A STATIONARY PHASE TYPE ESTIMATE

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Abstract. The purpose of this note is to prove a stationary phase estimate well adapted to parameter dependent phases. In particular, no discussion is made on the positions (and behaviour) of critical points, no lower or upper bound on the gradient of the phase is assumed, and the dependence of the constants with respect to derivatives of the phase and symbols is explicit.

For a fixed phase, the stationnary phase lemma (and its simplified version, the stationary phase estimate) is a very well understood tool which provides very good estimates for oscillatory integrals of the type

$$I_{\phi,b}(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(\xi)} b(\xi) d\xi \Rightarrow |I_{\Phi,b}(\lambda)| \leq C\lambda^{-\frac{d}{2}} \tag{1}$$

The method of proof is quite standard and follows the classical path:

(1) Using the non degeneracy of the hessian of the phase, one knows that the critical points are isolated, hence for a compactly supported symbol there are finitely many such critical points.
(2) Away from the critical points, the non stationary estimates (obtained for example by integrating by parts N times with the operator
$$L = \frac{\nabla_\xi \Phi \cdot \nabla_\xi}{i\lambda |\nabla_\xi \Phi|^2},$$
gives an estimate bounded by $C_N \lambda^{-N}$
(3) Near each critical point, performing first a change of variables (the Morse Lemma) to reduce to the case where the phase is quadratic, and then and exact calculation in Fourier variables gives the estimate (1)

When $d = 1$, Van der Corput Lemma provides a very robust estimate.

However, in higher dimensions, the situation is less simple, in particular when considering parameter dependent phases (with parameters living in a non-compact domain), where

(1) even away from the critical points, $\nabla_\xi \Phi$ can degenerate,
(2) the determinant of the hessian can degenerate,
(3) the number of critical points can blow-up.

In view of numerous applications (for example dispersion estimates for solutions to PDE's), a precise control of the behaviour, with respect to the phase and symbol, of the constant C in (1) is necessary. Many robust methods to prove (1) have been developped (see for example [2, 5, 3, 6]). However, it seems that none of these results

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gives an estimate directly applicable to general situations. This was the motivation for this note. Let

$$I(\lambda) = \int_{\mathbf{R}^d} e^{i\lambda\Phi(\xi)} b(\xi)\, d\xi$$

where $\Phi \in C^\infty(\mathbf{R}^d)$ is a real phase, $b \in C_0^\infty(\mathbf{R}^d)$ is a symbol.

We shall set $K = \text{supp}\, b$ and let V be a small open neighborhood of K. We shall assume that

$$(2)\qquad \begin{aligned} &(i) \quad \mathcal{M}_k := \sum_{2\leq|\alpha|\leq k} \sup_{\xi\in V}|D_\xi^\alpha \Phi(\xi)| < +\infty, \quad 2 \leq k \leq d+2,\\ &(ii) \quad \mathcal{N}_l := \sum_{|\alpha|\leq l} \sup_{\xi\in K}|D_\xi^\alpha b(\xi)| < +\infty, \quad l \leq d+1,\\ &(iii) \quad |\det \text{Hess}\ \Phi(\xi)| \geq a_0 > 0, \forall \xi \in V, \end{aligned}$$

where Hess Φ denotes the Hessian matrix of Φ.

Theorem 1. *There exists a constant C such that, for all (Φ, b) satisfying assumptions (2), and for all $\lambda \geq 1$,*

$$(3)\qquad |I(\lambda)| \leq \frac{C}{a_0^{1+d}}\big(1 + \mathcal{M}_{d+2}^{\frac{d}{2}+d^2}\big)\mathcal{N}_{d+1}\lambda^{-\frac{d}{2}}.$$

Remarks 2. 1. We notice that no upper bound (nor lower bound) on $\nabla\Phi$ is required. This is important in particular in the case were the phase Φ depends on parameters. For instance, in some cases the phase Φ is of the form $\Phi(x, y, \xi) = (x-y)\cdot\xi + \phi(x, y, \xi)$ where x, y are in $\mathbf{R}^d$. In these case $\nabla\Phi = x - y + \nabla\phi$ and there is no natural upper nor lower bound for it.

2. Here is another example (see [1]). Assume $\Phi(x, y, \xi) = (x - y)\cdot\xi + t\theta(x, y, \xi)$ where $t \in (0, T)$ and x, y are in $\mathbf{R}^d$. Assume that Φ and b satisfiy $(i), (ii)$ uniformly in (t, x, y) and that $|\det \text{Hess}\ \theta| \geq c > 0$ where c depends only on the dimension d. Then setting $X = \frac{x}{t}, Y = \frac{y}{t}$ we write $i\lambda\Phi = i\lambda t\big((X - Y)\cdot\xi + \theta(t, tX, tY, \xi)\big)$ and we may apply Proposition 1 with $a_0 = c$ and λ replaced by λt. We obtain an estimate of $I(\lambda)$ by $t^{-\frac{d}{2}}\lambda^{-\frac{d}{2}}$ as soon as $t \geq \lambda^{-1}$.

3. The term a_0^{1+d} in (3) can be written as $a_0 a_0^d$ where the factor a_0^d comes from the possible occurence of a_0^{-d} critical points of the phase on the support of b. In the case where Φ has only one non degenerate critical point this term could be avoided. In this direction we have the following result.

Theorem 3. *Assume that Φ and b satisfy the assumptions (2) and that the map*

$$(4)\qquad V \to \mathbf{R}^d, \quad \xi \mapsto \nabla\Phi(\xi), \quad \textit{is injective.}$$

Then one can find $C > 0$ depending only on the dimension d such that

$$(5)\qquad |I(\lambda)| \leq \frac{C}{a_0}\big(1 + \mathcal{M}_{d+2}^{\frac{d}{2}}\big)\mathcal{N}_{d+1}\lambda^{-\frac{d}{2}}.$$

Here are two examples where Theorem 3 appplies.

Examples 4. 1. Assume besides (2) that

$$(6)\qquad \langle \text{Hess}\ \Phi(\xi)X, X\rangle > 0, \quad \forall \xi \in V, \quad \forall X \in \mathbf{R}^d.$$

then (4) is satisfied.

For simplicity we shall assume that the neighborhood V of supp b appearing in (2) (i) is convex. First of all, since the symmetric matrix Hess Φ is a non negative, its eigenvalues are non negative. It follows from the hypothesis (2) (iii) (see (11)) that

$$\langle \text{Hess } \Phi(\xi)X, X\rangle \geq \frac{a_0}{(C_d M_2)^{d-1}}|X|^2, \quad \forall \xi \in V, \quad \forall X \in \mathbf{R}^d. \tag{7}$$

With $\xi, \eta \in V$ we write

$$\begin{aligned}\nabla\Phi(\xi) - \nabla\Phi(\eta) &= \int_0^1 \frac{d}{ds}\big[\nabla\Phi(s\xi + (1-s)\eta)\big]\, ds,\\ &= \int_0^1 \text{Hess } \Phi(s\xi + (1-s)\eta)\cdot(\xi-\eta)\, ds.\end{aligned}$$

It follows from (7) that

$$\langle \nabla\Phi(\xi) - \nabla\Phi(\eta), \xi - \eta\rangle \geq \frac{a_0}{(C_d M_2)^{d-1}}|\xi-\eta|^2$$

from which we deduce that

$$\frac{a_0}{(C_d M_2)^{d-1}}|\xi-\eta| \leq |\nabla\Phi(\xi) - \nabla\Phi(\eta)|$$

which completes the proof.

2. Let A be a real, symmetric, non singular $d\times d$ matrix and Ψ be a smooth phase such that $\mathcal{M}_{d+2}(\Psi) < +\infty$. Set $\Phi(\xi) = \frac{1}{2}\langle A\xi, \xi\rangle + \varepsilon\Psi(\xi)$. Then if ε is small enough the assumptions in Theorem 3 are satisfied.

Remark 5. Notice that the estimates (3), (5) do not seem to be optimal with respect to the power of a_0 since according to the usual stationnary phase method one could expect to have $a_0^{-\frac{1}{2}}$ in the right hand side.

Actually it is sufficient to prove the following weaker inequality.

Theorem 6. *1. Under the hypotheses of Theorem 1 there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for every $\lambda \geq 1$*

$$|I(\lambda)| \leq \mathcal{F}(\mathcal{M}_{d+2})\mathcal{N}_{d+1}\frac{1}{a_0^{1+d}}\lambda^{-\frac{d}{2}}. \tag{8}$$

2. Under the hypotheses of Theorem 3 there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that for every $\lambda \geq 1$

$$|I(\lambda)| \leq \mathcal{F}(\mathcal{M}_{d+2})\mathcal{N}_{d+1}\frac{1}{a_0}\lambda^{-\frac{d}{2}}. \tag{9}$$

Proof of Theorems 1 and 3 given Theorem 6. We assume that (5) is proved and our goal is to deduce that (3) holds with $C = \mathcal{F}(1)$. Set $t = 1 + \mathcal{M}_{d+2}$ and consider $\lambda \geq 1$. Since $t\lambda \geq 1$ we can apply (8) with (λ, Φ) replaced with $(t\lambda, \Phi(\xi)/t)$ to deduce that

$$\begin{aligned}\left|\int_{\mathbf{R}^d} e^{i\lambda\Phi(\xi)}b(\xi)\, d\xi\right| &= \left|\int_{\mathbf{R}^d} e^{it\lambda\frac{\Phi(\xi)}{t}}b(\xi)\, d\xi\right|\\ &\leq \mathcal{F}\left(\mathcal{M}_{d+2}\left(\frac{\Phi}{t}\right)\right)\mathcal{N}_{d+1}\frac{1}{(a_0/t^d)^{1+d}}(t\lambda)^{-\frac{d}{2}}\\ &\leq \mathcal{F}\left(\frac{\mathcal{M}_{d+2}(\Phi)}{t}\right)\mathcal{N}_{d+1}\frac{1}{a_0^{1+d}}\lambda^{-\frac{d}{2}}t^{\frac{d}{2}+d^2},\end{aligned}$$

which yields the wanted estimate. The case 2. is analogue. □

We are left with the proof of Theorem 6. We begin by some preliminaries.

0.1. **Preliminaries.** In that follows we shall denote by C_d a positive constant depending only on the dimension d and by $\mathcal{F}$ a non decreasing function from $\mathbf{R}^+$ to $\mathbf{R}^+$ which can change from line to line.

Point 1. First of all we may assume that

$$\lambda^{\frac{1}{2}} a_0 \geq 1. \tag{10}$$

Indeed if $\lambda^{\frac{1}{2}} a_0 \leq 1$ then $1 \leq (\lambda^{\frac{1}{2}} a_0)^{-d}$ and we write

$$|I(\lambda)| \leq \|b\|_{L^1(\mathbf{R}^d)} \leq \|b\|_{L^1(\mathbf{R}^d)} \frac{a_0}{a_0} \frac{1}{(\lambda^{\frac{1}{2}} a_0)^d} \leq C\mathcal{M}_2^d \|b\|_{L^1(\mathbf{R}^d)} \frac{1}{a_0^{1+d}} \lambda^{-\frac{d}{2}}$$

since by (2), (iii) we have $a_0 \leq C\mathcal{M}_2^d$.

Point 2. Set $H = \mathrm{Hess}\Phi$. By (i) the eingenvalues $(\lambda_j)_{j=1,\ldots,d}$ of H are bounded by $C_d\mathcal{M}_2$. It follows from (iii) that

$$|\lambda_j| \geq \frac{a_0}{(C_d\mathcal{M}_2)^{d-1}}, \quad 1 \leq j \leq d. \tag{11}$$

Therefore

$$|H(\xi)X| \geq \frac{a_0}{(C_d\mathcal{M}_2)^{d-1}} |X|, \quad \forall \xi \in V, \quad \forall X \in \mathbf{R}^d. \tag{12}$$

We shall use the Taylor formula

$$\nabla\Phi(\xi) = \nabla\Phi(\eta) + \mathrm{Hess}\ \Phi(\eta)(\xi - \eta) + R, \quad |R| \leq C_d'\mathcal{M}_3|\xi - \eta|^2. \tag{13}$$

Lemma 7. *Let $\delta > 0$ be defined by*

$$\delta = \frac{a_0}{12C_d'\mathcal{M}_3(C_d\mathcal{M}_2)^{d-1}}. \tag{14}$$

where C_d, C_d' have been defined above. Let $\xi^* \in$ supp b Then,

(15) *on the ball $B(\xi^*, \delta)$ the map $\xi \mapsto \nabla\Phi(\xi)$ is injective.*

Proof. Indeed by (13) and (12) if $\xi, \eta \in B(\xi^*, \delta)$ we can write

$$|\nabla\Phi(\xi) - \nabla\Phi(\eta)| \geq \Big(\frac{a_0}{(C_d\mathcal{M}_2)^{d-1}} - \frac{a_0}{6(C_d\mathcal{M}_2)^{d-1}}\Big)|\xi - \eta| \geq \frac{5a_0}{6(C_d\mathcal{M}_2)^{d-1}}|\xi - \eta|.$$

□

Let $(\xi_j^*)_{j=1,\ldots,J} \subset$ supp b, such that supp $b \subset \cup_{j=1}^J B(\xi_j^*, \delta)$. Taking a partition of unity (χ_j) and setting $b_j = \chi_j b$ we have

$$I(\lambda) = \sum_{j=1}^J I_j(\lambda), \quad I_j(\lambda) = \int_{\mathbf{R}^d} e^{i\lambda\Phi(\xi)} b_j(\xi)\, d\xi. \tag{16}$$

Notice that χ_j can be taken of the form $\chi_0(\frac{\xi - \xi_j^*}{\delta})$ so that

$$|\partial_\xi^\alpha \chi_j(\xi)| \leq C\delta^{-|\alpha|}. \tag{17}$$

Notice also that $J \leq C_d\delta^{-d}$.

Lemma 8. *Let $i \in \{1, \dots, d\}$ and $A_i = \frac{\partial_i \Phi}{|\nabla\Phi|^2}$. One can find $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ non decreasing such that*

$$|D_\xi^\alpha A_i(\xi)| \le \mathcal{F}(M_{1+|\alpha|}) \sum_{k=2}^{1+|\alpha|} \frac{1}{|\nabla\Phi(\xi)|^k}, \quad |\alpha| \ge 1. \tag{18}$$

Proof. We proceed by induction on $|\alpha|$. A simple computation shows that (18) is true for $|\alpha| = 1$. Assume it is true for $|\alpha| \le l$ and let $|\gamma| = l + 1 \ge 2$. Differentiating $|\gamma|$ times the equality $|\nabla\Phi|^2 A_i = \partial_i \Phi$ we obtain

$$\begin{aligned} &|\nabla\Phi|^2 D_\xi^\gamma A_i = (1) - (2) - (3), \quad \text{with} \quad (1) = D_\xi^\gamma \partial_i \Phi \\ &(2) = \sum_{|\beta|=1} \binom{\gamma}{\beta} (D_\xi^\beta |\nabla\Phi|^2) D^{\gamma-\beta} A_i, \quad (3) = \sum_{2 \le |\beta| \le |\gamma|} \binom{\gamma}{\beta} (D_\xi^\beta |\nabla\Phi|^2) D^{\gamma-\beta} A_i. \end{aligned} \tag{19}$$

We have $|(1)| \le \mathcal{M}_{l+2}$. By the induction, $|(2)| \le C\mathcal{M}_2 |\nabla\Phi| \mathcal{F}(M_{l+1}) \sum_{k=2}^{l+1} \frac{1}{|\nabla\Phi|^k}$. Now for $|\beta| \ge 2$ we have $|\gamma| - |\beta| \le l - 1$. Since $|D_\xi^\beta |\nabla\Phi|^2| \le C_2 \mathcal{M}_{|\beta|+1} |\nabla\Phi| + \mathcal{F}(\mathcal{M}_{|\beta|})$ the induction shows that $|(3)| \le C\mathcal{F}(\mathcal{M}_{l+2})(1 + |\nabla\Phi|) \sum_{k=2}^{l} \frac{1}{|\nabla\Phi|^k}$. Dividing both members of the first equation in (19) by $|\nabla\Phi|^2$ we obtain eventually

$$|D_\xi^\gamma A_i| \le \mathcal{F}(\mathcal{M}_{l+2}) \sum_{k=2}^{l+2} \frac{1}{|\nabla\Phi|^k}, \quad |\gamma| = l + 1.$$

This completes the proof of (18). □

We shall also need the following result.

Lemma 9. *On the set $\{\xi : 0 < |\nabla\Phi(\xi)| \le 2\}$ we have*

$$|\partial_\xi^\alpha |\nabla\Phi(\xi)|| \le \mathcal{F}(\mathcal{M}_{|\alpha|+1}) |\nabla\Phi(\xi)|^{1-|\alpha|}, \quad |\alpha| \ge 1. \tag{20}$$

Proof. The proof goes by induction on $|\alpha|$. For $|\alpha| = 1$ it is a simple computation. Assume this is true for $1 \le |\alpha| \le k$ let $|\gamma| = k + 1 \ge 2$. Set $F(\xi) = |\nabla\Phi(\xi)|$. Then we write

$$\partial_\xi^\gamma \big(F(\xi)F(\xi)\big)| = \partial_\xi^\gamma \sum_{j=1}^{d} (\partial_j \Phi)^2.$$

The right hand side is bounded by $\mathcal{F}(\mathcal{M}_{|\gamma|+1})(1 + F(\xi))$. By the Leibniz formula the left hand side can be written as $2F(\xi)\partial_\xi^\gamma F(\xi)$ plus a finite sum of terms of the form $\big(\partial_\xi^{\gamma_1} F(\xi)\big)\big(\partial_\xi^{\gamma_2} F(\xi)\big)$ where $1 \le |\gamma_j| \le k$ and $\gamma_1 + \gamma_2 = \gamma$. For these last terms we can use the induction and we obtain

$$F(\xi)\partial_\xi^\gamma F(\xi) \le \mathcal{F}(\mathcal{M}_{|\gamma|+1})(1 + F(\xi) + F(\xi)^{2-|\gamma|}).$$

Dividing both members by $F(\xi)$ we obtain

$$\begin{aligned} |\partial_\xi^\gamma F(\xi)| &\le \mathcal{F}(\mathcal{M}_{|\gamma|+1}) \Big(1 + \frac{1}{F(\xi)} + F(\xi)^{1-|\gamma|}\Big) \\ &\le \mathcal{F}(\mathcal{M}_{|\gamma|+1}) F(\xi)^{1-|\gamma|} \big(F(\xi)^{|\gamma|-1} + F(\xi)^{|\gamma|-2} + 1\big). \end{aligned}$$

Since $|\gamma| \ge 2$ and $F(\xi) \le 2$ we obtain the desired result. □

Lemma 10. *Consider the vector field $\mathcal{L} = A \cdot \nabla$ where $A_i = \frac{\partial_i \Phi}{|\nabla \Phi|^2}$. For any $N \in \mathbf{N}$ we have*

$$(21)\quad ({}^t\mathcal{L})^N = \sum_{|\alpha|\le N} c_{\alpha,N}\partial^\alpha, \quad \textit{with} \quad |\partial_\xi^\beta c_{\alpha,N}| \le \mathcal{F}(\mathcal{M}_{N-|\alpha|+|\beta|+1}) \sum_{k=N}^{2N-|\alpha|+|\beta|} \frac{1}{|\nabla\Phi|^k}.$$

(Here we set $\mathcal{M}_1 = 1$. It occurs when $\beta = 0, |\alpha| = N$.)

Proof. Again we proceed by induction on N. For $N = 1$ we have $c_{\alpha,N} = A_i$ if $|\alpha| = 1$ and $c_{\alpha,N} = \mathrm{div} A$ if $|\alpha| = 0$. Then (21) follows immediately from (18). Assume that (21) is true up to the order N and let us prove it for $N+1$. We write

$$\begin{aligned}
({}^t\mathcal{L})^{N+1} = {}^t\mathcal{L}({}^t\mathcal{L})^N = (\nabla\cdot A)({}^t\mathcal{L})^N &= \sum_{|\alpha|\le N}\sum_{i=1}^d \partial_i(A_i c_{\alpha,N}\partial^\alpha), \\
&= \sum_{|\alpha|\le N} (\mathrm{div}\ A) c_{\alpha,N}\partial^\alpha + \sum_{|\alpha|\le N} A\cdot\nabla c_{\alpha,N}\partial^\alpha + \sum_{|\alpha|\le N}\sum_{i=1}^d A_i c_{\alpha,N}\partial_i\partial^\alpha, \\
&= \sum_{|\gamma|\le N+1} c_{\gamma,N+1}\partial^\gamma,
\end{aligned}$$

where

$$\begin{aligned}
&c_{0,N+1} = (\mathrm{div}\ A)c_{0,N} + A\cdot\nabla c_{0,N}, \quad \text{if} \quad |\gamma| = 0, \\
&c_{\gamma,N+1} = (\mathrm{div}\ A)c_{\gamma,N} + A\cdot\nabla c_{\gamma,N} + A_i c_{\alpha,N} \quad |\alpha| = |\gamma|-1, \ \text{if} \quad 1\le|\gamma|\le N, \\
&c_{\gamma,N+1} = A_i c_{\alpha,N}, \quad \text{if } \partial^\gamma = \partial_i\partial^\alpha, |\alpha| = N, \ \text{if} \quad |\gamma| = N+1.
\end{aligned}$$

We estimate now each coefficient. First of all $\partial^\beta c_{0,N+1}$ is a finite sum of terms of the form $(\partial^{\beta_1}\partial_i A_i)(\partial^{\beta_2} c_{0,N})$ and $(\partial^{\beta_1} A_i)(\partial^{\beta_2}\partial_i c_{0,N})$ with $\beta = \beta_1 + \beta_2$. Using (18) and the induction the first term is bounded by

$$\mathcal{F}(\mathcal{M}_{|\beta_1|+2}) \sum_{k=2}^{|\beta_1|+2} |\nabla\Phi|^{-k} \mathcal{F}(\mathcal{M}_{N+|\beta_2|+1}) \sum_{l=N}^{2N+|\beta_2|} |\nabla\Phi|^{-l}.$$

Concerning the second term, if $\beta_1 = 0, \beta_2 = \beta$ it is bounded by

$$\frac{1}{|\nabla\phi|}\mathcal{F}(\mathcal{M}_{N+|\beta|+2}) \sum_{l=N}^{2N+|\beta|+1} |\nabla\Phi|^{-l} \le \mathcal{F}(\mathcal{M}_{N+1+|\beta|+1}) \sum_{l=N+1}^{2(N+1)+|\beta|} |\nabla\Phi|^{-l}.$$

If $\beta_1 \neq 0$ it is bounded by

$$\mathcal{F}(\mathcal{M}_{|\beta_1|+1}) \sum_{k=2}^{|\beta_1|+1} |\nabla\Phi|^{-k} \mathcal{F}(\mathcal{M}_{N+|\beta_2|+1}) \sum_{l=N}^{2N+|\beta_2|+1} |\nabla\Phi|^{-l}.$$

Since $N+2 \le k+l \le 2N+2+|\beta_1|+|\beta_2| = 2(N+1)+|\beta|$ we see that $\partial^\beta c_{0,N+1}$ satisfies the estimate in (21) with N replaced by $N+1$.

Let us look to the term $\partial^\beta c_{\gamma,N+1}$ with $|\gamma| = N+1$. This term is also a finite sum of terms of the form $(\partial^{\beta_1} A_i)(\partial^{\beta_2} c_{\alpha,N}), |\alpha| = |\gamma|-1$. As above, if $\beta_1 = 0$, using (18) and the induction it is bounded by

$$\frac{1}{|\nabla \phi|}\mathcal{F}(\mathcal{M}_{N-|\gamma|+1+|\beta|+1}) \sum_{l=N}^{2N-|\gamma|+1+|\beta|} |\nabla\Phi|^{-l}$$
$$\leq \mathcal{F}(\mathcal{M}_{N+1-|\gamma|+|\beta|+1}) \sum_{l=N+1}^{2(N+1)-|\gamma|+|\beta|} |\nabla\Phi|^{-l}.$$

If $\beta_1 \neq 0$ it is bounded by

$$\mathcal{F}(\mathcal{M}_{|\beta_1|+1}) \sum_{k=2}^{|\beta_1|+1} |\nabla\Phi|^{-k}\mathcal{F}(\mathcal{M}_{N-|\gamma|+1+|\beta_2|}) \sum_{l=N}^{2N-|\gamma|+1+|\beta_2} |\nabla\Phi|^{-l}.$$

Since $N+2 \leq k+l \leq 2N+2-|\gamma|+|\beta|$ we see that $\partial^\beta c_{\gamma,N+1}$ satisfies also the estimate in (21). The estimates of the other terms are similar and left to the reader. □

0.2. **Proof of Theorem 6.** Case 1. Let $\psi \in C_0^\infty(\mathbf{R})$ be such that $\psi(x) = 1$ if $|x| \leq 1$, $\psi(x) = 0$ if $|x| \geq 2$. With the notation in (16), j beeing fixed, we write

(22)
$$I_j(\lambda) = \int e^{i\lambda\Phi(\xi)}\psi\big(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|\big)\chi_j(\xi)b(\xi)\, d\xi$$
$$+ \int e^{i\lambda\Phi(\xi})(1-\psi\big(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|\big))\chi_j(\xi)b(\xi)\, d\xi =: K_j(\lambda) + L_j(\lambda).$$

We shall use (see (15)) the fact that on the support of χ_j the map $\xi \mapsto \nabla\Phi(\xi)$ is injective. Let us estimate K_j. We write

$$|K_j(\lambda)| \leq \int |\psi\big(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|\big)\chi_j(\xi)b(\xi)|\, d\xi$$

and we set $\eta = \lambda^{\frac{1}{2}}\nabla\Phi(\xi)$ then $d\eta = \lambda^{\frac{d}{2}}|\det \text{Hess}\ \Phi(\xi)|\, d\xi$. Then using (2) (iii) and the notations therein we obtain

$$|K_j(\lambda)| \leq \frac{C_d}{a_0}\mathcal{N}_0\lambda^{-\frac{d}{2}}. \tag{23}$$

To estimate L_j we introduce the vector field $X = \frac{1}{i\lambda}\frac{\nabla\Phi}{|\nabla\Phi|^2}\cdot\nabla$ which satisfies

$$Xe^{i\lambda\Phi} = e^{i\lambda\Phi}.$$

Now with $N \geq 1$ to be chosen we write

$$L_j(\lambda) = \int e^{i\lambda\Phi(\xi)}({}^tX)^N\Big\{(1-\psi\big(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|\big))\chi_j(\xi)b(\xi)\Big\}\, d\xi.$$

Since $X = \frac{1}{i\lambda}\mathcal{L}$ we can use (21) and we obtain

$$|L_j(\lambda)| \leq C_N \sum_{\substack{\alpha=\alpha_1+\alpha_2+\alpha_3\\ |\alpha|\leq N}} S_{\alpha,N}, \quad \text{where}$$

$$S_{\alpha,N} = \lambda^{-N}\int |c_{\alpha,N}|\left|\partial_\xi^{\alpha_1}\big[1-\psi\big(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|\big)\big]\right|\left|\partial_\xi^{\alpha_2}\chi_j(\xi)\right|\left|\partial_\xi^{\alpha_3}b(\xi)\right|\, d\xi.$$

Our aim is to prove that with an appropriate choice of N we have

$$|L_j(\lambda)| \leq \mathcal{F}(\mathcal{M}_{d+2})\mathcal{N}_{d+1}\frac{1}{a_0}\lambda^{-\frac{d}{2}}. \tag{24}$$

Step 1. $\alpha_1 = 0$. Here we integrate on the set $|\nabla\Phi(\xi)| \geq \lambda^{-\frac{1}{2}}$. We use (15), the bounds (17), (21), (2) (iii) and we make the change of variable $\eta = \nabla\Phi(\xi)$; then $d\eta = |\det \text{Hess } \Phi(\xi)|\, d\xi$ then

$$|S_{\alpha,N}| \leq \frac{\lambda^{-N}}{a_0}\delta^{-|\alpha_2|}\mathcal{N}_{|\alpha_3|}\mathcal{F}(\mathcal{M}_{N+1}) \sum_{k=N}^{2N-|\alpha|} \int_{|\eta|\geq\lambda^{-\frac{1}{2}}} \frac{d\eta}{|\eta|^k}$$
$$\leq \frac{C_d\lambda^{-N}}{a_0}\delta^{-|\alpha_2|}\mathcal{N}_{|\alpha_3|}\mathcal{F}(\mathcal{M}_{N+1}) \sum_{k=N}^{2N-|\alpha|} \int_{\lambda^{-\frac{1}{2}}}^{+\infty} r^{d-1-k}\, dr.$$

Taking $N = d+1$ since $|\alpha_j| \leq |\alpha| \leq N$ we see that

$$|S_{\alpha,N}| \leq \frac{\lambda^{-N}}{a_0}\delta^{-|\alpha|}\mathcal{N}_{d+1}\mathcal{F}(\mathcal{M}_{d+2})\lambda^{N-\frac{1}{2}|\alpha|-\frac{d}{2}}$$
$$\leq \frac{1}{a_0}\mathcal{N}_{d+1}\mathcal{F}(\mathcal{M}_{d+2})\lambda^{-\frac{d}{2}}\frac{1}{(\lambda^{\frac{1}{2}}\delta)^{|\alpha|}}.$$

Since by (14) δ is proportional to a_0 and by (10) we have assumed that $\lambda^{\frac{1}{2}}a_0 \geq 1$ we obtain eventually $|S_{\alpha,N}| \leq \frac{1}{a_0}\mathcal{N}_{d+1}\mathcal{F}(\mathcal{M}_{d+2})\lambda^{-\frac{d}{2}}$

Step 2. $\alpha_1 \neq 0$. Here, since we differentiate ψ we are integrating on the set $\lambda^{-\frac{1}{2}} \leq |\nabla\Phi(\xi)| \leq 2\lambda^{-\frac{1}{2}} \leq 1$. We can therefore use (20).

We have to estimate

$$S_{\alpha,N} = \lambda^{-N}\int |c_{\alpha,N}| \left|\partial_\xi^{\alpha_1}\left[\psi(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|)\right]\right| \left|\partial_\xi^{\alpha_2}\chi_j(\xi)\right| \left|\partial_\xi^{\alpha_3}b(\xi)\right|\, d\xi.$$

By the Faa-di-Bruno formula we have

$$\partial_\xi^{\alpha_1}\left[\psi(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|)\right] = \sum_{1\leq|\beta|\leq|\alpha_1|} a_{\alpha,\beta}\psi^{(\beta)}(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|)\prod_{i=1}^{s}(\lambda^{\frac{1}{2}}\partial_\xi^{l_i}|\nabla\Phi|)^{k_i}$$

where $a_{\alpha,\beta}$ are absolute constants, $\sum_{i=1}^s k_i = \beta, \sum_{i=1}^s k_i|l_i| = |\alpha_1|$. Using (20) we deduce that

$$\left|\partial_\xi^{\alpha_1}\left[\psi(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|)\right]\right| \leq \mathcal{F}(\mathcal{M}_{|\alpha_1|+1}) \sum_{1\leq|\beta|\leq|\alpha_1|} \lambda^{\frac{|\beta|}{2}} \left|\psi^{(\beta)}(\lambda^{\frac{1}{2}}|\nabla\Phi(\xi)|)\right| |\nabla\Phi|^{|\beta|-|\alpha_1|}.$$

Using (21) and the fact that $|\nabla\Phi(\xi)| \leq 1$ we have $|c_{\alpha,N}| \leq \mathcal{F}(\mathcal{M}_{N+1})|\nabla\Phi(\xi)|^{|\alpha|-2N}$. Eventually we have $\left|\partial_\xi^{\alpha_2}\chi_j(\xi)\right| \leq C_\alpha\delta^{-|\alpha_2|}$. Performing as above the change of variables $\eta = \nabla\Phi(\xi)$ we will have

$$|S_{\alpha,N}| \leq \frac{\lambda^{-N}\delta^{-|\alpha_2|}}{a_0}\mathcal{F}(M_{N+1})\mathcal{N}_{|\alpha|} \sum_{1\leq|\beta|\leq|\alpha_1|} \lambda^{\frac{|\beta|}{2}} \int_{\lambda^{-\frac{1}{2}}\leq|\eta|\leq 2\lambda^{-\frac{1}{2}}} |\eta|^{|\alpha|-2N+|\beta|-|\alpha_1|}\, d\eta$$
$$\leq \frac{1}{a_0}\mathcal{F}(\mathcal{M}_{N+1})\mathcal{N}_{|\alpha|}\lambda^{-\frac{d}{2}}(\lambda^{\frac{1}{2}}\delta)^{-|\alpha_2|}.$$

Since $\lambda^{\frac{1}{2}}\delta \geq \mathcal{F}(\mathcal{M}_3)$ we obtain $|S| \leq \frac{1}{a_0}\mathcal{F}(\mathcal{M}_{N+1})\mathcal{N}_N\lambda^{-\frac{d}{2}}$. Therefore (24) is proved. Using (22), (23), (24) since $N = d+1$ we obtain

$$|I_j| \leq \frac{1}{a_0}\mathcal{F}(\mathcal{M}_{d+2})\mathcal{N}_{d+1}\lambda^{-\frac{d}{2}}, \quad 1 \leq j \leq J.$$

Now since $J \leq C_d \delta^{-d} \leq \mathcal{F}(\mathcal{M}_3) a_0^{-d}$ using (16) we obtain eventually

$$|I(\lambda)| \leq \frac{1}{a_0^{1+d}} \mathcal{F}(\mathcal{M}_{d+2}) \mathcal{N}_{d+1} \lambda^{-\frac{d}{2}}$$

which completes the proof of the first case of the theorem.

We prove now the second part of Theorem 6.

In that case it is not necessary to make a localization of $I(\lambda)$ in small balls of size δ as in the first case.

Then as before we write

$$\begin{aligned} I(\lambda) &= \int e^{i\lambda\Phi(\xi)} \psi\big(\lambda^{\frac{1}{2}} |\nabla\Phi(\xi)|\big) b(\xi)\, d\xi \\ &\quad + \int e^{i\lambda\Phi(\xi)} (1 - \psi\big(\lambda^{\frac{1}{2}} |\nabla\Phi(\xi)|\big)) b(\xi)\, d\xi =: K(\lambda) + L(\lambda) \end{aligned} \tag{25}$$

and the final estimate follows from (23) and (24).

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