The peak of the solution of elliptic equations

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Abstract

The location of critical points corresponding to both Dirichlet and Neumann boundary condition are concerned. A counter example of inheritance of convexity of domain is given; then using Pohozeav identity locally, the location of critical points of Neumann boundary value problem and the uniqueness of local extrema of Dirichlet boundary value problem are studied.

Keywords: Critical points, inheritance of convexity, Neumann boundary value problems, hot spot problem.

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1 Introduction

The solution of elliptic equations are the steady states of both heat transferring problem and the standing wave of sound propagation. For heat transferring problem, the maximum of the solution is expected to occur at the boundary but not necessary for solution of standing wave problem. The location of maximum of heat transferring problem is related to the hot spot conjecture of Rauch [1]. The answer to this question is unknown but there are many counter examples [2, 3] and example 1 of this article. This problem then adjusts with an additional condition of requiring that the domain is convex. For nonlinear equations, solution with such a kind of phenomenon is called spike layer [4].
The location of the maximum of the solution then attracts many attentions [5, 6, 7, 8, 9]. In this article, the location of the maximum of the solution of Neumann boundary value and the uniqueness of local extrema of Dirichlet boundary value problem are studied and an interesting counter example will be given. It is interesting because it is a counter example of inheritance of convexity of positive solution as well. It was believed that the positive solution of Dirichlet boundary value problem of equation (3) will inherit the convexity of the domain. However, counter examples were found by Koreeva [11], Cabré and Chanillo [12] and Hamel et al. [13]. The variation of inheritance of convexity problem is then to consider the uniqueness of local maximum which will be studied at the end of this article.

The domain of the counter example that we consider is a disk. Since symmetry to origin is the best geometry condition of a convex domain could have, it means that other condition is needed to support the inheritance of convexity problem and the hot spot of heat transferring problem of convex domain.

**Example 1.** \( u = \frac{(x^2+y^2)^2}{4} - (x^2 + y^2)^{\frac{3}{2}} + (x^2 + y^2), \) which is the symmetry solution over domain \( D = B_{r_0}(0) \) of (1), where \( B_r(p) \) is the ball of radius \( r \) centered at point \( p \).

\[ -\Delta u = 3 - \sqrt{1 + 2\sqrt{u} + 8\sqrt{u}}, \quad x \in D. \]  

(1)

If domain \( D = B_1(0) \) then \( u \) satisfies Neumann boundary condition

\[ \frac{\partial u}{\partial n} \big|_{\partial D} = 0, \]  

(2)

and all points of the boundary are maximum of \( u \) which does confirm hot spot conjecture. However, if \( D = B_2(0) \) then \( u \) satisfies both Neumann and Dirichlet boundary condition; moreover, \( u \) is positive but not convex and still \( \nabla u = 0 \) along \( B_1(0) \). Thus \( u \) disagree with both hot spot of heat transferring on convex domain and the inheritance of convexity of positive solution.

In this article, the following problem is studied,

\[ -\Delta u = f(u), \quad x \in D \]  

(3)
with Neumann boundary condition (2), or Dirichlet boundary value problem
\[ u|_{\partial D} = 0. \] (4)

To classify the degenerate critical point such as \((0, 0)\) of \(x^3 - y^3\) or the collection of critical points \(\partial B_1(0)\) as example 1, Hessian matrix is insufficient. Furthermore, the definition of critical point at boundary of Neumann boundary value problem is not clear therefore it is difficult to apply Hessian matrix to exam. To deal with such a situation, new definition is given below.

In this article, we use \(S_t\) to denote the level curve of a function. However, it is possible that the level curve might contain many components. Therefore, we use \(s^j_t \subset S_t = \bigcup_{j=1}^k s^j_t\) to denote the simply connected component of \(S_t\). Here \(s^j_t\) may be a singleton \(\{x_0\}\), if \(x_0\) is a local extrema.

**Definition 2.** An isolated critical point \(p\) of function \(u\) is a local maximum (minimum) if for any \(\epsilon > 0\), there exists a \(\delta > 0\) such that \(\nabla u(x) \cdot (x - p) > 0\) \((<)\) and \(0 < \|\nabla u(x)\| \leq \epsilon\) for \(x \in B_\delta(p) \cap \bar{D}\backslash\{p\}\); otherwise, \(p\) is a saddle point.

**Definition 3.** If \(s^j_t \subset S_t\) is a simply connected component of level curve \(S_t\) and \(\nabla u(x) = 0\) for all \(x \in s^j_t\) then \(x\) is called a non-isolated critical point of function \(u\).

By the definition of \(\nabla\), it is clear that \(u(p) \geq u(x)\) \((\leq)\) for all \(x \in B_\delta\), if \(p\) is a local maximum (minimum) and we call \(p\) a local extrema if \(p\) is either maximum or minimum. Moreover, if \(p\) is a local maximum of the solution \(u\) of equation (3) then \(f(u(p)) \geq 0\). To see this, we integrate equation (3) over \(B_\delta(p)\) and it yields,
\[-\int_{\partial B_\delta(p)\cap \bar{D}} \frac{\partial u}{\partial n} ds = \int_{B_\delta(p)\cap \bar{D}} f(u(x)) dx.\]

Since \(p\) is a local maximum and \(n(x) = -\frac{x-p}{\delta}\), it gives \(\frac{\partial u}{\partial n} \leq 0\) where \(n(x)\) is the outward normal of \(B_\delta(p)\) at \(x\). If \(p \in \partial D\) then \(\frac{\partial u}{\partial n} = 0\). Notice that, \(\delta\) is arbitrary, therefore \(f(u(p)) \geq 0\) and hence the following conclusion is derived.

**Proposition 4.** If \(p\) is a local maximum (minimum) of the solution \(u\) of (3) then \(f(u(p)) \geq 0\) \((f(u(p)) \leq 0)\).
2 Main results

Throughout this article we assume that $D$ is a smooth open bounded simply connected convex domain of $\mathbb{R}^2$ satisfying interior spherical condition and $f$ is monotone with respect to $u$ or $\frac{df}{du} > 0$. As usual, we say that $D$ is smooth if $\partial D$ is smooth. $\bar{D}$ is the closure of domain $D$ and $\mathring{D}$ is the interior of $D$. We shall note that all proofs of this article are dimensionless except the term with $\Delta \|x\|^2$. However, $\Delta \|x\|^2$ of $\mathbb{R}^2$ is the minimum amount $\mathbb{R}^N$; therefore, all results are expected to be true for higher dimension $N > 2$.

In the proof below, it involves with the critical point at boundary, therefore we introduce the following notations

$$B_\delta(p) = B_\delta(p) \cap D,$$

$$B_\delta^+(p) = \{x \in B_\delta(p) : \nabla u(x) \cdot (x - p) > 0\},$$

and

$$B_\delta^-(p) = \{x \in B_\delta(p) : \nabla u(x) \cdot (x - p) < 0\}.$$

Without ambiguity, $p \in \partial D$ and $\nabla u(p) = 0$ if and only if $\lim_{x \to p} \nabla u = 0$ where $x \in \bar{D}$. In particular, if $p \in \partial D$ but $\nabla u(x) \cdot (x - p)$ remains constant sign, for all $x \in \bar{D}$, then we still say that $p$ is a local extrema.

From Neumann boundary condition, we have a natural constraint:

$$\int_D f(u)dx = 0, \quad (5)$$

therefore $f(u)$ must change its sign, say at $u_0$. As usual, we let $F(u) = \int f(u)du$. The assumption $\frac{df}{du} > 0$ implies that $F(t)$ concaves upward with respect to $t$ and hence $u_0$ is the absolute minimum of $F$. To explore the behavior of solution $u$ of Neumann boundary value problem, we denote $D^+ = \{x \in D : f(u(x)) > 0\}$, $D^- = \{x \in D : f(u(x)) < 0\}$ and $m = \min_{x \in \bar{D}} u(x)$ and $M = \max_{x \in \bar{D}} u(x)$.

Most of the results of this article are based on the following hypothesis:

$$u \cdot f(u) - 2F(u) > 0, \quad \frac{df}{du} > 0. \quad (A)$$
2.1 Neumann boundary value problem

First, we consider equation (3) with Neumann boundary value problem.

Lemma 5. If $u$ is the smooth solution of $(3)$ satisfying hypothesis $(A)$, $F(t) > 0$ and if $p \in \bar{D}$ is an isolated critical point then $p$ is a local extrema.

Proof. The lemma will be proved by deriving a contradiction. Without loss of generality, we assume that $p \in \bar{D}^+$ such that $\nabla u \cdot (x - p)$ changes its sign.

To derive the results, we apply Phozeav identity locally. Multiplying $\nabla u(x) \cdot (x - p)$ to equation (3) and integrating over $\mathcal{B}_x^+(p)$, it yields

$$\int_{\mathcal{B}_x^+(p)} -\Delta u(\nabla u(x) \cdot (x - p))dx = -\int_{\partial \mathcal{B}_x^+(p)} \frac{\partial u}{\partial n}(\nabla u(x) \cdot (x - p))ds + \int_{\mathcal{B}_x^+(p)} \nabla u \cdot \nabla(\nabla u(x) \cdot (x - p))dx.$$  

Replacing $-\Delta u$ by $f(u)$, the left hand side of (6) yields

$$\int_{\mathcal{B}_x^+(p)} -\Delta u(\nabla u(x) \cdot (x - p))dx = \int_{\mathcal{B}_x^+(p)} f(u)(\nabla u \cdot (x - p))dx,$$

where

$$\int_{\mathcal{B}_x^+(p)} f(u) \cdot (\nabla u \cdot (x - p) dx = \int_{\mathcal{B}_x^+(p)} \nabla F(u) \cdot (x - p)dx,$$

and

$$\int_{\mathcal{B}_x^+(p)} \nabla F(u) \cdot (x - p) dx = \int_{\partial \mathcal{B}_x^+(p)} F(u) \cdot ((x - p) \cdot n)ds - 2 \int_{\mathcal{B}_x^+(p)} F(u)dx.$$

Calculating the right hand side of (6) and by $D \subset \mathbb{R}^2$ it yields

$$\int_{\mathcal{B}_x^+(p)} \nabla u \cdot \nabla(\nabla u(x) \cdot (x - p))dx = \int_{\partial \mathcal{B}_x^+(p)} \frac{|\nabla u|^2}{2}((x - p) \cdot n)ds - \int_{\mathcal{B}_x^+(p)} \|\nabla u\|^2 dx.$$  

Replacing $\int_{\mathcal{B}_x^+(p)} \|\nabla u\|^2 dx$ by $\int_{\partial \mathcal{B}_x^+(p)} \frac{\partial u}{\partial n}uds + \int_{\mathcal{B}_x^+(p)} f(u)udx$ and then adding all together, we get

$$0 < \int_{\mathcal{B}_x^+(p)} -2F(u) + f(u)udx$$

$$= \int_{\partial \mathcal{B}_x^+(p)}(\frac{|\nabla u|^2}{2} - F(u))((x - p) \cdot n) - \frac{\partial u}{\partial n}(\nabla u(x) \cdot (x - p) + u)ds.$$

Let $\partial B_0^+(p) = \mathcal{N} \cup \mathcal{D} \cup B$ where $\mathcal{N} = \{ x \in \mathcal{B}_x^+(p) | \nabla u(x) \cdot (x - p) = 0 \}$, $\mathcal{D} = \mathcal{B}_x^+(p) \cap \partial \mathcal{D}$ and $B = \partial B_0(p) \cap \partial \mathcal{B}_x^+(p)$. If $p \in \partial \mathcal{D}$ then $\mathcal{D} \neq \emptyset$ otherwise it is an empty set.
Along $\mathcal{N}$, $x - p$ is parallel to the tangent of the curve therefore $(x - p) \cdot n = 0$. On the other hand, $\nabla u \cdot (x - p) \geq 0$ over $B_0^+(p)$ therefore $\frac{\partial u}{\partial n} \geq 0$. Along $B$, $x - p \cdot n = \delta$ and $\nabla u \cdot (x - p) \geq 0$ therefore $\frac{\partial u}{\partial n} \geq 0$. Along $D$, $\frac{\partial u}{\partial n} = 0$ ($u = 0$ for Dirichlet boundary value problem) and $x - p \cdot n = \delta$. Since $F(u(p)) > 0$, \( (11) \) yields

$$0 < \int_{B_0^+(p)} -2F(u) + f(u)u dx = \int_{\partial B_0^+(p)} -F(u)((x - p) \cdot n) - \frac{\partial u}{\partial n} u - \frac{\delta \| \nabla u \|^2}{2} ds = \int_B -\frac{\delta \| \nabla u \|^2}{2} - \delta F(u) - \frac{\partial u}{\partial n} u ds - \int_{\mathcal{N}} \frac{\partial u}{\partial n} u dx, \quad (12)$$

Every terms on the right hand side of equation \( (12) \) are negative, a contradiction. Therefore $p$ must be a local extrema. The proof is completed. $\square$

From the conclusion of Lemma 5 and implicit function theorem, the level curves of $u$ are either the union of disjoint simply connected smooth curves or singletons. Thus we have the following conclusion.

**Corollary 6.** If hypothesis (A) holds, $F > 0$ and if $S_t = \bigcup_{j=1}^k s_t^j$ then for all $t$ $s_t^i \cap s_t^j = \emptyset$.

**Theorem 7.** If hypothesis (A) holds, $F > 0$, and if $S_{u_0} = s_{u_0}$ contains only one component then $S_{u_0} \cap \partial D \neq \emptyset$, $\nabla u \neq 0$ along $S_{u_0}$ and there exists a unique $p_{\pm} \in \partial D^\pm$ such that $u(p_{\pm}) = \max_{x \in \partial D^\pm} u(x)$ and $u(p_{\pm}) = \min_{x \in \partial D^-} u(x)$, respectively.

**Proof.** If on the contrary $S_{u_0} \cap \partial D = \emptyset$ then the sign of $f(u)$ along $\partial D$ remains constant. If $u|_{\partial D} = C$ then $\nabla u(x) \cdot T(x) = 0$ where $T(x)$ is the unit tangent vector at $x$ along $\partial D$. Thus $\nabla u(x) = 0$ which contradicts Lemma 5. Hence $u$ cannot be a constant along $\partial D$. Let $u(p) = \max_{x \in \partial D} u(x)$ and $u(q) = \min_{x \in \partial D} u(x)$ then by Proposition 4, $f(u(q)) \leq 0$, a contradiction. Thus $S_{u_0} \cap \partial D = \emptyset$. By Lemma 5, $\nabla u \neq 0$ along $S_{u_0}$.

To prove the uniqueness of local maximum along $\partial D^+$, we let $p_i \in \partial D^+$ such that $u(p_i) = \max_{x \in \partial D^+} u(x)$. Let $\xi \subset \partial D^+$ be the arc containing all the points lie in between $p_i$. By mean value theorem, there exists at least a critical point $p_0$ lies in between $p_i$. By Lemma 5, $p_0$ cannot be a saddle point. Therefore $p_0$ is a local minimum which contradicts proposition 4. $\square$
If the condition $F(u) > 0$ may relax to $F(u) \geq 0$ but with assumption $F(t) = 0$ only when $t = u_0$, then Lemma 5 still holds. Thus $u$ contains interior maximum provided that $S_{u_0}$ contains more than one component.

2.2 Dirichlet boundary value problem

Lemma 5 is a local property of the solution of equation (3) therefore it remains true for Dirichlet boundary value problem. With the conclusion of Lemma 5 and mean value theorem, we may derive that the positive solution of (3) has a unique local maximum provided that the domain is convex. From the proof of Lemma 5, we see that the necessary condition of it is that $F > 0$. The positiveness of the solution and the conditions $\frac{df}{du} > 0$ and $f(t) > 0$ for $t > 0$ imply that $F(u) > 0$ over $\hat{D}$. Thus the following conclusion holds.

**Theorem 8.** If $u$ is the smooth positive solution of (3) satisfying Dirichlet boundary condition with hypothesis (A), $f(0) \geq 0$ then $u$ has a unique local maximum.

**Proof.** If $p, q$ are both local maximum of $u$ then by mean value theorem there must another critical point $p_0$ which is either a saddle or a local minimum. If $p_0 \in \hat{D}$ then $p_0$ cannot be a local minimum because $f(u) > 0$ which contradicts proposition 4. $p_0$ cannot be a saddle because that will contradict Lemma 5. Next, if $p_0 \in \partial D$ and if it is a saddle or local minimum then there is a subset $B^+ \subset B_\delta(p) \cap D$ such that $\nabla u \cdot (x - p_0) > 0$, if $x \neq p_0$, which contradicts Lemma 5. Thus the interior local extrema is unique. The proof is completed.

**Remark 9.** The assumptions $f(u)u > 2F(u)$ and $F(u) > 0$ of hypothesis (A) indicate that if $f(u) = u^p$ then $p > 1$ which coincides with counter example 1. The condition $\frac{df}{du} > 0$ does fit the first non constant eigenfunction of Laplacian with Neumann boundary condition. However, excluding the constant eigenfunction, the second non-constant eigenfunction $\cos(x)\cos(y)$ on $[0, 2\pi] \times [0, 2\pi]$ has an interior critical point and level curve $S_0$ contains two components. Therefore, the location of the local extrema
seems not only depends on the convexity of the domain but the number of the level curve of \( u_0 \) as well.

References


