Existence and uniqueness of positive solutions for a class of logistic type elliptic equations in $\mathbb{R}^N$ involving fractional Laplacian

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Abstract

In this paper, we study the existence and uniqueness of positive solutions for the following nonlinear fractional elliptic equation:

$$(-\Delta)^{\alpha} u = \lambda a(x)u - b(x)u^p \quad \text{in } \mathbb{R}^N,$$

where $\alpha \in (0,1)$, $N \geq 2$, $\lambda > 0$, a and b are positive smooth function in $\mathbb{R}^N$ satisfying

$$a(x) \to a^{\infty} > 0 \quad \text{and} \quad b(x) \to b^{\infty} > 0 \quad \text{as } |x| \to \infty.$$

Our proof is based on a comparison principle and existence, uniqueness and asymptotic behaviors of various boundary blow-up solutions for a class of elliptic equations involving the fractional Laplacian.

1 Introduction

A celebrated result of Du and Ma [10] asserts that the uniqueness positive solution of

$$-\Delta u = \lambda u - u^p \quad \text{in } \mathbb{R}^N$$

for $N \geq 1$, $\lambda > 0$ and $p > 1$, is $u \equiv \lambda^{\frac{1}{p-1}}$. Moreover, in [10], the authors also consider the following logistic type equation:

$$-\Delta u = \lambda a(x)u - b(x)u^p \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p > 1$, a and b are positive smooth function in $\mathbb{R}^N$ satisfying

$$a(x) \to a^{\infty} > 0 \quad \text{and} \quad b(x) \to b^{\infty} > 0 \quad \text{as } |x| \to \infty.$$

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Then they proved that problem (1.1) has a unique positive solution for each $\lambda > 0$. A similar problem for quasi-linear operator has been studied by Du and Guo [9].

In the present work, we are interested in understanding whether similar results hold for equations involving a nonlocal diffusion operator, the simplest of which is perhaps the fractional Laplacian. For $\alpha \in (0, 1)$, we study the following fractional elliptic problem:

$$(-\Delta)^{\alpha} u = \lambda u - u^p \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $N \geq 2$. The fractional Laplacian is defined, up to a normalization constant, by

$$(-\Delta)^{\alpha} u(x) = \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2\alpha}} dy, \quad \forall\, x \in \mathbb{R}^N.$$

Our first main result is

Theorem 1.1 *Let $\lambda > 0$. Suppose $u \in C^{2\alpha+\beta}_{loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, \omega)$ for some $\beta > 0$ and $\omega = 1/(1 + |y|^{N+2\alpha})$ is a nonnegative solution of (1.2). Then u must be a constant if p verifies*

$$1 + 2\alpha < p < \frac{1+\alpha}{1-\alpha}. \tag{1.3}$$

Remark 1.1 *We notice that*

$$\frac{N+2\alpha}{N-2\alpha} \leq \frac{1+\alpha}{1-\alpha},$$

if $N \geq 2$.

As in [10] and [9], our proof of this result based on a comparison principle for concave sublinear problems (see Lemma 2.1) and involves boundary blow-up solutions. We use a rather intuitive squeezing method to proof Theorem 1.1 as follows. Denote B_R as a ball centered at the origin with radius R. Then problem

$$\begin{cases} (-\Delta)^{\alpha} v = \lambda v - v^p & \text{in } B_R, \\ v = 0 & \text{in } \mathbb{R}^N \setminus B_R, \end{cases}$$

has a unique positive solution v_R if R is large enough for any fixed $\lambda > 0$. On the other hand, the following boundary blow-up propblem

$$\begin{cases} (-\Delta)^{\alpha} w = \lambda w - w^p & \text{in } B_R, \\ \lim_{x \in B_R, x \to \partial B_R} w(x) = +\infty, & \\ w = g & \text{in } \mathbb{R}^N \setminus \bar{B}_R, \end{cases} \tag{1.4}$$

for some $g \in L^1(\mathbb{R}^N \setminus \bar{B}_R, \omega)$ and $\lambda > 0$, has a positive solution w_R for any $R > 0$. The comparison principle implies that any entire positive solution of (1.2) satisfies $v_R \leq u \leq w_R$ in B_R. Moreover, one can show (see Lemmas 2.2 and 2.3 in Section 2) that both v_R and w_R converge locally uniformly to $\lambda^{\frac{1}{p-1}}$ as $R \to +\infty$. Therefore, $u \equiv \lambda^{\frac{1}{p-1}}$ in $\mathbb{R}^N$.

Next, we make use of Theorem 1.1 to study logistic type fractional elliptic problems with variable coefficients that are asymptotically positive constants. More precisely, we study the following problem

$$(-\Delta)^\alpha u = \lambda a(x)u - b(x)u^p \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where a and b are positive smooth function in $\mathbb{R}^N$. Moreover, we suppose that

$$a(x) \to a^\infty > 0 \quad \text{and} \quad b(x) \to b^\infty > 0 \quad \text{as } |x| \to \infty. \tag{1.6}$$

We can prove that

Theorem 1.2 *Let $\lambda > 0$. Suppose a and b are positive smooth function in $\mathbb{R}^N$ and satisfying (1.6). Then equation (1.5) has a unique positive solution if p verifies (1.3).*

We prove Theorem 1.2 by a similar argument as in the proof of Theorem 1.2, we consider the Dirichlet problem and the boundary blow-up problem in a ball B_R. When R is large, these problems have positive solutions v_R and w_R respectively. By comparison principle, as $R \to \infty$, v_R increase to a minimal positive solution of (1.5) and w_R decrease to a maximal positive solution of (1.5). Therefore, when (1.5) has a unique positive solution, v_R and w_R approximate this unique solution from below and above, respectively.

We mentioned that, in [10] and [9], the existence and uniqueness results hold provided $p > 1$, but in our Theorems 1.1 and 1.2 we require p satisfying (1.3). This is because we will use Perron's method (we refer the reader to User's guide [6] for the presentation of Perron's method which extends to the case of nonlocal equations, see for example [3, 4, 11]) to construct solution of problem 1.4 by applying Proposition 2.2 and choosing

$$\tau = -\frac{2\alpha}{p-1} \in (-1, \tau_0(\alpha))$$

in $V_\tau(x)$ (see (2.13)). This implies

$$p < 1 - \frac{2\alpha}{\tau_0(\alpha)}.$$

Moreover, in [5], the authors proved that $\tau_0(\alpha)$ has a simplicity formula, that is, $\tau_0(\alpha) = \alpha - 1$. Thus, we have

$$p < 1 - \frac{2\alpha}{\tau_0(\alpha)} = \frac{1+\alpha}{1-\alpha}.$$

This article is organized as follows. In Section 2 we present some preliminary lemmas to prove a comparison principle involving the fractional Laplacian, existence and asymptotic behaviors of boundary blow-up solutions. Section 3 is devoted to prove the existence and uniqueness results of problems (1.2) and (1.5), i.e., Theorems 1.1 and 1.2.

2 Preliminary lemmas

In this section, we introduce some lemmas which are useful in the proof of our main results. The first important ingredient is the comparison principle involving the fractional Laplacian which is useful in dealing with boundary blow-up problems.

Lemma 2.1 *(Comparison principle) Suppose that Ω is a bounded domain in $\mathbb{R}^N$, $a(x)$ and $b(x)$ are continuous functions in Ω with $\|a\|_{L^\infty(\Omega)} < \infty$ and $b(x)$ nonnegative and not identity zero. Suppose $u_1, u_2 \in C^{2\alpha+\beta}(\Omega)$ for some $\beta > 0$ are positive in Ω and satisfy*

$$(-\Delta)^\alpha u_1 - a(x)u_1 + b(x)u_1^p \geq 0 \geq (-\Delta)^\alpha u_2 - a(x)u_2 + b(x)u_2^p \quad \text{in } \Omega \tag{2.1}$$

and $\limsup_{x\to\partial\Omega}(u_2 - u_1) \leq 0$ with $u_2 - u_1 \leq 0$ in $\mathbb{R}^N \setminus \bar{\Omega}$, where $p > 1$. Then $u_2 \leq u_1$ in Ω.

In order to prove Lemma 2.1, we need the following proposition.

Proposition 2.1 *For $u \geq 0$ and $v > 0$, we have*

$$L(u,v) \geq 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N,$$

where

$$L(u,v)(x,y) = (u(x) - u(y))^2 - (v(y) - v(x))\left(\frac{u(y)^2}{v(y)} - \frac{u(x)^2}{v(x)}\right).$$

Moreover, the equality holds if and only if $u = kv$ a.e. for some contant k.

We note that Proposition 2.1 is a special case ($p=2$) of Lemma 4.6 in [13] and we omit the proof here.

Proof of Lemma 2.1. Let ϕ_1 and ϕ_2 be nonnegative functions in $C_0^\infty(\Omega)$. By (2.1), we obtain that

$$\int_{\mathbb{R}^{2N}} \frac{(u_1(x)-u_1(y))(\phi_1(x)-\phi_1(y))}{|x-y|^{N+2\alpha}} - \frac{(u_2(x)-u_2(y))(\phi_2(x)-\phi_2(y))}{|x-y|^{N+2\alpha}} dxdy$$
$$\geq \int_\Omega b(x)[u_2^p\phi_2 - u_1^p\phi_1]dx + \int_\Omega a(x)(u_1\phi_1 - u_2\phi_2)dx \quad (2.2)$$

For $\varepsilon > 0$, we denote $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = \varepsilon/2$ and let

$$v_i = \frac{[(u_2+\varepsilon_2)^2 - (u_1+\varepsilon_1)^2]^+}{u_i+\varepsilon_i}, \quad i = 1, 2.$$

By our our assumption, v_i is zero near $\partial\Omega$ and in $\mathbb{R}^N \setminus \bar{\Omega}$. Hence $v_i \in X_0^\alpha(D_0)$, where $D_0 \subset\subset \Omega$ and $X_0^\alpha(D_0) = \{w \in H^\alpha(\mathbb{R}^N) : w = 0 \ a.e \text{ in } \mathbb{R}^N \setminus D_0\}$. In fact, it is clear that $\|v_1\|_{L^2(\mathbb{R}^N)} = \|v_1\|_{L^2(D_0)} \leq C$ and thus it remains to verify that the Gagliardo norm of v_1 in $\mathbb{R}^N$ is bounded by a constant. Using the symmetry of the integral in the Gagliardo norm with respect to x and y and the fact that $v_1 = 0$ in $\mathbb{R}^N \setminus \Omega$, we can split as follows

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_1(x)-v_1(y)|^2}{|x-y|^{N+2\alpha}}dxdy &= \int_\Omega\int_\Omega \frac{|v_1(x)-v_1(y)|^2}{|x-y|^{N+2\alpha}}dxdy \\ &+ 2\int_\Omega \left(\int_{\mathbb{R}^N\setminus\Omega} \frac{|v_1(x)|^2}{|x-y|^{N+2\alpha}}dy\right)dx. \quad (2.3)\end{aligned}$$

Next, we estimate both integrals in the right hand side of (2.3) is finite. We first notice that, for any $y \in \mathbb{R}^N \setminus D_0$,

$$\frac{|v_1(x)|^2}{|x-y|^{N+2\alpha}} = \frac{\chi_{D_0}(x)|v_1(x)|^2}{|x-y|^{N+2\alpha}} \leq \chi_{D_0}(x)|v_1(x)|^2 \sup_{x\in D_0} \frac{1}{|x-y|^{N+2\alpha}}.$$

This implies that

$$\int_\Omega \left(\int_{\mathbb{R}^N\setminus\Omega} \frac{|v_1(x)|^2}{|x-y|^{N+2\alpha}}dy\right)dx \leq \left(\int_{\mathbb{R}^N\setminus\Omega} \frac{1}{dist(y,\partial D_0)^{N+2\alpha}}dy\right)\|v_1\|^2_{L^2(D_0)} < +\infty$$

since $dist(\partial\Omega, \partial D_0) \geq \gamma > 0$ and $N+2\alpha > N$. Hence, the second term in the right hand side of (2.3) is finite by the above inequality. In order to show the first term in the right hand side of (2.3) is also finite, we need the following estimates

$$\begin{aligned}
&\left|\frac{(u_2(x)+\varepsilon_2)^2-(u_1(x)+\varepsilon_1)^2}{u_1(x)+\varepsilon_1}-\frac{(u_2(y)+\varepsilon_2)^2-(u_1(y)+\varepsilon_1)^2}{u_1(y)+\varepsilon_1}\right| \\
&\quad=\left|\frac{(u_2(x)+\varepsilon_2)^2}{u_1(x)+\varepsilon_1}-\frac{(u_2(y)+\varepsilon_2)^2}{u_1(y)+\varepsilon_1}+(u_1(y)-u_1(x))\right| \\
&\quad\le\left|\frac{(u_2(x)+\varepsilon_2)^2}{u_1(x)+\varepsilon_1}-\frac{(u_2(y)+\varepsilon_2)^2}{u_1(y)+\varepsilon_1}\right|+|u_1(y)-u_1(x)| \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{(u_2(x)+\varepsilon_2)^2}{u_1(x)+\varepsilon_1}-\frac{(u_2(y)+\varepsilon_2)^2}{u_1(y)+\varepsilon_1}\right| \\
&=\left|\frac{(u_2(x)+\varepsilon_2)^2-(u_2(y)+\varepsilon_2)^2}{u_1(x)+\varepsilon_1}+\frac{(u_2(y)+\varepsilon_2)^2(u_1(y)-u_1(x))}{(u_1(x)+\varepsilon_1)(u_1(y)+\varepsilon_1)}\right| \\
&\le\frac{u_2(x)+u_2(y)+2\varepsilon_2}{u_1(x)+\varepsilon_1}|u_2(x)-u_2(y)|+\frac{(u_2(x)+\varepsilon_2)^2}{(u_1(x)+\varepsilon_1)(u_1(y)+\varepsilon_1)}|u_1(y)-u_1(x)| \qquad (2.5)
\end{aligned}$$

Combining (2.4) and (2.5), we have

$$\begin{aligned}
&\left|\frac{(u_2(x)+\varepsilon_2)^2-(u_1(x)+\varepsilon_1)^2}{u_1(x)+\varepsilon_1}-\frac{(u_2(y)+\varepsilon_2)^2-(u_1(y)+\varepsilon_1)^2}{u_1(y)+\varepsilon_1}\right| \\
&\le C(\varepsilon_1,\varepsilon_2,\|u_1\|_{L^\infty(\Omega)},\|u_2\|_{L^\infty(\Omega)})(|u_1(y)-u_1(x)|-|u_2(y)-u_2(x)|) \\
&\le \tilde{C}|x-y|^{2\alpha+\beta}.
\end{aligned}$$

In the last inequality of above estimate, we have used the fact $u_1, u_2 \in C^{2\alpha+\beta}(\Omega)$. This implies

$$\int_\Omega\int_\Omega \frac{|v_1(x)-v_1(y)|^2}{|x-y|^{N+2\alpha}}dxdy<+\infty$$

since the following inequality

$$\left|w+(x)-w^+(y)\right|\le|w(x)-w(y)|$$

for all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ and function $w:\mathbb{R}^N\to\mathbb{R}$. Therefore, $v_1\in X_0^\alpha(D_0)$. Similarly, we can show $v_2\in X_0^\alpha(D_0)$. On the other hand, by Theorem 6 in [12], we know that v_i can be approximate arbitrarily closely in the $X_0^\alpha(D_0)$ norm by $C_0^\infty(D_0)$ functions. Hence, we see that (2.2) holds when ϕ_i is replaced by v_i for $i=1,2$.

Denote

$$D(\varepsilon)=\{x\in\Omega:u_2(x)+\varepsilon_2>u_1(x)+\varepsilon_1\}.$$

We notice that the integrands in the right hand side of (2.2) (with $\phi_i = v_i$) vanishing outside $D(\varepsilon)$. Next, we prove the left hand side of (2.2) in nonpositive. We first divide $\mathbb{R}^{2N}$ into four disjoint region as:

$$\begin{aligned}\mathbb{R}^{2N} &= \left[\mathbb{R}^N \setminus D(\varepsilon) \times \mathbb{R}^N \setminus D(\varepsilon)\right] \cup \left[D(\varepsilon) \times \mathbb{R}^N \setminus D(\varepsilon)\right] \\ &\cup \left[\mathbb{R}^N \setminus D(\varepsilon) \times D(\varepsilon)\right] \cup \left[D(\varepsilon) \times D(\varepsilon)\right].\end{aligned}$$

For $(x, y) \in \mathbb{R}^N \setminus D(\varepsilon) \times \mathbb{R}^N \setminus D(\varepsilon)$, we know that $v_i(x) = v_i(y) = 0$, $i = 1, 2$. Therefore,

$$A_1 := \int_{\mathbb{R}^N \setminus D(\varepsilon)} \int_{\mathbb{R}^N \setminus D(\varepsilon)} \frac{\mathcal{L}(u_1, u_2)}{|x-y|^{N+2\alpha}} dxdy = 0,$$

where

$$\mathcal{L}(u_1, u_2) = (u_1(x) - u_1(y))(v_1(x) - v_1(y)) - (u_2(x) - u_2(y))(v_2(x) - v_2(y)).$$

For $(x, y) \in D(\varepsilon) \times \mathbb{R}^N \setminus D(\varepsilon)$, we notice that $v_1(y) = v_2(y) = 0$ and, by the definition of $D(\varepsilon)$,

$$u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1 \quad \text{and} \quad u_2(y) + \varepsilon_2 \leq u_1(y) + \varepsilon_1. \tag{2.6}$$

It follows that

$$\begin{aligned}&\mathcal{L}(u_1, u_2) = [u_1(x) - u_1(y)]\, v_1(x) - [u_2(x) - u_2(y)]\, v_2(x) \\ &= [(u_1(x) + \varepsilon_1) - (u_1(y) + \varepsilon_1)]\, v_1(x) - [(u_2(x) + \varepsilon_2) - (u_2(y) + \varepsilon_2)]\, v_2(x) \\ &= \tfrac{[(u_2(x)+\varepsilon_2)^2 - (u_1(x)+\varepsilon_1)^2]}{(u_1(x)+\varepsilon_1)(u_2(x)+\varepsilon_2)} \cdot [(u_1(x) + \varepsilon_1)(u_2(y) + \varepsilon_2) - (u_1(y) + \varepsilon_1)(u_2(x) + \varepsilon_2)] \\ &\leq 0.\end{aligned}$$

Hence,

$$A_2 = \int_{D(\varepsilon)} \int_{\mathbb{R}^N \setminus D(\varepsilon)} \frac{\mathcal{L}(u_1, u_2)}{|x-y|^{N+2\alpha}} dydx \leq 0.$$

A similar argument implies that

$$A_3 = \int_{R^N \setminus D(\varepsilon)} \int_{D(\varepsilon)} \frac{\mathcal{L}(u_1, u_2)}{|x-y|^{N+2\alpha}} dydx \leq 0.$$

Finally, if $(x, y) \in D(\varepsilon) \times D(\varepsilon)$, it is easy to check that

$$\begin{aligned}\mathcal{L}(u_1, u_2) &= (u_1(x) - u_1(y))(v_1(x) - v_1(y)) - (u_2(x) - u_2(y))(v_2(x) - v_2(y)) \\ &= -(u_1(x) - u_1(y))^2 + (u_1(y) - u_1(x)) \left(\frac{(u_2(y) + \varepsilon_2)^2}{u_1(y) + \varepsilon_1} - \frac{(u_2(x) + \varepsilon_2)^2}{u_1(x) + \varepsilon_1} \right) \\ &- (u_2(x) - u_2(y))^2 + (u_2(y) - u_2(x)) \left(\frac{(u_1(y) + \varepsilon_1)^2}{u_2(y) + \varepsilon_2} - \frac{(u_1(x) + \varepsilon_1)^2}{u_2(x) + \varepsilon_2} \right).\end{aligned}$$

By Proposition 2.1, we know that $\mathcal{L}(u_1, u_2)(x, y) \leq 0$ in $D(\varepsilon) \times D(\varepsilon)$. Therefore,

$$A_4 = \int_{D(\varepsilon)} \int_{D(\varepsilon)} \frac{\mathcal{L}(u_1, u_2)}{|x - y|^{N+2\alpha}} dxdy \leq 0.$$

Summing up these estimates from A_1 to A_4, we know that the left hand side of (2.2) is nonpositive.

On the other hand, as $\varepsilon \to 0$, the first term in the right hand side of (2.2) converges to

$$\int_{D(0)} b(x) \left(u_2^{p-1} - u_1^{p-1}\right) (u_2^2 - u_1^2) dx,$$

while the last term in the right side of (2.2) converges to 0.

Next, we show that $D(0) = \emptyset$. Suppose to the contrary that $D(0) \neq \emptyset$. Since the left side of (2.2) is nonpositive by the estimates from A_1 to A_4 and right hand side of (2.2) tends to 0 as $\varepsilon \to 0$, we easy deduce

$$\int_{\mathbb{R}^{2N}} \frac{\mathbb{L}(u_1, u_2)}{|x - y|^{N+2\alpha}} dxdy = 0,$$

where $\mathbb{L}(u_1, u_2) = \lim_{\varepsilon \to 0} \mathcal{L}(u_1, u_2)$ and

$$\int_{D(0)} b(x) \left(u_2^{p-1} - u_1^{p-1}\right) (u_2^2 - u_1^2) dx = 0.$$

This imply that

$$b \equiv 0 \quad \text{in } D(0)$$

and

$$\mathbb{L}(u_1, u_2) \equiv 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N.$$

Hence, by Proposition 2.1, we know $u_1 = ku_2$ in $D(0)$ for some constant k. Since $b \not\equiv 0$ in Ω, it follows from the above that $D(0) \neq \Omega$. Thus, $D(0) \subset \Omega$, $\partial D(0) \cap \Omega \neq \emptyset$. It follows that the open set $D(0)$ has connected component $\mathcal{G}$ such that $\partial \mathcal{G} \cap \Omega \neq \emptyset$. Now on $\mathcal{G}$, $u_1 = ku_2$. On the other hand, we have $u_1|_{\partial \mathcal{G} \cap \Omega} = u_2|_{\partial \mathcal{G} \cap \Omega} > 0$. Thus, $k = 1$. So we have $u_1 = u_2$ in $\mathcal{G}$, which contradicts $\mathcal{G} \subset D(0)$. Therefore, we must have $D(0) = \emptyset$ and thus $u_1 \geq u_2$ in Ω. We complete the proof of Lemma 2.1. □

By applying this comparison principle together with the Perron's method for the nonlocal equation, we can obtain the following two lemmas.

Lemma 2.2 *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary and $p > 1$. Suppose a and b are smooth positive functions in $\bar{\Omega}$, and let μ_1 denote* *the first eigenvalue of* $(-\Delta)^\alpha u = \mu a(x)u$ *in* Ω *with* $u = 0$ *in* $\mathbb{R}^N \setminus \Omega$. *Then equation*

$$\begin{cases} (-\Delta)^\alpha u = \mu u[a(x) - b(x)u^{p-1}] & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

has a unique positive solution for every $\mu > \mu_1$. *Furthermore, the unique solution* u_μ *satisfies* $u_\mu \to [a(x)/b(x)]^{1/p-1}$ *uniformly in amy compact subset of* Ω *as* $\mu \to +\infty$.

Proof. (Existence) The existence follows from a simple sub- and super-solution argument. In fact, any constant great than or equal to $M = \max_{\bar{\Omega}}[a(x)/b(x)]^{1/(p-1)}$ is a super-solution. Let ϕ be a positive eigenfunction corresponding to μ_1 (for the existence of the first eigenvalue and corresponding eigenfunction has been obtained in [13] and [15]), then for each fixed $\mu > \mu_1$ and small positive ε, $\varepsilon\phi < M$ and is a sub-solution. Therefore, by the sub- and super-solution method (see [14]), there exist at least one positive solution.

(Uniqueness) If u_1 and u_2 are two positive solutions, by Lemma 2.1, we have $u_1 \leq u_2$ and $u_2 \leq u_1$ both hold in Ω. Hence, $u_1 = u_2$. This proves the uniqueness.

(Asymptotic behaviour) Given any compact subset K of Ω and any small $\varepsilon > 0$ such that $\varepsilon < v_0(x) = [a(x)/b(x)]^{1/(p-1)}$ in Ω. Let

$$v_\varepsilon(x) = \begin{cases} v_0(x) + \varepsilon & \text{in K}, \\ l(x) & \text{in } \Omega \setminus \text{K}, \\ 0, & \text{in } \mathbb{R}^{\text{N}} \setminus \Omega, \end{cases}$$

where $l(x)$ is nonnegative function such that v_ε is smooth in Ω and satisfying $D_0 := \text{supp}(\text{v}_\varepsilon) \subset\subset \Omega$. Thus, for any $x \in \Omega$,

$$\begin{aligned} |(-\Delta)^\alpha v_\varepsilon(x)| &\leq \int_{\mathbb{R}^N} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy \\ &= \int_\Omega \frac{|v_\varepsilon(x) - v_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{|v_\varepsilon(x)|}{|x-y|^{N+2\alpha}} dy \\ &\leq \int_\Omega \frac{|v_\varepsilon(x) - v_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy + \left(\int_{\mathbb{R}^N \setminus \Omega} \frac{1}{dist(y, \partial D_0)^{N+2\alpha}} dy \right) \|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \\ &\leq C \end{aligned}$$

for some positive constant $C = C(\varepsilon)$ since v_ε is smooth and $dist(\partial\Omega, \partial D_0) \geq \gamma > 0$. On the other hand, we notice that $v_\varepsilon(a(x) - b(x)v_\varepsilon^{p-1}) \leq -\delta$ in Ω for some positive constant $\delta = \delta(\varepsilon)$. Hence, for all large μ, v_ε is a super-solution of our problem.

On the other hand, let ϕ be a positive eigenfunction corresponding to μ_1. Then we can find a small neighborhood of $\partial\Omega$ in Ω, say U, such that ϕ is very small in U. Therefore, for all $\mu > \mu_1 + 1$, we have

$$(-\Delta)^\alpha \phi = \mu_1 a(x)\phi \le \mu\phi[a(x) - b(x)\phi^{p-1}] \quad \text{in } U. \tag{2.7}$$

By shrinking U further if necessary, we can assume that $\bar{U} \cap K = \emptyset$ and $\phi < v_0 - \varepsilon$ in U. Next, we choose smooth function w_ε as

$$w_\varepsilon(x) = \begin{cases} v_0(x) - \varepsilon & \text{in K}, \\ \phi(x) & \text{in U}, \\ l(x) & \text{in the rest of } \Omega, \\ 0, & \text{in } \mathbb{R}^{\mathrm{N}} \setminus \Omega, \end{cases}$$

where l is a positive function such that w_ε is smooth in Ω and satisfying $l \le v_0 - \varepsilon/2$. Moreover, we let

$$\phi(x) \le w_\varepsilon(x) \quad \text{in } \Omega \tag{2.8}$$

otherwise we choose $\tilde{\phi} = \phi/C$ for some constant $C > 0$ large replace ϕ. Then we can see that, for $x \in \Omega \setminus U$,

$$\begin{aligned} |(-\Delta)^\alpha w_\varepsilon(x)| &\le \int_{\mathbb{R}^N} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy \\ &= \int_\Omega \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy + \int_{\mathbb{R}^N\setminus\Omega} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy \\ &= \int_\Omega \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy + \int_{\mathbb{R}^N\setminus\Omega} \frac{|w_\varepsilon(x)|}{|x-y|^{N+2\alpha}} dy \\ &= \int_\Omega \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{N+2\alpha}} dy + \left(\int_{\mathbb{R}^N\setminus\Omega} \frac{1}{dist(y, \partial U \cap \Omega)^{N+2\alpha}} dy \right) \|w_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \\ &\le C, \end{aligned}$$

for some positive constant $C = C(\varepsilon)$ since $dist(\partial\Omega, \partial U \cap \Omega) \ge \gamma > 0$. Moreover, we know $w_\varepsilon(a(x) - b(x)w_\varepsilon^{p-1}) \ge \delta$ in $\Omega \setminus U$ for some positive constant $\delta = \delta(\varepsilon)$. Therefore,

$$(-\Delta)^\alpha w_\varepsilon \le \mu w_\varepsilon(a(x) - b(x)w_\varepsilon^{p-1}) \quad \text{in } \Omega \setminus U \tag{2.9}$$

for all large μ. For $x \in U$, by (2.7) and (3.7), we have

$$(-\Delta)^\alpha w_\varepsilon(x) = \int_\Omega \frac{w_\varepsilon(x) - w_\varepsilon(y)}{|x-y|^{N+2\alpha}} dy + \int_{\mathbb{R}^N\setminus\Omega} \frac{w_\varepsilon(x)}{|x-y|^{N+2\alpha}} dy$$

$$
\begin{aligned}
&= \int_\Omega \frac{\phi(x)-w_\varepsilon(y)}{|x-y|^{N+2\alpha}}dy + \int_{\mathbb{R}^N\setminus\Omega} \frac{\phi(x)}{|x-y|^{N+2\alpha}}dy \\
&\le \int_\Omega \frac{\phi(x)-\phi(y)}{|x-y|^{N+2\alpha}}dy + \int_{\mathbb{R}^N\setminus\Omega} \frac{\phi(x)}{|x-y|^{N+2\alpha}}dy \\
&= (-\Delta)^\alpha \phi(x) \\
&\le \mu\phi[a(x)-b(x)\phi^{p-1}] \\
&= \mu w_\varepsilon[a(x)-b(x)w_\varepsilon^{p-1}], \qquad (2.10)
\end{aligned}
$$

for $\mu > \mu_1 + 1$. Finally, combining (2.9) and (2.10), we know w_ε is a subsolution of our problem for all large μ.

Since $w_\varepsilon < v_\varepsilon$, we deduce that $w_\varepsilon \le u_\mu < v_\varepsilon$ in Ω. In particular,

$$[a(x)/b(x)]^{1/(p-1)} - \varepsilon \le u_\mu \le [a(x)/b(x)]^{1/(p-1)} + \varepsilon$$

in K for all large μ. Hence, $u_\mu \to [a(x)/b(x)]^{1/p-1}$ as $\mu \to +\infty$ in K, as required. □

Lemma 2.3 *Let Ω, a and b be as in Lemma 2.2. Suppose p verifies (1.3), then equation*

$$
\begin{cases}
(-\Delta)^\alpha u = \mu u[a(x)-b(x)u^{p-1}] & \text{in } \Omega, \\
\lim_{x\in\Omega, x\to\partial\Omega} u = +\infty, & \\
u = g_\mu & \text{in } \mathbb{R}^N\setminus\Omega
\end{cases}
\qquad (2.11)
$$

has at least one positive solution for each $\mu > 0$ if the measurable function g_μ satisfying

$$\int_{\mathbb{R}^N\setminus\Omega} \frac{g_\mu(y)}{1+|y|^{N+2\alpha}}dy \le C, \qquad (2.12)$$

where positive constant C is independent of μ. Furthermore, suppose u_μ is a positive solution of (2.11), then u_μ satisfies $u_\mu \to [a(x)/b(x)]^{1/(p-1)}$ uniformly in amny compact subset of Ω as $\mu \to +\infty$.

We first recall the following result in [4]. Assume that $\delta > 0$ such that the distance function $d(x) = dist(x, \partial\Omega)$ is of C^2 in $A_\delta = \{x \in \Omega : d(x) < \delta\}$ and define

$$
V_\tau(x) = \begin{cases}
l(x), & x \in \Omega \setminus A_\delta, \\
d(x)^\tau, & x \in A_\delta, \\
0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\qquad (2.13)
$$

where τ is a parameter in $(-1, 0)$ and the function l is positive such that V_τ is C^2 in Ω.

Proposition 2.2 *([4], Proposition 3.2) Assume that Ω is a bounded, open subset of $\mathbb{R}^N$ with a C^2 boundary. Then there exists $\delta_1 \in (0,\delta)$ and s constant $C > 1$ shch that if $\tau \in (-1, \tau_0(\alpha))$ where $\tau_0(\alpha)$ is the unique solution of*

$$C(\tau) = \int_0^{+\infty} \frac{\chi_{(0,1)}|1-t|^\tau + (1+t)^\tau - 2}{t^{1+2\alpha}} dt$$

for $\tau \in (-1,0)$ and $\chi_{(0,1)}$ is the characteristic function of the interval $(0,1)$, then

$$\frac{1}{C} d(x)^{\tau-2\alpha} \le -(-\Delta)^\alpha V_\tau(x) \le C d(x)^{\tau-2\alpha}, \quad for\ all\ x \in A_{\delta_1}.$$

Next, we will the existence result in Lemma 2.3 by applying Perrod's method and thus we need to find ordered sub and super-solution of (2.11). As in [4], we begin with a simple lemma that reduce the problem to find them only in A_δ.

Lemma 2.4 *Let Ω, a and b be as in Lemma 2.2. Suppose U and W are order super and sub-solution of (2.11) in the sub-domain A_δ. Then there exists λ large such that $U_\lambda = U + \lambda\eta$ and $W_\lambda = W - \lambda\eta$ are ordered super and sub-solution of (2.11), where $\eta \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \le \eta \le 1$ and $supp(\eta) \subset \Omega \setminus A_\delta$.*

Proof. The proof is similar as Lemma 4.1in [4] and we just need replace $\bar{V}$ in Lemma 4.1in [4] to η for our lemma. So we omit the proof here. □

Now we in position to prove Lemma 2.3.
Proof of Lemma 2.3. (Existence) We define

$$G_\mu(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{\tilde{g}_\mu(x+y)}{|y|^{N+2\alpha}} dy \quad \text{for } x \in \Omega,$$

where

$$\tilde{g}_\mu(x) = \begin{cases} 0, & x \in \Omega, \\ g_\mu(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

We observe that

$$G(x) = -(-\Delta)^\alpha \tilde{g}_\mu(x) \quad \text{for } x \in \Omega.$$

Moreover, we know that G_μ is continuous (see Lemma 2.1 in [4]) and nonnegative in Ω. Therefore, if u is a solution of (2.11), then $u - \tilde{g}_\mu$ is the solution of

$$\begin{cases} (-\Delta)^\alpha u = \mu u[a(x) - b(x)u^{p-1}] + G_\mu(x) & \text{in } \Omega, \\ \lim_{x\in\Omega, x\to\partial\Omega} u = +\infty, & \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{2.14}$$

and vice versa, if u is a solution of (2.14), then $u+\tilde{g}_\mu$ is a solution of (2.11). Next, we will look for solution of (2.14) instead of (2.11).

Define

$$U_\lambda(x)=\lambda V_\tau(x) \quad \text{and} \quad W_\lambda(x)=\lambda W_\tau(x),$$

where $\tau=-2\alpha/(p-1)$. Notice that $\tau=-2\alpha/(p-1)\in(-1,\alpha-1)$, $\tau p=\tau-2\alpha$ and $\tau p<\tau<0$.

By Proposition 2.2, we find that for $x\in A_\delta$ and $\delta>0$ small

$$\begin{aligned}(-\Delta)^\alpha U_\lambda &+ \mu b(x)U_\lambda^p-\mu a(x)U_\lambda-G_\mu(x)\\ &\geq -C\lambda d(x)^{\tau-2\alpha}+\mu b(x)\lambda^p d(x)^{\tau p}-\mu a(x)\lambda d(x)^{\tau}-G_\mu(x)\\ &\geq -C\lambda d(x)^{\tau-2\alpha}+\mu b(x)\lambda^p d(x)^{\tau p}-\mu a(x)\lambda d(x)^{\tau p}-G_\mu(x),\end{aligned}$$

for some $C>0$. Then there exists a large $\lambda>0$ such that U_λ is a super-solution of (2.14) with the first equation in A_δ since G_μ is continuous in Ω. Similarly, by Proposition 2.2, we have that for $x\in A_\delta$ and $\delta>0$ small

$$\begin{aligned}(-\Delta)^\alpha W_\lambda &+ \mu b(x)W_\lambda^p-\mu a(x)W_\lambda-G_\mu(x)\\ &\leq -\frac{\lambda}{C}d(x)^{\tau-2\alpha}+\mu b(x)\lambda^p d(x)^{\tau p}-\mu a(x)\lambda d(x)^{\tau}-G_\mu(x)\\ &\leq -\frac{\lambda}{C}d(x)^{\tau-2\alpha}+\mu b(x)\lambda^p d(x)^{\tau p}\\ &\leq 0,\end{aligned}$$

if $\lambda>0$ small. Here we have used the fact G_μ is nonnegative.

Finally, by using Lemma 2.4, there exists a solution $\tilde{u}_\mu$ of problem (2.14) and thus a solution $u_\mu=\tilde{u}_\mu+\tilde{g}_\mu$ is a solution of (2.11). Moreover, $u_\mu>0$ in Ω. Indeed, since 0 is a sub-solution of (2.11), by Lemma 2.1, we have $u_\mu\geq 0$ in Ω. If $u_\mu(x')=0$ for some points $x'\in\Omega$ and $u_\mu\not\equiv 0$ in $\mathbb{R}^N$, then by the definition of fractional Laplacian $(-\Delta)^\alpha u_\mu(x')<0$ which is a contradiction. Therefore, $u_\mu>0$ in Ω.

(Asymptotic behaviour) Let K be an arbitrary compact subset of Ω, $v_0(x)=[a(x)/b(x)]^{1/(p-1)}$ in Ω and $\varepsilon>0$ any small positive number satisfies $v_0>\varepsilon$ in $\bar{\Omega}$. Define

$$\tilde{w}_\varepsilon(x)=\begin{cases} v_0(x)-\varepsilon+\lambda\eta(x) & \text{in } K,\\ \mu^{-1}d(x)^\tau & \text{in } A_\delta,\\ l(x) & \text{in the rest of } \Omega,\\ 0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

where τ is a parameter in $(-1,0)$, λ and η defined as in Lemma 2.4 and the function l is positive such that w_ε is C^2 in Ω. By a similar argument as

Proposition 3.2 in [4], there exists $\delta_1 \in (0, \delta)$ and constants $c > 0$ and $C > 0$ such that

$$c(1 + \mu^{-1} d(x)^{\tau - 2\alpha}) \le -(-\Delta)^\alpha \tilde{w}_\varepsilon(x) \le C(1 + \mu^{-1} d(x)^{\tau - 2\alpha})$$

for all $x \in A_{\delta_1}$ and $\tau \in (-1, \alpha - 1)$. Hence, for $x \in A_\delta$ and $\delta > 0$ small,

$$\begin{aligned}
(-\Delta)^\alpha \tilde{w}_\varepsilon \;&+\; \mu b(x) \tilde{w}_\varepsilon^p - \mu a(x) \tilde{w}_\varepsilon - G_\mu(x) \\
&\le\; -c\mu^{-1} d(x)^{\tau - 2\alpha} + \mu^{1-p} b(x) d(x)^{\tau p} - a(x) \lambda d(x)^\tau - G_\mu(x) - c \\
&\le\; -c\mu^{-1} d(x)^{\tau - 2\alpha} + \mu^{1-p} b(x) d(x)^{\tau p} \\
&\le\; 0,
\end{aligned}$$

if μ is large enough. Hence, $\tilde{w}_\varepsilon$ is sub-solution in A_δ. By applying Lemma 2.4, we know that $w_\varepsilon = \tilde{w}_\varepsilon - \lambda\eta + \tilde{g}_\mu$ is a sub-solution of problem (2.11) for all large $\mu > 0$.

On the other hand, we define choose a function

$$\tilde{v}_\varepsilon(x) = \begin{cases} v_0(x) + \varepsilon - \lambda\eta & \text{in K}, \\ \mu d(x)^\tau & \text{in } \mathrm{A}_\delta, \\ l(x) & \text{in the rest of } \Omega, \\ 0, & \text{in } \mathbb{R}^{\mathrm{N}} \setminus \Omega, \end{cases}$$

where τ is a parameter in $(-1, 0)$, λ and η defined as in Lemma 2.4 and the function l is positive such that v_ε is C^2 in Ω. By a similar argument as Proposition 3.2 in [4], there exists $\delta_1 \in (0, \delta)$ and constants $c > 0$ and $C > 0$ such that

$$c(1 + \mu d(x)^{\tau - 2\alpha}) \le -(-\Delta)^\alpha \tilde{v}_\varepsilon(x) \le C(1 + \mu d(x)^{\tau - 2\alpha})$$

for all $x \in A_{\delta_1}$ and $\tau \in (-1, \alpha - 1)$. Hence, for $x \in A_\delta$ and $\delta > 0$ small,

$$\begin{aligned}
(-\Delta)^\alpha \tilde{v}_\varepsilon \;&+\; \mu b(x) \tilde{v}_\varepsilon^p - \mu a(x) \tilde{v}_\varepsilon - G_\mu(x) \\
&\ge\; -C\mu d(x)^{\tau - 2\alpha} + \mu^{p+1} b(x) d(x)^{\tau p} - \mu^2 a(x) \lambda d(x)^\tau - G_\mu(x) - C \\
&\ge\; -C\mu d(x)^{\tau - 2\alpha} + \mu^{p+1} b(x) d(x)^{\tau p} - \mu^2 a(x) \lambda d(x)^{\tau p} - G_\mu(x) - C \\
&\ge\; 0,
\end{aligned}$$

if μ is large enough since $\|G_\mu\|_{L^\infty(\bar{\Omega})} \le C$ for some constant $C > 0$ independent of μ by (2.14). Hence, $\tilde{v}_\varepsilon$ is sub-solution in A_δ. By applying Lemma 2.4, we know that $v_\varepsilon = \tilde{v}_\varepsilon + \lambda\eta + \tilde{g}_\mu$ is a super-solution of problem (2.11) for all large $\mu > 0$.

As $w_\varepsilon < v_\varepsilon$ in Ω, we must have $w_\varepsilon \le u_\mu \le v_\varepsilon$ in Ω. This implies that $u_\mu \to v_0$ in K as $\mu \to \infty$, as required. We complete the proof of this Lemma. □

3 Proofs

The main purpose of this section is to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let us first observe that a nonnegative entire solution of (1.2) is either identically zero or positive everywhere. Indeed, if $u(x') = 0$ for some points $x' \in \mathbb{R}^N$ and $u \not\equiv 0$ in $\mathbb{R}^N$, then by the definition of fractional Laplacian $(-\Delta)^\alpha u(x') < 0$ which is a contradiction. Therefore, we only consider positive solution.

Suppose $\lambda > 0$ and let u be an arbitrary positive entire solution of (1.2). We will show that $u(x_0) = \lambda^{1/(p-1)}$ for any point $x_0 \in \mathbb{R}^N$ by using pointwise convergence of Lemmas 2.2 and 2.3.

For any $t > 0$, define

$$u_t(x) = u[x_0 + t(x - x_0)].$$

Then u_t satisfies

$$(-\Delta)^\alpha u = t^{2\alpha}(\lambda u - u^p) \quad \text{in } \mathbb{R}^N.$$

Let B denote the unit ball with center x_0. By Lemma 2.2, for all large t, the problem

$$\begin{cases} (-\Delta)^\alpha v = t^{2\alpha} u(\lambda - u^{p-1}) & \text{in } B, \\ u = 0 & \text{in } \mathbb{R}^N \setminus B \end{cases}$$

has a unique positive solution v_t and $v_t \to \lambda^{1/(p-1)}$ as $t \to \infty$ at $x = x_0 \in B$. By Lemma 2.1, we have that $u_t \geq v_t$ in B and thus

$$u(x_0) = u_t(x_0) \geq v_t(x_0).$$

Letting $t \to \infty$ in the above inequality we conclude that $u(x_0) \geq \lambda^{1/(p-1)}$.

Let w_t be a positive solution of

$$\begin{cases} (-\Delta)^\alpha w = t^{2\alpha} w(\lambda - w^{p-1}) & \text{in } B, \\ \lim_{x \in B, x \to \partial B} w = +\infty, & \\ w = u_t & \text{in } \mathbb{R}^N \setminus \bar{B}. \end{cases}$$

By our assumption, we know that

$$\int_{\mathbb{R}^N} \frac{u_t(x)}{1 + |x|^{N+2\alpha}} dx \leq C, \tag{3.1}$$

where constant $C > 0$ independent of t for t large enough. In fact

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{u_t(x)}{1 + |x|^{N+2\alpha}} dx &= \int_{\mathbb{R}^N} \frac{u(x_0 + t(x - x_0))}{1 + |x|^{N+2\alpha}} dx \\ &= \int_{\mathbb{R}^N} \frac{u(x)}{t^N \left(1 + \left|\frac{x + (t-1)x_0}{t}\right|^{N+2\alpha}\right)} dx. \end{aligned} \tag{3.2}$$

Define function

$$f(t) = t^N \left(1 + \left|\frac{x+(t-1)x_0}{t}\right|^{N+2\alpha}\right),$$

we know $f(1) = 1 + |x|^{N+2\alpha}$ and $f(t) \to +\infty$ as $t \to +\infty$. Then, we can choose t large enough such that $f(t) \geq f(1)$. So by (3.2), for t large enough, we have

$$\int_{\mathbb{R}^N} \frac{u_t(x)}{1+|x|^{N+2\alpha}}dx \leq \int_{\mathbb{R}^N} \frac{u(x)}{1+|x|^{N+2\alpha}}dx \leq C,$$

since $u \in L^1(\mathbb{R}^N, \omega)$.

Then, applying Lemma 2.3, we see that $w_t \to \lambda^{1/(p-1)}$ as $t \to \infty$ at $x = x_0 \in B$. Applying Lemma 2.1, we have that $u_t \leq w_t$ in B and thus

$$u(x_0) = u_t(x_0) \leq w_t(x_0).$$

Letting $t \to \infty$ in the above inequality we conclude that $u(x_0) \leq \lambda^{1/(p-1)}$. Therefore, $u(x_0) = \lambda^{1/(p-1)}$. Since x_0 is arbitrary, we conclude that $u \equiv \lambda^{1/(p-1)}$ in $\mathbb{R}^N$ for $\lambda > 0$, the unique constant solution of (1.2). □

Next, we will extend Theorem 1.1 to similar problem with variable coefficients, that is, Theorem 1.2. We first consider the following equation which is more general than (1.5):

$$(-\Delta)^\alpha u = a(x)u - b(x)u^p, \quad x \in \mathbb{R}^N, \tag{3.3}$$

where $a(x)$ and $b(x)$ are continuous functions in $\mathbb{R}^N$ and satisfying

$$\lim_{|x|\to\infty} a(x) = a^\infty > 0, \quad \lim_{|x|\to\infty} b(x) = b^\infty > 0. \tag{3.4}$$

Here we allow a and b can be change sign which is more general than (1.5).

Theorem 3.1 *Under the above assumptions, if $u \in C^{2\alpha+\beta}_{loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, \omega)$ for some $\beta > 0$ is a positive solution of (3.3) with p verifies (1.3), then*

$$\lim_{|x|\to\infty} u(x) = \left(\frac{a^\infty}{b^\infty}\right)^{\frac{1}{p-1}}.$$

We postpone the proof of Theorem 3.1 and first we use it to prove the following result.

Corollary 3.1 *Under the assumptions in Theorem 3.1, if we further assume that b is a nonnegative, then problem (3.3) has at most one positive solution.*

Proof. Suppose u_1 and u_2 are two positive solutions of (3.3). By Theorem 3.1, we have

$$\lim_{|x|\to\infty}[(1+\varepsilon)u_1 - u_2] = \varepsilon(a^\infty/b^\infty)^{1/(p-1)} > 0$$

for any positive constant ε.

Since b is nonnegative, then $(1+\varepsilon)u_1$ is a super solution of (3.3). Therefore, applying Lemma 2.1 in a large ball to conclude that $(1+\varepsilon)u_1 \geq u_2$ in a large ball. It follows that this is true in all of $\mathbb{R}^N$. Hence, $u_1 \geq u_2$ in $\mathbb{R}^N$ since ε is arbitrary. Similarly, we also can deduce $u_2 \geq u_1$ in $\mathbb{R}^N$. Finally, we must have $u_1 = u_2$ in $\mathbb{R}^N$, that is, (3.3) has at most one positive solution. □

Now we are in the position to prove Theorem 3.1.

Proof of Theorem 3.1. We prove it by a contradiction argument. Assume that there exists a sequence points $x_n \in \mathbb{R}^N$ satisfying $|x_n| \to \infty$ such that $|u(x_n) - (a^\infty/b^\infty)^{1/(p-1)}| \geq \varepsilon_0$ for some constant $\varepsilon_0 > 0$.

We define

$$a_n(x) = a(x_n + x), \quad b_n(x) = b(x_n + x) \quad \text{and } u_n(x) = u(x_n + x).$$

Then u_n satisfies

$$(-\Delta)^\alpha u_n = a_n(x)u_n - b_n(x)u_n^p \quad \text{in } \mathbb{R}^N. \tag{3.5}$$

If we let

$$L_\alpha^n u = -(-\Delta)^\alpha u - (a^\infty - a_n)u,$$

then we can rewrite (3.5) as

$$-L_\alpha^n u_n = a^\infty u_n - b_n(x)u_n^p \quad \text{in } \mathbb{R}^N.$$

Next, we fix a ball $B_r = \{x \in \mathbb{R}^N \mid |x| < r\}$ and consider the following problem

$$\begin{cases} -L_\alpha^n w = a^\infty w - b_n w^p & \text{in } B_r, \\ w = 0 & \text{in } \mathbb{R}^N \setminus B_r. \end{cases} \tag{3.6}$$

By using the variational characterization of the first eigenvalue and (3.4), we see that $\lambda_1(-L_\alpha^n, B_r) \to \lambda_1((-\Delta)^\alpha, B_r)$ as $n \to \infty$, where $\lambda_1(-L_\alpha^n, B_r)$, $\lambda_1((-\Delta)^\alpha, B_r)$ denote the first eigenvalues of $-L_\alpha^n$ and $(-\Delta)^\alpha$ in B_r with Dirichlet boundary conditions in $\mathbb{R}^N \setminus B_r$, respectively. Since we can choose r large enough such that $\lambda_1((-\Delta)^\alpha, B_r) < a^\infty$, then we may assume that $\lambda_1(-L_\alpha^n, B_r) < a^\infty$ for all n. On the other hand, we know that $b_n \to b^\infty$

uniformly in B_r and thus we may also assume that $b_n \geq b^\infty/2$ in B_r for all n.

Let $\phi_n \in X_0^\alpha(B_r)$ be the first eigenfunction corresponding to $\lambda_1(-L_\alpha^n, B_r)$, that is,

$$\begin{cases} (-\Delta)^\alpha \phi_n + (a^\infty - a_n)\phi_n = \lambda_1(-L_\alpha^n, B_r)\phi_n & \text{in } B_r, \\ \phi_n = 0 & \text{in } \mathbb{R}^N \setminus B_r, \end{cases} \tag{3.7}$$

with $\|\phi_n\|_{L^\infty(B_r)} = 1$. By Theorems 1 and 2 in [16] and using (3.4), we know ϕ_n is also a viscosity solution of (3.7). Then by Theorem 2.6 in [8], we have $\phi_n \in C_{loc}^\beta(B_r)$. Then, by Corollary 4.6 in [7], ϕ_n converges uniformly to a ϕ_∞ and ϕ_∞ satisfies

$$\begin{cases} (-\Delta)^\alpha \phi_\infty = \lambda_1((-\Delta)^\alpha, B_r)\phi_\infty & \text{in } B_r, \\ \phi_\infty = 0 & \text{in } \mathbb{R}^N \setminus B_r. \end{cases}$$

in viscosity sense. Next, by a similar argument as Theorem 2.1 in [4], we know $\phi_\infty \in C_{loc}^{2\alpha+\beta}(B_r)$ and is a classical solution. Then ϕ_∞ is the normalized positive eigenfunction corresponding to $\lambda_1((-\Delta)^\alpha, B_r)$.

It is easily to check that $\varepsilon\phi_n$ is a subsolution of (3.6) for every n if we choose ε small enough. Furthermore, $(2a^\infty/b^\infty)^{1/(p-1)}$ is a supersolution of (3.6) for all n. Then (3.6) has a positive solution w_n satisfies $\varepsilon\phi_n \leq w_n \leq (2a^\infty/b^\infty)^{1/(p-1)}$. Then, using the regularity results again, we know w_n converges in $C_{loc}^{2\alpha+\beta}(B_r)$ to some function w satisfying $\varepsilon\phi_\infty \leq v \leq (2a^\infty/b^\infty)^{1/(p-1)}$ and

$$\begin{cases} (-\Delta)^\alpha w = a^\infty w - b^\infty w^p & \text{in } B_r, \\ w = 0 & \text{in } \mathbb{R}^N \setminus B_r. \end{cases}$$

Applying Lemma 2.2, we know the above problem has a unique positive solution. Therefore, $w = w_r$ is uniquely determined and the whole sequence w_n converges to w_r.

By the comparison principle (see Lemma 2.1), we know that

$$u_n \geq w_n \to w_r \quad \text{in } B_r. \tag{3.8}$$

Next, we show u_n has a uniformly bounded in $\mathbb{R}^N$ for all n large enough, that is, there exists a positive constant C independent of n such that $u_n(x_0) \leq C$ for any $x_0 \in \mathbb{R}^N$. We define, for any $t > 0$,

$$u_{t,n}(x) = u_n[x_0 + t(x - x_0)].$$

Then $u_{t,n}$ satisfying

$$(-\Delta)^\alpha u = t^{2\alpha}(\tilde{a}_n u - \tilde{b}_n u^p) \quad \text{in } \mathbb{R}^N,$$

where $\tilde{a}_n(x) = a_n(x_0 + t(x - x_0))$ and $\tilde{b}_n(x) = b_n(x_0 + t(x - x_0))$. On the other hand, since $\tilde{a}_n \to a^\infty$ and $\tilde{b}_n \to b^\infty$ uniformly in B where B denote the unit ball with center x_0, we may assume $\tilde{a}_n \leq 2a^\infty$ and $\tilde{b}_n \geq b^\infty/2$ in B for all n.

We consider the following problem

$$\begin{cases} (-\Delta)^\alpha v = t^{2\alpha}(2a^\infty v - (b^\infty/2)v^p) & \text{in } B, \\ \lim_{x\in B, x\to\partial B} v = +\infty, & \\ v = u_{t,n} & \text{in } \mathbb{R}^N \setminus B. \end{cases} \tag{3.9}$$

As a argument before, we know $u_{t,n} \in L^1(\mathbb{R}^N, \omega)$ for t and n large enough. Thus, by applying Lemma 2.3, we know this problem has at least one positive solution. Let v_t is a solution of (3.9), then $v_t \to (4a^\infty/b^\infty)^{1/(p-1)}$ as $t \to \infty$ at $x = x_0 \in B$. Then the comparison principle deduce that $u_{t,n} \leq v_t$ in B and thus

$$u_n(x_0) = u_{t,n}(x_0) \leq v_t(x_0).$$

Letting $t \to \infty$ in the above inequality we conclude that $u_n(x_0) \leq (4a^\infty/b^\infty)^{1/(p-1)}$ as we required.

Hence, $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for all n large enough, where constant $C > 0$ independent of n. On the other hand, $u_n \in C^{2\alpha+\beta}_{loc}(\mathbb{R}^N)$ implies that u_n converges uniformly to some function u_∞ and

$$(-\Delta)^\alpha u_n \to (-\Delta)^\alpha u_\infty \quad \text{in } B_r$$

is strongly as $n \to +\infty$. Hence, u_∞ is nonnegative and satisfies

$$(-\Delta)^\alpha u = a^\infty u - b^\infty u^p \quad \text{in } B_r.$$

Furthermore, $u_\infty \geq w_r > 0$. Thus u_∞ is a positive solution and $|u_\infty(0) - (a^\infty/b^\infty)^{1/(p-1)}| \geq \varepsilon_0$ due to the choice of x_n.

Choose a sequence $r = r_1 \leq r_2 \leq \cdots \leq r_m \to \infty$ as $m \to \infty$. We can apply the above argument to each r_m and then use a diagonal process to obtain a positive solution U of

$$(-\Delta)^\alpha u = a^\infty u - b^\infty u^p \quad \text{in } \mathbb{R}^N, \tag{3.10}$$

which satisfies $U(0) \geq w_{r_m}(0)$ and $|U(0) - (a^\infty/b^\infty)^{1/(p-1)}| \geq \varepsilon_0$. By changing of variables of the form $x = \theta y$, $\theta \in \mathbb{R}$, then (3.10) can write as

$$(-\Delta)^\alpha v = (a^\infty/b^\infty)v - v^p \quad \text{in } \mathbb{R}^N,$$

where $v(y) = u(x) = u(\theta y)$. In fact, we can choose $\theta = (b^\infty)^{-1/(2\alpha)}$. Then applying Theorem 1.1 to the above equation, we have $v \equiv (a^\infty/b^\infty)^{1/(p-1)}$.

Hence, $u \equiv (a^\infty/b^\infty)^{1/(p-1)}$. This a contradiction. We complete the proof. $\square$

Proof of Theorem 1.2. First, we let $\lambda > 0$. We consider the following eigenvalue problem with weight function:

$$\begin{cases} (-\Delta)^\alpha u = \lambda a(x) u & \text{in } B_r, \\ u = 0 & \text{in } \mathbb{R}^N \setminus B_r. \end{cases}$$

We denote μ_1 be the first eigenvalue of this problem. Since $\mu_1 \to 0$ as $r \to \infty$, we can choose $r_1 > 0$ large enough such that $\mu_1 \le \lambda$ when $r \ge r_1$. So we can choose an increasing sequence $r_1 < r_2 < \cdots < r_n \to \infty$ and consider the following problem

$$\begin{cases} (-\Delta)^\alpha u = \lambda a(x) u - b(x) u^p & \text{in } B_n, \\ u = 0 & \text{in } \mathbb{R}^N \setminus B_n, \end{cases} \tag{3.11}$$

where $B_n = B_{r_n}$. By Lemma 2.2, problem (3.11) has a unique positive solution u_n for each n. Furthermore, by the comparison principle (see Lemma 2.1), we know $u_n \le u_{n+1}$. On the other hand, any positive constant M satisfying $M^{p-1} \ge M_0^{p-1} = \lambda \sup_{\mathbb{R}^N} a(x) / \inf_{\mathbb{R}^N} b(x)$ is a supersolution of (3.11). It follows that $u_n \le M_0$ for all n. Therefore, u_n is increasing in n and $u_\infty(x) = \lim_{n\to\infty} u_n(x)$ is well defined in $\mathbb{R}^N$. Then, u_∞ satisfying (1.5). Since $u_\infty \ge u_n > 0$ in B_n for each n, we know that u_∞ is a positive solution of (1.5). Moreover, by Corollary 3.1, u_∞ is the unique solution of (1.5). We complete the proof. $\square$

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