exactly two components of $\mathbb{S}^n \setminus \mathcal{L}(\Sigma)$, which we denote by $U^\pm$. Let $\phi : \mathbb{S}^n \to \mathbb{S}^n$ be the homeomorphism given by Proposition 5.2 so $\phi(U^-) = U^+$. Pick a regular tubular neighborhood $T \subset \mathbb{S}^n$ of $\mathcal{L}(\Sigma)$. We let $V^\pm = U^\pm \cup T$ and observe that $\bar{U}^\pm$, the closure of $U^\pm$, is a retract of $V^\pm$ and that $\mathcal{L}(\Sigma)$ is a retraction of $T = V^- \cap V^+$.

As $\bar{U}^\pm$ is a retraction of $V^\pm$ and $\mathcal{L}(\Sigma)$ is a retraction of T, the natural inclusion maps induce isomorphisms between $\tilde{H}_k(\bar{U}^\pm)$ and $\tilde{H}_k(V^\pm)$ and between $\tilde{H}_k(\mathcal{L}(\Sigma))$ and $\tilde{H}_k(T)$.

As such there is a natural map $\Phi : \tilde{H}_k(V^-) \to \tilde{H}_k(V^+)$ defined by the following diagram,



where here $i^\pm : \mathcal{L}(\Sigma) \to U^\pm$ and $j^\pm : T \to V^\pm$ denote the natural inclusion maps and we used that $\phi \circ i^- = i^+$. As ϕ is a homeomorphism, both ϕ_* and Φ are isomorphisms. This implies that the map

$$J = (j_*^-, -j_*^+) : \tilde{H}_k(T) \to \tilde{H}_k(V^-) \oplus \tilde{H}_k(V^+)$$

is surjective if and only if $\tilde{H}_k(V^-) = \tilde{H}_k(V^+) = \{0\}$. Indeed, if the map is surjective, then for any element $\alpha \in \tilde{H}_k(V^-)$ there is an element $\beta \in \tilde{H}_k(T)$ so that $J(\beta) = (\alpha, 0)$. That is, $j_*^-(\beta) = \alpha$ and $j_*^+(\beta) = 0$. Hence, $0 = j_*^+(\beta) = \Phi(j_*^-(\beta)) = \Phi(\alpha)$. In other words, as Φ is an isomorphism, $\alpha \in \ker(\Phi) = \{0\}$ and so $\tilde{H}_k(V^-) = \{0\}$. The proof that $\tilde{H}_k(V^+) = \{0\}$ is the same. The converse is immediate.

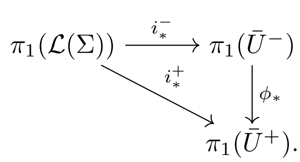
We next recall several standard facts about the reduced homology of manifolds and of manifolds with boundary. First of all, as $\mathcal{L}(\Sigma)$ is a connected, oriented $(n-1)$-dimensional manifold, $\tilde{H}_k(\mathcal{L}(\Sigma)) = \tilde{H}_k(T) = \{0\}$ for $k = 0$ and $k \geq n$ and $\tilde{H}_{n-1}(\mathcal{L}(\Sigma)) = \tilde{H}_{n-1}(T) = \mathbb{Z}$. Likewise, as the $\bar{U}^\pm$ are connected oriented n-manifolds with boundary, $\tilde{H}_k(\bar{U}^\pm) = \tilde{H}_k(V^\pm) = 0$ for $k = 0$ and $k \geq n$.

In order to compute the remaining reduced homology groups, we use the Mayer-Vietoris long exact sequence for the reduced homology of $(V^-, V^+, \mathbb{S}^n)$. This gives the following exact sequences for $k \geq 0$

$$\tilde{H}_{k+1}(\mathbb{S}^n) \longrightarrow \tilde{H}_k(T) \xrightarrow{J} \tilde{H}_k(V^-) \oplus \tilde{H}_k(V^+) \longrightarrow \tilde{H}_k(\mathbb{S}^n). \tag{5.9}$$

As $\tilde{H}_k(\mathbb{S}^n) = \mathbb{Z}$ for $k = n$ and is otherwise $\{0\}$, (5.9) implies that J is surjective for $0 \leq k \leq n-1$. Hence, for these k $\tilde{H}_k(\bar{U}^\pm) = \tilde{H}_k(V^\pm) = \{0\}$ and so the $U^\pm$ are homology n-balls as claimed. As such, (5.9) further implies that $\tilde{H}_k(\mathcal{L}(\Sigma)) = \tilde{H}_k(T) = \{0\}$ for $0 \leq k \leq n-2$ completing the verification that $\mathcal{L}(\Sigma)$ is a homology $(n-1)$-sphere.

To conclude the proof, it is enough, by the Hurewicz theorem, to show that $\pi_1(U^\pm) = 0$. To that end we observe that again, as $\phi \circ i^- = i^+$, the following is commutative



By the Van Kampen theorem and the fact that $\pi_1(\mathbb{S}^n) = \{1\}$ for $n \geq 2$,

$$\{1\} = \pi_1(\bar{U}^+) \star \pi_1(\bar{U}^-)/N \tag{5.10}$$

where $\pi_1(\bar{U}^+) \star \pi_1(\bar{U}^-)$ is the free group generated by $\pi_1(\bar{U}^+)$ and $\pi_1(\bar{U}^-)$ and N is the normal subgroup generated by the set

$$S = \left\{ i_*^+(\alpha)(i_*^-(\alpha))^{-1} : \alpha \in \pi_1(\mathcal{L}(\Sigma)) \right\} \subset \pi_1(\bar{U}^+) \star \pi_1(\bar{U}^-).$$

Pick $\beta \in \pi_1(\bar{U}^+)$, which we think of as an element $\pi_1(\bar{U}^+) \star \pi_1(\bar{U}^-)$. By (5.10),

$$\beta = \prod_{i=1}^{N} i_*^+(\alpha_i)(i_*^-(\alpha_i))^{-1}.$$

where $\alpha_i \in \pi_1(\mathcal{L}(\Sigma))$. Clearly, this can occur if and only if $i_*^-(\alpha_i) = 1$ for all $1 \leq i \leq N$. Hence, $i_*^+(\alpha_i) = \phi_*(i_*^-(\alpha_i)) = 1$, and so $\beta = 1$. That is, $\pi_1(\bar{U}^+) = \{1\}$ and, for the same reason, $\pi_1(\bar{U}^-) = \{1\}$. □

Proof. (of Corollary 1.3)

By Theorem 5.2, $\mathcal{L}(\Sigma)$ is a homology 2-sphere. By the classification of surfaces this means that $\mathcal{L}(\Sigma)$ is diffeomorphic to $\mathbb{S}^2$ and so Alexander's Theorem [1] implies that both components of $\mathbb{S}^3 \setminus \mathcal{L}(\Sigma)$ are diffeomorphic to $\mathbb{R}^3$ proving the claim. □

6. Surgery Procedure

We prove Theorem 1.1 using Corollary 1.3 and Theorem 4.5.

Proof. (of Theorem 1.1)

We first observe that $(\star_{3,\lambda_2})$ holds by [28, Theorem B] and that $(\star\star_{3,\lambda_2})$ holds by [4, Corollary 1.2]. If Σ is (after a translation and dilation) a self-shrinker, then, by [10, Theorem 0.7], Σ is diffeomorphic to $\mathbb{S}^3$, proving the theorem. Otherwise, flow Σ for a small amount of time by the mean curvature flow (using short time existence of for smooth closed initial hypersurfaces) to obtain a hypersurface, Σ', diffeomorphic to Σ and, by Huisken's monotonicity formula, with $\lambda[\Sigma'] < \lambda[\Sigma]$. On the one hand, if the level set flow of Σ' is non-fattening set $\Sigma_0 = \Sigma'$. On the other hand, if the level set flow of Σ' is fattening, then we can take Σ_0 to be a small normal graph over Σ' so $\lambda[\Sigma_0] < \lambda[\Sigma]$ and, because the non-fattening condition is generic, the level set flow of Σ_0 is non-fattening.

Hence, the hypotheses of Section 4 hold and we may apply Theorem 4.5 unconditionally to obtain a family of hypersurfaces $\Sigma^1, \ldots, \Sigma^N$ in $\mathbb{R}^4$. As Σ^N is diffeomorphic to $\mathbb{S}^3$, if $N = 1$, then there is nothing more to show and so we may assume that $N > 1$. We will now show that Σ^{N-1} is diffeomorphic to Σ^N and hence to $\mathbb{S}^3$.

Let us denote by $V = \cup_{j=1}^{m(N-1)} V_j^{N-1}$ and by $\hat{\Sigma}^N = \Sigma^N \setminus V$ and let $U = \cup_{j=1}^{m(N-1)} U_j^{N-1}$ and $\hat{\Sigma}^{N-1} = \Sigma^{N-1} \setminus U$ so $\hat{\Phi}^{N-1} : \hat{\Sigma}^N \to \hat{\Sigma}^{N-1}$ is the orientation preserving diffeomorpism given by Theorem 4.5. By Corollary 1.3, each component of $\bar{U}$ is diffeomorphic to a closed three-ball $\bar{B}^3$. Hence, each component of $\partial\hat{\Sigma}^{N-1}$ and $\partial\hat{\Sigma}^N$ is diffeomorphic to $\mathbb{S}^2$. That is, for $1 \leq i \leq m(N-1)$, ∂V_i^{N-1} is diffeomorphic to $\mathbb{S}^2$ and so, as Σ^N is diffeomorphic to the three-sphere, Alexander's theorem [1] implies that each $\bar{V}_i^{N-1}$ is diffeomorphic to the closed three-ball. Hence, there are orientation preserving diffeomorphisms $\Psi_i^{N-1} : \bar{V}_i^{N-1} \to \bar{U}_i^{N-1}$.

Denote by $\hat{\phi}_i^{N-1} : \partial V_i^{N-1} \to \partial U_i^{N-1}$ the diffeomorphism given by restricting $\hat{\Phi}^{N-1}$ and, likewise, let $\psi_i^{N-1} : \partial V_i^{N-1} \to \partial U_i^{N-1}$ denote the diffeomorphisms given by restricting Ψ_i^{N-1}. Observe, that the orientation of $\hat{\Sigma}^N$ and the orientation on $\bar{V}$ induce opposite orientations on $\partial\bar{V}$. Likewise, the orientation of $\hat{\Sigma}^{N-1}$ and that of $\bar{U}$ induce opposite orientations on $\partial\bar{U}$. By construction, the $\hat{\phi}_i^{N-1}$ preserve the orientations induced from $\hat{\Sigma}^N$ and $\hat{\Sigma}^{N-1}$. Hence, as the orientations induced by $\bar{V}_i^{N-1}$ and $\bar{U}_i^{N-1}$ are opposite to those induced by $\hat{\Sigma}^N$ and $\hat{\Sigma}^{N-1}$, the $\hat{\phi}_i^{N-1}$ also preserve these orientations. The same is true of the ψ_i^{N-1}. As such, $\xi_i^{N-1} = (\psi_i^{N-1})^{-1} \circ \hat{\phi}_i^{N-1} \in \mathrm{Diff}_+(\partial V_i^{N-1})$, where $\mathrm{Diff}_+(M)$ is the space of orientation preserving self-diffeomorphisms of an oriented manifold M (here we may use the orientation on ∂V_i^{N-1} induced by either V_i or $\hat{\Sigma}^N$). By [29] – see also [33] and [7] – the space $\mathrm{Diff}_+(\mathbb{S}^2)$ is path-connected and so any element of $\mathrm{Diff}_+(\mathbb{S}^2)$ extends to an element of $\mathrm{Diff}_+(\bar{B}^3)$. That is, there are diffeomorphism $\Xi_i^{N-1} \in \mathrm{Diff}_+(\bar{V}_i^{N-1})$ that restrict to ξ_i^{N-1} on ∂V_i^{N-1}. Thus, the maps $\hat{\Psi}_i^{N-1} = \Psi_i^{N-1} \circ \Xi_i^{N-1} : \bar{V}_i^{N-1} \to \bar{U}_i^{N-1}$ are diffeomorphisms that agree with $\hat{\Phi}^{N-1}$ on the common boundary.

Define $\Phi^{N-1} : \Sigma^N \to \Sigma^{N-1}$ by

$$\Phi^{N-1}(p) = \begin{cases} \hat{\Phi}^{N-1}(p) & p \in \hat{\Sigma}^N \\ \hat{\Psi}_i^{N-1}(p) & p \in V_i^{N-1}. \end{cases}$$

By construction, this map is a homeomorphism. Moreover, it is a standard procedure to construct a diffeomorphism between Σ^N and Σ^{N-1} by smoothing this map out (see for instance [17, Theorem 8.1.9]). Hence, Σ^{N-1} is diffeomorphic to $\mathbb{S}^3$ and iterating this argument shows that $\Sigma = \Sigma^1$ is diffeomorphic to $\mathbb{S}^3$ as claimed. □

Theorem 1.4 follows from Theorem 1.2, Theorem 4.5 and the Mayer-Vietoris long exact sequence for reduced homology. For completeness, we include a proof of the following standard topological fact which we will need to use.

Lemma 6.1. *Let M be closed n-dimensional manifold and $\Sigma \subset M$ a closed hypersurface. If M is a homology n-sphere and Σ is a homology $(n-1)$-sphere, then each component of $M \backslash \Sigma$ is a homology n-ball.*

Proof. Our hypotheses ensure that both M and Σ are oriented. Hence, Σ is two-sided and so there is an open $U^+ \subset M$ so that $\Sigma = \partial U^+$. Let $U^- = M \backslash \bar{U}^+$. To prove the lemma we will need to compute the Mayer-Vietoris long exact sequence for $(\bar{U}^-, \bar{U}^+, M)$. Strictly speaking, we should "thicken" $\bar{U}^+$ and $\bar{U}^-$ up with a regular tubular neighborhood of $\Sigma = \partial \bar{U}^\pm$ as in the proof of Theorem 1.2, but we leave the details of this to the reader.

The Mayer-Vietoris long exact sequence and the fact that M is a homology n-sphere and Σ is a homology $(n-1)$ sphere gives the sequences

$$\begin{array}{ccccccc} \tilde{H}_{k+1}(M) & \xrightarrow{\partial} & \tilde{H}_k(\Sigma) & \longrightarrow & \tilde{H}_k(\bar{U}^-) \oplus \tilde{H}_k(\bar{U}^-) & \longrightarrow & \tilde{H}_k(M) \\ \updownarrow = & & \updownarrow = & & \updownarrow = & & \updownarrow = \\ \tilde{H}_{k+1}(\mathbb{S}^n) & \xrightarrow{\partial} & \tilde{H}_k(\mathbb{S}^{n-1}) & \longrightarrow & \tilde{H}_k(\bar{U}^-) \oplus \tilde{H}_k(\bar{U}^+) & \longrightarrow & \tilde{H}_k(\mathbb{S}^n). \end{array}$$

For $0 \leq k \leq n-2$ and $k \geq n+1$ this immediately gives that $\tilde{H}_k(\bar{U}_\pm) = \{0\}$. When $k = n-1$, the map ∂ is necessarily generated by $[M] \mapsto [\Sigma]$ where $[M]$ is the fundamental class of M and $[\Sigma]$ is the fundamental class of Σ. In particular, this map is an isomorphism and so we conclude that $\tilde{H}_{n-1}(\bar{U}^\pm) = \{0\}$. For the same reason, $\tilde{H}_n(\bar{U}^\pm) = \{0\}$, which verifies the claim. □

Proof. (of Theorem 1.4)

Arguing as in the first paragraph of the proof of Theorem 1.1, we obtain $\Sigma^1, \ldots, \Sigma^N$ the hypersurfaces given by Theorem 4.5. As Σ^N is diffeomorphic to $\mathbb{S}^n$, it is a homology n-sphere. In particular, if $N = 1$, then there is nothing further to show. As such, we may assume that $N > 1$.

Let us show that Σ^{N-1} is a homology n-sphere. First, set $V = \cup_{j=1}^{m(N-1)} V_j^{N-1}$ and $\hat{\Sigma}^N = \Sigma^N \backslash V$ and let $U = \cup_{j=1}^{m(N-1)} U_j^{N-1}$ and $\hat{\Sigma}^{N-1} = \Sigma^{N-1} \backslash U$. Next observe that, as $\partial U_j^{N-1} = \mathcal{L}(\Gamma_j^{N-1})$ for some $\Gamma_j^{N-1} \in \mathcal{ACS}_n^*(\Lambda)$, Theorem 1.2 implies that each component of $\partial \hat{\Sigma}^{N-1}$ is a homology $(n-1)$-sphere. Hence, as $\partial U = \partial \hat{\Sigma}^{N-1}$ is diffeomorphic to $\partial\hat{\Sigma}^N = \partial V$, we see that each component of $\partial V = \partial\hat{\Sigma}^N$ is a homology $(n-1)$-sphere and so Lemma 6.1 implies that each component of $\bar{V}$ is a homology n-ball.

We may now use the Mayer-Vietoris long exact sequence to compute that $\tilde{H}_k(\hat{\Sigma}^N) = \{0\}$ for $k \neq n-1$ and $\tilde{H}_{n-1}(\hat{\Sigma}^N) = \mathbb{Z}^{m(N-1)-1}$. To see this, consider the Mayer-Vietoris long exact sequence of $(\bar{V}, \hat{\Sigma}^N, \Sigma^N)$. This long exact sequence and the fact that $\bar{V}$ is the union of homology n-balls gives, for $k > 0$, the exact sequences

$$\begin{array}{ccccccc}
\tilde{H}_{k+1}(\Sigma^N) & \xrightarrow{\partial} & \tilde{H}_k(\partial V) & \longrightarrow & \tilde{H}_k(\bar{V}) \oplus \tilde{H}_k(\hat{\Sigma}^N) & \longrightarrow & \tilde{H}_k(\Sigma^N) \\
\updownarrow = & & \updownarrow = & & \updownarrow = & & \updownarrow = \\
\tilde{H}_{k+1}(\mathbb{S}^n) & \xrightarrow{\partial} & \bigoplus_{j=1}^{m(N-1)} \tilde{H}_k(\mathbb{S}^{n-1}) & \longrightarrow & \tilde{H}_k(\hat{\Sigma}^N) & \longrightarrow & \tilde{H}_k(\mathbb{S}^n).
\end{array}$$

Hence, for $1 \leq k \leq n-2$ and $k \geq n+1$, $\tilde{H}_k(\hat{\Sigma}^N) = \{0\}$. When $k = n-1$, the map ∂ is generated by $[\Sigma^N] \mapsto ([\partial V_1^{N-1}], \ldots, [\partial V_{m(N-1)}^{N-1}])$ where $[\Sigma^N]$ is the fundamental class of Σ^N and $[\partial V_j^{N-1}]$ is the fundamental class of ∂V_j^{N-1}. It follows that $\tilde{H}_{n-1}(\hat{\Sigma}^N) = \mathbb{Z}^{m(N-1)-1}$ and, as this map is injective, that $\tilde{H}_n(\hat{\Sigma}^N) = \{0\}$. Finally, as $\hat{\Sigma}^N$ is connected, $\tilde{H}_0(\hat{\Sigma}^N) = \{0\}$ which completes the computation.

By Theorem 4.5, $\hat{\Sigma}^N$ is diffeomorphic to $\hat{\Sigma}^{N-1}$ and so $\tilde{H}_k(\hat{\Sigma}^{N-1}) = 0$ for $k \neq n-1$ and $\tilde{H}_{n-1}(\hat{\Sigma}^{N-1}) = \mathbb{Z}^{m(N-1)-1}$. Furthermore, Theorem 1.2 implies that each component of $\bar{U}$ is contractible. Hence, applying the Mayer-Vietoris long exact sequence to $(\hat{\Sigma}^{N-1}, \bar{U}, \Sigma^{N-1})$ gives, for $k > 0$,

$$\begin{array}{ccccccc}
\tilde{H}_k(\partial\bar{U}) & \longrightarrow & \tilde{H}_k(\bar{U}) \oplus \tilde{H}_k(\hat{\Sigma}^{N-1}) & \longrightarrow & \tilde{H}_k(\Sigma^{N-1}) & \longrightarrow & \tilde{H}_{k-1}(\partial\bar{U}) \\
\updownarrow = & & \updownarrow = & & \updownarrow = & & \updownarrow = \\
\bigoplus_{j=1}^{m(N-1)} \tilde{H}_k(\mathbb{S}^{n-1}) & \longrightarrow & \tilde{H}_k(\hat{\Sigma}^{N-1}) & \longrightarrow & \tilde{H}_k(\Sigma^{N-1}) & \longrightarrow & \bigoplus_{j=1}^{m(N-1)} \tilde{H}_{k-1}(\mathbb{S}^{n-1})
\end{array}$$

In particular, for $1 \leq k \leq n-2$ and $k \geq n+1$, we obtain that $\tilde{H}_k(\Sigma^{N-1}) = \{0\}$. The Mayer-Vietoris long exact sequence further gives the exact sequences

$$\begin{array}{ccccccc}
\tilde{H}_{n-1}(\partial\bar{U}) & \xrightarrow{\delta} & \tilde{H}_k(\bar{U}) \oplus \tilde{H}_k(\hat{\Sigma}^{N-1}) & \longrightarrow & \tilde{H}_{n-1}(\Sigma^{N-1}) & \longrightarrow & \tilde{H}_{n-2}(\partial\bar{U}) \\
\updownarrow = & & \updownarrow = & & \updownarrow = & & \updownarrow = \\
\mathbb{Z}^{m(N-1)} & \xrightarrow{\delta} & \mathbb{Z}^{m(N-1)-1} & \longrightarrow & \tilde{H}_{n-1}(\Sigma^{N-1}) & \longrightarrow & \{0\}.
\end{array}$$

Here δ is given by $(l_1, \ldots, l_{m(N-1)}) \mapsto (l_1 - l_{m(N-1)}, \ldots, l_{m(N-1)-1} - l_{m(N-1)})$. As δ is surjective, it follows that $\tilde{H}_{n-1}(\Sigma^{N-1}) = \{0\}$. Finally, as Σ^{N-1} is an oriented, connected n-dimensional manifold $\tilde{H}_n(\Sigma^{N-1}) = \mathbb{Z}$ and $\tilde{H}_0(\Sigma^{N-1}) = \{0\}$. Hence, Σ^{N-1} is a homology n-sphere.

As our argument only used that Σ^N was a homology n-sphere, we may repeat it to see that each of the Σ^i are homology n-spheres and so conclude that Σ is one as well. □

APPENDIX A.

Fix an open subset $U \subset \mathbb{R}^{n+1}$. A *hypersurface in* U, Σ, is a proper, codimension-one submanifold of U. A *smooth mean curvature flow in* U, S, is a collection of hypersurfaces in U, $\{\Sigma_t\}_{t \in I}$, I an interval, so that

(1) For all $t_0 \in I$ and $p_0 \in \Sigma_{t_0}$ there is a $r_0 = r_0(p_0, t_0)$ and an interval $I_0 = I_0(p_0, t_0)$ with $(p_0, t_0) \in B_{r_0}^{n+1}(p_0) \times I_0 \subset U \times I$;

(2) There is a smooth map $F : B_1^n \times I_0 \to \mathbb{R}^{n+1}$ so $F_t(p) = F(p,t) : B_1^n \to \mathbb{R}^{n+1}$ is a parameterization of $B_{r_0}^{n+1}(p_0) \cap \Sigma_t$; and

(3) $\left(\frac{\partial}{\partial t}F(p,t)\right)^\perp = \mathbf{H}_{\Sigma_t}(F(p,t))$.

It is convenient to consider the *space-time track* of S (also denoted by S):

$$S = \left\{(\mathbf{x}(p), t) \in \mathbb{R}^{n+1} \times \mathbb{R} : p \in \Sigma_t\right\} \subset U \times I. \tag{A.1}$$

This is a smooth submanifold of space-time and is transverse to each constant time hyperplane $\mathbb{R}^{n+1} \times \{t_0\}$. Along the space-time track S, let $\frac{d}{dt}$ be the smooth vector field

$$\left.\frac{d}{dt}\right|_{(p,t)} = \frac{\partial}{\partial t} + \mathbf{H}_{\Sigma_t}(p). \tag{A.2}$$

It is not hard to see that this vector field is tangent to S and the position vector satisfies

$$\frac{d}{dt}\mathbf{x}(p,t) = \mathbf{H}_{\Sigma_t}(p). \tag{A.3}$$

It is a standard fact that if each Σ_t in S is closed, i.e. is compact and without boundary, then there is a smooth map

$$F : M \times I \to \mathbb{R}^{n+1}$$

so each $F_t = F(\cdot, t) : M \to \mathbb{R}^{n+1}$ is a parameterization of Σ_t a closed n-dimensional manifold M. As a consequence, each Σ_t is diffeomorphic to M.

We will need the following generalization of this last fact to manifolds with boundary

Proposition A.1. *Let $\left\{\bar{B}_{2r_1}(\mathbf{x}_1), \ldots, B_{2r_n}(\mathbf{x}_n)\right\}$ be a collection of pairwise disjoint balls in $\mathbb{R}^{n+1}$ and let $U = \mathbb{R}^{n+1} \setminus \bigcup_{i=1}^n \bar{B}_{r_i}(\mathbf{x}_i)$. If $\{\Sigma_t\}_{t\in(-\tau,\tau)}$ is a smooth mean curvature flow in U with the property that*

(1) Each $\hat{\Sigma}_t = \Sigma_t \setminus \bigcup_{i=1}^n B_{2r_i}(\mathbf{x}_i)$ is compact,

(2) For each $1 \le i \le n$, $\partial B_{2r_i}(\mathbf{x}_i)$ intersects Σ_t transversally and non-trivially for all $t \in (-\tau, \tau)$,

then, for any $t_1, t_2 \in (-\tau, \tau)$, $\hat{\Sigma}_{t_1}$ and $\hat{\Sigma}_{t_2}$ are diffeomorphic as compact manifolds with boundary.

Proof. For simplicity, we consider only $n = 1$, $\mathbf{x}_1 = \mathbf{0}$ and $r_1 = \frac{1}{2}$. It is straightforward to extend this argument to the general case. Let S be the space-time track of the flow, so S is a smooth hypersurface in $(\mathbb{R}^{n+1} \setminus \bar{B}_{1/2}) \times (-\tau, \tau)$. As each Σ_t intersects ∂B_1 transversally, it is clear that S meets $\partial B_1 \times (-\tau, \tau)$ transversally. In particular, $\tilde{S} = S \setminus (B_1 \times (-\tau, \tau))$ is a smooth hypersurface with boundary. Let $\tilde{B} = \partial \tilde{S} = \{(p,t) : p \in \partial B_1 \cap \Sigma_t, t \in (-\tau, \tau)\}$.

Without loss of generality we may assume that the given t_1, t_2 satisfy $t_1 < t_2$. Let $\hat{S} = \tilde{S} \cap \left(\mathbb{R}^{n+1} \times [t_1, t_2]\right)$ and $\hat{B} = \tilde{B} \cap \left(\mathbb{R}^{n+1} \times [t_1, t_2]\right)$. Observe that $\hat{S}$ is a compact manifold with corners and $\hat{B}$ is one of its boundary strata. The other two boundary strata are $\hat{\Sigma}_{t_1} \times \{t_1\}$ and $\hat{\Sigma}_{t_2} \times \{t_2\}$.

As ∂B_1 meets each Σ_t transversally (and non-trivially) and $\hat{B}$ is compact, there is an $\epsilon > 0$ so that, for $(p,t) \in \hat{B}$, $|\mathbf{x}^\top(p)| \ge 2\epsilon$. By continuity there is a $\frac{1}{2} > \delta > 0$ so that, for any $t \in [t_1, t_2]$ and $p \in \left(\bar{B}_{1+\delta} \setminus B_{1-\delta}\right) \cap \Sigma_t$, $|\mathbf{x}^\top(p)| \ge \epsilon$. Now let $\eta \in C_0^\infty(\mathbb{R}^{n+1})$ be smooth function with $0 \le \eta \le 1$, $\eta = 1$ on ∂B_1 and $\mathrm{spt}(\eta) \subset \bar{B}_{1+\delta} \setminus B_{1-\delta}$. For $(p,t) \in \hat{S}$ consider the vector

$$\mathbf{V}(p,t) = -\eta(\mathbf{x}(p,t)) \frac{(\mathbf{x}^\perp(p,t) \cdot \mathbf{H}_{\Sigma_t}(p))}{|\mathbf{x}^\top(p,t)|^2} \mathbf{x}^\top(p,t)$$

and observe this gives a smooth vector field on S that restricts to a smooth compactly supported vector field on each Σ_t. Let $\mathbf{W} = \frac{d}{dt} + \mathbf{V}$ which is a smooth vector field on S.

We claim that $\mathbf{W}$ is tangent to $\hat{B}$ and transverse to $\hat{\Sigma}_{t_1} \times \{t_1\} \cup \hat{\Sigma}_{t_2} \times \{t_2\}$. As $\mathbf{V}$ is tangent to $\Sigma_t \times \{t\}$, the transversality of $\mathbf{W}$ follows from the transversality of $\frac{d}{dt}$. This transversality follows immediately from the definition of $\frac{d}{dt}$. To see the tangency note that, by construction, $\hat{B} = \left\{(p,t) \in \hat{S} : |\mathbf{x}(p,t)|^2 = 1\right\}$. For $(p,t) \in \hat{B}$, one computes

$$\begin{aligned}\mathbf{W} \cdot |\mathbf{x}(p,t)|^2 &= 2\mathbf{x}(p,t) \cdot \mathbf{W} \cdot \mathbf{x}(p,t) \\ &= 2\mathbf{x}(p,t) \cdot \mathbf{H}_{\Sigma_t}(p) - 2\eta(\mathbf{x}(p,t)) \frac{(\mathbf{x}^\perp(p,t) \cdot \mathbf{H}_{\Sigma_t}(p))}{|\mathbf{x}^\top(p,t)|^2} \mathbf{x}(p,t) \cdot \mathbf{x}^\top(p,t) \\ &= 0\end{aligned}$$

where the last equality used that $(p,t) \in \hat{B}$ so $\eta(\mathbf{x}(p,t)) = 1$. This verifies the claim.

To conclude the proof observe that, as $\hat{S}$ is compact and $\mathbf{W}$ is tangent to $\hat{B}$ and transverse to $\hat{\Sigma}_{t_1} \times \{t_1\} \cup \hat{\Sigma}_{t_2} \times \{t_2\}$, standard ODE theory gives that for any $P_0 = (p_0, t_0) \in \hat{S}$ the initial value problem

$$\begin{cases} \dot{\gamma}(s) = \mathbf{W}(\gamma(s)) \\ \gamma_{P_0}(0) = P_0 \end{cases}$$

has a unique smooth solution $\gamma_{P_0} : [t_1 - t_0, t_2 - t_0] \to \hat{S}$ which depends smoothly on P_0. These solutions satisfy $t(\gamma_{P_0}(s)) = s + t_0$ and so there is a diffeomorphism $\phi : \Sigma_{t_1} \to \Sigma_{t_2}$ given by $(\phi(p), t_2) = \gamma_{(p,t_1)}(t_2 - t_1)$. □

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