

LI-YAU GRADIENT BOUNDS UNDER NEARLY OPTIMAL CURVATURE CONDITIONS

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ABSTRACT. We prove Li-Yau type gradient bounds for the heat equation either on manifolds with fixed metric or under the Ricci flow. In the former case the curvature condition is $|Ric^-| \in L^p$ for some $p > n/2$, or $\sup_{\mathbf{M}} \int_{\mathbf{M}} |Ric^-|^2(y) d^{2-n}(x, y) dy < \infty$, where n is the dimension of the manifold. In the later case, one only needs scalar curvature being bounded. We will explain why the conditions are nearly optimal and give an application. The Li-Yau bound for the heat equation on manifolds with fixed metric seems to be the first one allowing Ricci curvature not bounded from below.

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1. INTRODUCTION

Let (\mathbf{M}^n, g_{ij}) be a complete Riemannian manifold. In [LY], P. Li and S.T. Yau discovered the following celebrated Li-Yau bound, for positive solutions of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u. \quad (1.1)$$

Suppose $Ric \geq -K$, where $K \geq 0$ and Ric is the Ricci curvature of \mathbf{M} . Then any positive solution of (1.1) satisfies

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha-1)} + \frac{n\alpha^2}{2t}, \quad \forall \alpha > 1. \quad (1.2)$$

In the special case where $Ric \geq 0$, one has the optimal Li-Yau bound

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}. \quad (1.3)$$

In the same paper, many applications of (1.2) and (1.3) have also been demonstrated by the authors, including the classical parabolic Harnack inequality, optimal Gaussian estimates of the heat kernel, estimates of eigenvalues of the Laplace operator, and estimates of the Green's function. Moreover, (1.2) and (1.3) can even imply the Laplacian Comparison Theorem (see e.g. [Chowetc] page 394).

The Li-Yau bound (1.2) was later improved for small time by Hamilton in [Ha3], where he proved under the same assumptions as above that

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{u_t}{u} \leq e^{4Kt} \frac{n}{2t}. \quad (1.4)$$

Hamilton [Ha3] further showed a matrix Li-Yau bound for the heat equation. Similar matrix Li-Yau bound was subsequently obtained by Cao-Ni [CaNi] on Kähler manifolds.

For the past three decades, many Li-Yau type bounds have been proved not only for the heat equation, but more generally, for other linear and semi-linear parabolic equations on manifolds with or without weights. Let us mention the result by Bakry and Ledoux [BL] who derived the Li-Yau bound for weighted manifolds by an ordinary differential inequality involving the entropy and energy of the backward heat equation. For most recent development, see the papers [CTZ], [Dav], [GM], [LX], [QZZ], [Wan], [WanJ] and the latest [BBG] and its references. In all of these results, the essential assumption is that the Ricci curvature or the corresponding Bakry-Emery Ricci curvature is bounded from below by a constant. In many situations, it is highly desirable to weaken this assumption.

On the other hand, Li-Yau bounds have been extended to situations with moving metrics. Let $g_{ij}(t)$, $t \in [0, T]$, be a family of Riemannian metrics on \mathbf{M} which solves the Ricci flow:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t), \quad (1.5)$$

where $R_{ij}(t)$ is the Ricci curvature tensor of $g_{ij}(t)$. One may still consider linear and semi-linear parabolic equations under the Ricci flow in the sense that in the heat operator $\frac{\partial}{\partial t} - \Delta$, we have $\Delta = \Delta_t$ which is the Laplace operator with respect to the metric $g_{ij}(t)$ at time t . The two most prominent examples are the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right)u = 0, \quad \partial_t g_{ij} = -2R_{ij} \quad (1.6)$$

and the conjugate heat equation

$$\left(\Delta - R + \frac{\partial}{\partial t}\right)u = 0, \quad \partial_t g_{ij} = -2R_{ij}. \quad (1.7)$$

The study of Li-Yau bound for heat type equations under the Ricci flow was initiated by Hamilton. In [Ha4], he obtained a Li-Yau bound for the scalar curvature along the Ricci flow on 2-sphere. This result was later improved by Chow [Chow]. In higher dimensions, both matrix and trace Li-Yau bounds for curvature tensors, also known as Li-Yau-Hamilton inequalities, were obtained by Hamilton [Ha5] for the Ricci flow with bounded curvature and nonnegative curvature operator. These estimates played a crucial role in the study of singularity formations of the Ricci flow on three-manifolds and solution to the Poincaré conjecture. We remark that Brendle [Bre] has generalized Li-Yau-Hamilton inequalities under weaker curvature assumptions. The Li-Yau-Hamilton inequality for the Kähler-Ricci flow with nonnegative holomorphic bisectional curvature was obtained by H.-D. Cao [Cao]. In addition, in [P1], Perelman showed a Li-Yau type bound for the fundamental solution of the conjugate heat equation (1.7) under the Ricci flow (see also [Ni]). Recently, there have been many results on Li-Yau bounds for positive solutions of the heat or conjugate heat equations under the Ricci flow. For example authors of [KuZh] and [Cx] proved Li-Yau type bound for all positive solutions of the conjugate heat equation with out any curvature condition, just like Perelman's aforementioned result for the

fundamental solution. In [CH] and [BCP] the authors proved various Li-Yau type bounds for positive solutions of (1.6) under either positivity condition of the curvature tensor or boundedness of the Ricci curvature. So there is a marked difference between these results on the conjugate heat equation and the heat equation in the curvature conditions. In view of the absence of curvature condition for the conjugate heat equation, one would hope that the curvature conditions for the heat equation can be weakened.

Recently, in [BZ], the authors proved the following gradient estimate for bounded positive solutions u of the heat equation (1.6),

$$|\Delta u| + \frac{|\nabla u|^2}{u} - aR \leq \frac{Ba}{t}, \quad (1.8)$$

where $R = R(x, t)$ is the scalar curvature of the manifold at time t , and B is a constant and a is an upper bound of u on $M \times [0, T]$. Although this result requires no curvature condition and it has some other applications, it is not a Li-Yau type bound.

The goal of this paper is to prove Li-Yau bounds for positive solutions for both the fixed metric case (1.1) and the Ricci flow case (1.6) under essentially optimal curvature conditions.

The first theorem is for the fixed metric case, we will have two independent conditions and two conclusions. The conditions are motivated by different problems such as studying manifolds with integral Ricci curvature bound and the Kähler-Ricci flow. The conclusions range from long time bound with necessarily worse constants, to short time bound with better constants.

Theorem 1.1. *Let (M, g_{ij}) be a compact n dimensional Riemannian manifold, and u a positive solution of (1.1). Suppose either one of the following conditions holds.*

(a) $\int_M |Ric^-|^p dy \equiv \sigma < \infty$ for some $p > \frac{n}{2}$, where Ric^- denotes the nonpositive part of the Ricci curvature; and the manifold is noncollapsed under scale 1, i.e., $|B(x, r)| \geq \rho r^n$ for $0 < r \leq 1$ and some $\rho > 0$;

(b) $\sup_M \int_M |Ric^-|^2 d^{-(n-2)}(x, y) dy \equiv \sigma < \infty$ and the heat kernel of (1.1) satisfies (1.12): the Gaussian upper bound. Here $d(x, y)$ is the distance from x to y .

Then,

(1) for any constant $\alpha \in (0, 1)$, we have

$$\alpha \underline{J}(t) \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{(2 - \delta)\alpha \underline{J}(t)} \frac{1}{t} \quad (1.9)$$

for $t \in (0, \infty)$, where

$$\delta = \frac{2(1 - \alpha)^2}{n + (1 - \alpha)^2},$$

and

$$\underline{J}(t) = \begin{cases} 2^{-1/(5\delta-1)} e^{-(5\delta-1)\frac{n}{2p-n} [4\sigma\hat{C}(t)^{1/p}]^{\frac{2p}{2p-n}} t}, & \text{under condition (a);} \\ 2^{-1/(10\delta-2)} e^{-C_0(5\delta-1)\sigma\hat{C}(t)t}, & \text{under condition (b),} \end{cases} \quad (1.10)$$

with C_0 being a constant depending only on n , p and ρ , and $\hat{C}(t)$ the increasing function on the right hand side of (1.12).

(2) in particular, for any $\beta \in (0, 1)$, there is a $T_0 = T_0(\beta, \sigma, p, n, \rho)$ such that

$$\beta \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2\beta} \frac{1}{t} \quad (1.11)$$

for $t \in (0, T_0]$. Here $T_0 = c(1 - \beta)^{4p/(2p-n)}$ and $c(1 - \beta)^4$ under conditions (a) and (b), respectively; and c is a positive constant depending only on the parameters of conditions (a) and (b), i.e., $c = c(\sigma, p, n, \rho)$.

Remark 1.2. Condition (a) actually implies that the heat kernel of (1.1) has a Gaussian upper bound for all time,

$$G(x, t; y, 0) \leq \frac{\hat{C}(t)}{t^{n/2}} e^{-\bar{c}d^2(x,y)/t}, \quad t \in (0, \infty) \quad (1.12)$$

for some positive constant \bar{c} and positive increasing function $\hat{C}(t)$ which grows to infinity as $t \rightarrow \infty$. For short time interval $(0, 1]$, the function $\hat{C}(t)$ can be replaced by a constant \hat{C} . This is proven in [TZz] Section 2 and [TZq1]. Longer time bound follows from the reproducing formula of heat kernels.

In addition, a volume upper bound $|B(x, r)| \leq Cr^n$ follows from Petersen-Wei [PW1]. We will use these facts during the proof of the theorem.

The condition $|\text{Ric}^-| \in L^p$ for some $p > \frac{n}{2}$ is nearly optimal in the sense that it is not clear $|\nabla u|$ will stay bounded when $|\text{Ric}^-| \in L^{\frac{n}{2}}$, which is the well known border line condition where regularity may fail.

Professor Guofang Wei kindly informed us that the noncollapsing condition in (a) may possibly be removed by a recent result of Dai-Wei-Z. L. Zhang.

Remark 1.3. The motivation of assuming (b) is that it is preserved under the Kähler-Ricci flow as proved in [TZq1] and [TZq2].

The Li-Yau bound in the above theorem seems to be the first one allowing Ricci curvature not bounded from below. Moreover, as an application, we use (1.11) to extend some results in [CoNa] on the parabolic approximations of the distance functions to the case where $|\text{Ric}^-| \in L^p$ for some $p > \frac{n}{2}$. The main extension results were first proved in [TZz].

Next we turn to the heat equation coupled with the Ricci flow (1.6) for which we prove

Theorem 1.4. Let \mathbf{M} be a compact n dimensional Riemannian manifold, and $g_{ij}(t)$, $t \in [0, T)$, a solution of the Ricci flow (1.5) on \mathbf{M} . Denote by R the scalar curvature of \mathbf{M} at t , and R_1 a positive constant. Suppose that $-1 \leq R \leq R_1$ for all time t , and u is a positive solution of the heat equation (1.6). Then, for any $\delta \in [\frac{1}{2}, 1)$, we have

$$\delta \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \delta \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} - \alpha R + \frac{\beta}{R+2} \leq \frac{1}{t} \left(\frac{n}{2\delta} + \frac{4n\beta T}{\delta(1-\delta)} \right) \quad (1.13)$$

for $t \in (0, T)$, where $\alpha = \frac{n}{2\delta(1-\delta)^2}$ and $\beta = \alpha(R_1 + 2)^2$.

Remark 1.5. Note that the curvature assumption is made only on the scalar curvature rather than on the Ricci or curvature tensor. In this sense, this assumption is essentially optimal. Under suitable assumptions, the result in the theorem still holds when \mathbf{M} is complete noncompact.

For the Ricci flow on a compact manifold \mathbf{M} , one can always rescale a solution so that the scalar curvature to be bounded from below by -1 .

The Li-Yau bound in the above theorem actually is scaling invariant. Readers can refer to Theorem 3.2 in section 3 for the corresponding version before rescaling the metrics. In case the scalar curvature is 0, the Ricci curvature is also 0 by the maximum principle. Then, by scaling, we can let $\delta = 1$ in the theorem and the bound becomes the optimal Li-Yau bound for Ricci flat case.

Remark 1.6. This theorem clearly implies a Harnack inequality for positive solutions of (1.6) if the scalar curvature is bounded.

This paper is organized as follows: the main Theorems 1.1 and 1.4 are proved in sections 2 and 3, respectively. The main technical hurdle is to construct certain auxiliary functions to cancel various curvature terms arising from commutation formulas. For example, in order to prove Theorem 1.1, one needs to deal with the bad term $|Ric^-| \frac{|\nabla u|^2}{u}$. If one only imposes integral conditions on $|Ric^-|$, then this term can not be bounded by good terms coming out of the Bochner's formula. The auxiliary functions for Theorem 1.1 are obtained by solving a nonlinear evolution equation, which is used to cancel the bad term. When proving Theorem 1.4 in section 3, an additional bad term $\langle \nabla^2 u, Ric \rangle$ appears. We will use the good terms coming from the equation of $\frac{\beta}{R}$ to control it. In section 4, we deduce from Theorem 1.1 the extended parabolic approximations of the distance functions.

2. FIXED METRIC CASE

In this section, we work on a compact Riemannian manifold \mathbf{M} with a fixed metric g . For the Ricci curvature, we assume either

$$|Ric^-| \in L^p(\mathbf{M}), \quad p > n/2, \quad \text{or} \quad \sup_{x \in \mathbf{M}} \int_{\mathbf{M}} \frac{|Ric^-(y)|^2}{d^{n-2}(x, y)} dy < \infty. \quad (2.1)$$

Proof of Theorem 1.1: By direct computation, we have

$$(\Delta - \partial_t) \frac{|\nabla u|^2}{u} = \frac{2}{u} \left| u_{ij} - \frac{u_i u_j}{u} \right|^2 + 2R_{ij} \frac{u_i u_j}{u}.$$

Let $J = J(x, t)$ be a smooth positive function and $\alpha \in (0, 1)$ be a parameter. Then

$$\begin{aligned} & (\Delta - \partial_t) \left[\alpha J \frac{|\nabla u|^2}{u} - \partial_t u \right] \\ &= \alpha \left[\frac{2}{u} \left| u_{ij} - \frac{u_i u_j}{u} \right|^2 J + 2R_{ij} \frac{u_i u_j}{u} J + \Delta J \frac{|\nabla u|^2}{u} + 2\nabla J \nabla \frac{|\nabla u|^2}{u} - (\partial_t J) \frac{|\nabla u|^2}{u} \right]. \end{aligned}$$

Denote the heat operator $\Delta - \frac{\partial}{\partial t}$ by \mathcal{L} . Recall the quotient formula for the heat operator.

$$\mathcal{L} \left(\frac{F}{G} \right) + 2\nabla \ln G \nabla \frac{F}{G} = \frac{\mathcal{L}F}{G} - \frac{F\mathcal{L}G}{G^2}.$$

Take

$$F = \alpha J \frac{|\nabla u|^2}{u} - \partial_t u, \quad G = u,$$

and

$$Q \equiv \alpha J \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}. \quad (2.2)$$

We find that

$$\begin{aligned} & (\Delta - \partial_t)Q + 2\frac{\nabla u}{u}\nabla Q \\ &= \alpha \left[\frac{2}{u^2} \left| u_{ij} - \frac{u_i u_j}{u} \right|^2 J + 2R_{ij} \frac{u_i u_j}{u^2} J + \Delta J \frac{|\nabla u|^2}{u^2} + 2\nabla J \nabla \left(\frac{|\nabla u|^2}{u} \right) \frac{1}{u} - (\partial_t J) \frac{|\nabla u|^2}{u^2} \right]. \end{aligned} \quad (2.3)$$

Let $f = \ln u$. Using the identities

$$\frac{1}{u^2} \left| u_{ij} - \frac{u_i u_j}{u} \right|^2 = |f_{ij}|^2,$$

and

$$\nabla \left(\frac{|\nabla u|^2}{u} \right) \frac{1}{u} = \nabla \left(\frac{|\nabla u|^2}{u^2} \right) + \frac{|\nabla u|^2}{u} \frac{\nabla u}{u^2} = \nabla (|\nabla f|^2) + \frac{|\nabla u|^2}{u} \frac{\nabla u}{u^2},$$

we can turn (2.3) into

$$\begin{aligned} & (\Delta - \partial_t)Q + 2\frac{\nabla u}{u}\nabla Q \\ &= \alpha \left[2|f_{ij}|^2 J + 2R_{ij} \frac{u_i u_j}{u^2} J + \Delta J \frac{|\nabla u|^2}{u^2} + 2\nabla J \nabla |\nabla f|^2 + 2\nabla J \frac{\nabla u}{u} \left(\frac{|\nabla u|^2}{u^2} \right) - (\partial_t J) \frac{|\nabla u|^2}{u^2} \right]. \end{aligned} \quad (2.4)$$

Observe that, in local coordinates,

$$2\nabla J \nabla |\nabla f|^2 = 2J_i (f_j^2)_i = 4J_i f_{ji} f_j \geq -\delta |f_{ij}|^2 J - \frac{|\nabla J|^2}{J} |\nabla f|^2 4\delta^{-1}.$$

Therefore, we deduce the following inequality

$$\begin{aligned} & (\Delta - \partial_t)Q + 2\frac{\nabla u}{u}\nabla Q \\ & \geq \alpha \left[(2J - \delta J) |f_{ij}|^2 - 2|Ric^-| \frac{|\nabla u|^2}{u^2} J + \Delta J \frac{|\nabla u|^2}{u^2} - \frac{|\nabla J|^2}{J} |\nabla f|^2 4\delta^{-1} \right. \\ & \quad \left. - 2|\nabla J| \frac{|\nabla u|^3}{u^3} - (\partial_t J) \frac{|\nabla u|^2}{u^2} \right]. \end{aligned} \quad (2.5)$$

Using the inequality, for any $\delta > 0$,

$$2|\nabla J| \frac{|\nabla u|^3}{u^3} = 2|\nabla J| |\nabla f|^3 \leq \delta J |\nabla f|^4 + \delta^{-1} \frac{|\nabla J|^2}{J} |\nabla f|^2,$$

we can turn the above inequality into

$$\begin{aligned} & (\Delta - \partial_t)Q + 2\frac{\nabla u}{u}\nabla Q \\ & \geq \alpha (2J - \delta J) |f_{ij}|^2 + \alpha \left[\Delta J - 2|Ric^-| J - 5\delta^{-1} \frac{|\nabla J|^2}{J} - \partial_t J \right] |\nabla f|^2 - \delta \alpha J |\nabla f|^4. \end{aligned}$$

From (1.1), we know

$$\Delta f - \partial_t f = |\nabla f|^2.$$

Hence,

$$\begin{aligned}
& (\Delta - \partial_t)(tQ) + 2\frac{\nabla u}{u}\nabla(tQ) \\
& \geq \alpha t(2J - \delta J)\frac{1}{n}(|\nabla f|^2 - \partial_t f)^2 + \alpha \left[\Delta J - 2VJ - 5\delta^{-1}\frac{|\nabla J|^2}{J} - \partial_t J \right] t|\nabla f|^2 \\
& \quad - \delta\alpha tJ|\nabla f|^4 - Q,
\end{aligned} \tag{2.6}$$

where we have written $|\text{Ric}^-| = V$.

For any given parameter $\delta > 0$ such that $5\delta^{-1} > 1$, we make the following

Claim 2.1. *the problem*

$$\begin{cases} \Delta J - 2VJ - 5\delta^{-1}\frac{|\nabla J|^2}{J} - \partial_t J = 0, & \text{on } \mathbf{M} \times (0, \infty); \\ J(\cdot, 0) = 1, \end{cases} \tag{2.7}$$

has a unique solution for $t \in [0, \infty)$, which satisfies

$$\underline{J}(t) \leq J(x, t) \leq 1, \tag{2.8}$$

where

$$\underline{J}(t) = \begin{cases} 2^{-1/(5\delta^{-1}-1)} e^{-(5\delta^{-1}-1)\frac{n}{2p-n}[4\sigma\hat{C}(t)^{1/p}]^{\frac{2p}{2p-n}}t}, & \text{under condition (a);} \\ 2^{-1/(10\delta^{-1}-2)} e^{-C_0(5\delta^{-1}-1)\sigma\hat{C}(t)t}, & \text{under condition (b),} \end{cases} \tag{2.9}$$

with C_0 being a constant depending only on n, p and ρ , and $\hat{C}(t)$ the increasing function in (1.12).

In the following steps, we will prove the claim.

step 1. Conversion into an integral equation.

Let $a = 5\delta^{-1}$, and

$$w = J^{-(a-1)}. \tag{2.10}$$

It is straightforward to check that w satisfies

$$\begin{cases} \Delta w - \partial_t w + 2(a-1)Vw = 0, & \text{on } \mathbf{M} \times (0, \infty); \\ w(\cdot, 0) = 1. \end{cases} \tag{2.11}$$

Since V is a smooth function, (2.11) has a long time solution.

To show that J exists for all time and derive the bounds for J , we derive the bounds for w first. Via the Duhamel's formula, (2.11) can be transformed to the following integral equation,

$$w(x, t) = 1 + 2(a-1) \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) w(y, s) dy ds. \tag{2.12}$$

Here $G(x, t; y, s) = G(y, t; x, s)$ is the heat kernel on \mathbf{M} .

step 2. long time bounds

Here we prove long time bounds for solutions of (2.11).

Let w be a solution of (2.11). For a lower bound of w , we can show that

$$w \geq 1 \tag{2.13}$$

for any $t > 0$.

In fact, let $\epsilon > 0$ be a small positive number, which will be taken to 0 eventually. Then the function $Z_\epsilon = e^{\epsilon t} w$ satisfies the equation

$$\Delta Z_\epsilon + 2(a-1)VZ_\epsilon - \partial_t Z_\epsilon = -\epsilon Z_\epsilon. \quad (2.14)$$

First, by continuity, since $w(\cdot, 0) = 1$, we know that $w \geq 0$ at least for a short time. Applying the maximum principle on (2.11), we see that (2.13) holds at least for a short time. So $Z_\epsilon > 1$ at least for a short time. We now show that

$$Z_\epsilon > 1 \quad (2.15)$$

for all time $t > 0$ as long as the solution exists. Suppose not. Then there exists a first time t_0 and point $x_0 \in \mathbf{M}$ such that $Z_\epsilon(x_0, t_0) = 1$. At this point (x_0, t_0) , the following holds

$$\Delta Z_\epsilon \geq 0, \quad \partial_t Z_\epsilon \leq 0, \quad 2(a-1)VZ_\epsilon \geq 0.$$

This is a contradiction to equation (2.14). Letting $\epsilon \rightarrow 0$ in (2.15), we know (2.13) holds for all time.

Notice that (2.13) implies that $J \leq 1$ as long as the solution exists, which is not obvious to see from (2.7).

Next, for any fixed $T > 0$, we derive an upper bound for $w(x, t)$ on $[0, T]$. We will treat condition (a) and (b) separately.

Let

$$m(t) = \sup_{\mathbf{M} \times [0, t]} w(x, s).$$

First, under condition (b), since

$$\begin{aligned} w(x, t) &= 1 + 2(a-1) \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) w(y, s) dy ds \\ &\leq 1 + 2(a-1) \left(\int_0^t \int_{\mathbf{M}} G(x, t; y, s) V^2(y) dy ds \right)^{1/2} \left(\int_0^t \int_{\mathbf{M}} G(x, t; y, s) w^2(y, s) dy ds \right)^{1/2} \\ &\leq 1 + 2(a-1) \left(\int_0^t \int_{\mathbf{M}} \frac{\hat{C}(t)}{(t-s)^{n/2}} e^{-\frac{\bar{c}d^2(x,y)}{t-s}} V^2(y) dy ds \right)^{1/2} \left(\int_0^t \int_{\mathbf{M}} G(x, t; y, s) m^2(s) dy ds \right)^{1/2} \\ &\leq 1 + C_0(a-1) \sqrt{\hat{C}(t)} \left(\int_{\mathbf{M}} \frac{V^2(y)}{d^{n-2}(x, y)} dy \right)^{1/2} \left(\int_0^t m^2(s) ds \right)^{1/2} \\ &\leq 1 + C_0(a-1) \sqrt{\sigma \hat{C}(T)} \left(\int_0^t m^2(s) ds \right)^{1/2} \end{aligned}$$

for all $t \in [0, T]$ and $x \in \mathbf{M}$, we have

$$m(t) \leq 1 + C_0(a-1) \sqrt{\sigma \hat{C}(T)} \left(\int_0^t m^2(s) ds \right)^{1/2},$$

and hence

$$m^2(t) \leq 2 + 2C_0(a-1)^2 \sigma \hat{C}(T) \int_0^t m^2(s) ds,$$

which is the Grönwall inequality.

Therefore, we get

$$m^2(t) \leq 2e^{2C_0(a-1)^2\sigma\hat{C}(T)t}, \quad (2.16)$$

i.e.,

$$m(t) \leq \sqrt{2}e^{C_0(a-1)^2\sigma\hat{C}(T)t}. \quad (2.17)$$

Especially, we have shown

$$w(x, t) \leq \sqrt{2}e^{C_0(a-1)^2\sigma\hat{C}(t)t}, \quad (2.18)$$

for any $t \in [0, \infty)$. From (2.10), we have

$$2^{-1/(2a-2)}e^{-C_0(a-1)\sigma\hat{C}(t)t} \leq J(x, t) \leq 1.$$

Under condition (a),

$$\begin{aligned} w(x, t) &= 1 + 2(a-1) \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) w(y, s) dy ds \\ &\leq 1 + 2(a-1) \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) m(s) dy ds. \end{aligned}$$

Thus,

$$\begin{aligned} m(t) &\leq 1 + 2(a-1) \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) m(s) dy ds \\ &= 1 + 2(a-1) \int_0^{t-\epsilon} \int_{\mathbf{M}} G(x, t; y, s) V(y) m(s) dy ds \\ &\quad + 2(a-1) \int_{t-\epsilon}^t \int_{\mathbf{M}} G(x, t; y, s) V(y) m(s) dy ds. \end{aligned} \quad (2.19)$$

Notice that

$$\begin{aligned} \int_{\mathbf{M}} G(x, t; y, s) V(y) dy &\leq \|V\|_{L^p} \left(\int_{\mathbf{M}} G^{\frac{p}{p-1}}(x, t; y, s) dy \right)^{(p-1)/p} \\ &= \|V\|_{L^p} \left(\int_{\mathbf{M}} G^{\frac{1}{p-1}} \cdot G dy \right)^{(p-1)/p} \\ &\leq \sigma\hat{C}(t)^{1/p} \frac{1}{(t-s)^{\frac{n}{2p}}}. \end{aligned}$$

Therefore, (2.19) can be further written as

$$m(t) \leq 1 + 2(a-1)\sigma\hat{C}(T)^{1/p} \int_0^{t-\epsilon} \frac{1}{(t-s)^{\frac{n}{2p}}} m(s) ds + 2(a-1)\sigma\hat{C}(T)^{1/p} m(t) \epsilon^{1-\frac{n}{2p}}.$$

Moving the third term on the right hand side to the left hand side, we get

$$\begin{aligned} \left[1 - 2(a-1)\sigma\hat{C}(T)^{1/p} \epsilon^{1-\frac{n}{2p}} \right] m(t) &\leq 1 + 2(a-1)\sigma\hat{C}(T)^{1/p} \int_0^{t-\epsilon} \frac{1}{(t-s)^{\frac{n}{2p}}} m(s) ds \\ &\leq 1 + 2(a-1)\sigma\hat{C}(T)^{1/p} \epsilon^{-\frac{n}{2p}} \int_0^{t-\epsilon} m(s) ds. \end{aligned} \quad (2.20)$$

Setting

$$\epsilon = \left(4(a-1)\sigma\hat{C}(T)^{1/p}\right)^{-\frac{2p}{2p-n}},$$

we have

$$1 - 2(a-1)\sigma\hat{C}(T)^{1/p}\epsilon^{1-\frac{n}{2p}} = \frac{1}{2}.$$

Therefore, (2.20) becomes

$$m(t) \leq 2 + \left(4(a-1)\sigma\hat{C}(T)^{1/p}\right)^{\frac{2p}{2p-n}} \int_0^t m(s)ds,$$

which again is the Grönwall inequality.

Hence, we have

$$m(t) \leq 2 \exp \left\{ \left(4(a-1)\sigma\hat{C}(T)^{1/p}\right)^{\frac{2p}{2p-n}} t \right\}, \quad (2.21)$$

for all $t \in [0, T]$. Especially, we have

$$w(x, t) \leq 2 \exp \left\{ \left(4(a-1)\sigma\hat{C}(t)^{1/p}\right)^{\frac{2p}{2p-n}} t \right\}, \quad (2.22)$$

for any $t \in [0, \infty)$, i.e.,

$$2^{-1/(a-1)} e^{-(a-1)\frac{n}{2p-n} [4\sigma\hat{C}(t)^{1/p}]^{\frac{2p}{2p-n}} t} \leq J(x, t) \leq 1.$$

Therefore, J exists for $t \in [0, \infty)$. This completes the proof of the claim 2.1.

Now we continue with the proof of part (1). In (2.6), choosing J as in Claim 2.1, then we deduce

$$(\Delta - \partial_t)(tQ) + 2\frac{\nabla u}{u}\nabla(tQ) \geq \alpha t(2J - \delta J)\frac{1}{n} (|\nabla f|^2 - \partial_t f)^2 - \delta \alpha t J |\nabla f|^4 - Q \quad (2.23)$$

For any $T > 0$, let (x_0, t_0) be a maximum point of $tQ = t\left(\alpha J \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}\right)$ in $\mathbf{M} \times [0, T]$. Then at this point, the above inequality induces

$$0 \geq \alpha t(2J - \delta J)\frac{1}{n} (|\nabla f|^2 - \partial_t f)^2 - \delta \alpha t J |\nabla f|^4 - Q.$$

Clearly we can assume $Q \geq 0$ at (x_0, t_0) since the result is already proven otherwise. Then

$$(|\nabla f|^2 - \partial_t f)^2 \geq \left(\alpha J \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u}\right)^2 + (1 - \alpha J)^2 |\nabla f|^4.$$

Plugging this into the previous inequality, we find that

$$0 \geq \alpha \frac{2J - \delta J}{n} t Q^2 + \left[\frac{2 - \delta}{n} (1 - \alpha J)^2 - \delta \right] \alpha t J |\nabla f|^4 - Q.$$

By choosing

$$\delta = \frac{2(1 - \alpha)^2}{n + (1 - \alpha)^2}, \quad (2.24)$$

one has

$$\frac{2 - \delta}{n} (1 - \alpha)^2 - \delta = 0. \quad (2.25)$$

Since $J \leq 1$, we derive from above that

$$\frac{2-\delta}{n}(1-\alpha J)^2 - \delta \geq 0 \quad \text{on } \mathbf{M} \times [0, \infty).$$

Therefore, at (x_0, t_0) ,

$$0 \geq \alpha \frac{2J - \delta J}{n} t^2 Q^2 - tQ,$$

which infers

$$tQ \leq tQ|_{(x_0, t_0)} \leq \frac{n}{(2-\delta)\alpha J} \leq \frac{n}{(2-\delta)\alpha \underline{J}(T)},$$

i.e.,

$$\alpha \underline{J}(t) \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{(2-\delta)\alpha \underline{J}(t)} \frac{1}{t}. \quad (2.26)$$

This proves part (1) of the theorem.

For part (2), we first prove an improved short time bound for J . Consider the closed ball in $L^\infty(\mathbf{M} \times [0, T_0])$

$$X = \{w \in L^\infty(\mathbf{M} \times [0, T_0]) \mid 1 \leq w \leq 1 + \eta\}. \quad (2.27)$$

Here η is a positive number in $(0, 1)$, and T_0 is a constant to be determined. Let $w_0 = w(\cdot, 0) = 1$, and P the map

$$Pw = w_0 + 2(a-1) \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) w(y, s) dy ds. \quad (2.28)$$

For any $w \in X$, since $w \geq w_0 = 1$, we have

$$Pw \geq w_0.$$

Moreover,

$$\begin{aligned} Pw - w_0 &\leq 2(a-1) \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) w(y, s) dy ds \\ &\leq 2(a-1)(1+\eta)w_0 \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) dy ds. \end{aligned} \quad (2.29)$$

Notice that, under condition (b), we have $\sup_x \int_{\mathbf{M}} \frac{|\text{Ric}^-(y)|^2}{d(x, y)^{n-2}} dy < \infty$. Then by using the Gaussian upper bound of G ,

$$\begin{aligned}
& \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) dy ds \\
& \leq \left(\int_0^t \int_{\mathbf{M}} G(x, t; y, s) dy ds \right)^{1/2} \left(\int_0^t \int_{\mathbf{M}} G(x, t; y, s) V^2(y) dy \right)^{1/2} \\
& \leq C\sqrt{t} \left(\int_0^t \int_{\mathbf{M}} \frac{1}{(t-s)^{n/2}} e^{-cd^2(x,y)/(t-s)} V^2(y) dy ds \right)^{1/2} \\
& \leq C\sqrt{t} \left(\int_{\mathbf{M}} \frac{1}{d^{n-2}(x, y)} V^2(y) dy \right)^{1/2} \\
& = C\sqrt{t}\sqrt{\sigma} \equiv C_0\sqrt{t}.
\end{aligned}$$

Hence

$$\int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) dy ds \leq C_0\sqrt{t}. \quad (2.30)$$

If $|Ric^-| \in L^p$ with $p > n/2$, the $1/2$ power on t on the right hand side above should be replaced by $1 - \frac{n}{2p}$. Here is a quick proof. By Remark 1.2, the heat kernel G also has an Gaussian upper bound and $|B(x, r)| \leq Cr^n$. So

$$\begin{aligned}
& \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) dy ds \\
& \leq C \int_0^t \int_{\mathbf{M}} \frac{1}{(t-s)^{n/2}} e^{-cd^2(x,y)/(t-s)} V(y) dy ds \\
& \leq C \int_0^t \left(\int_{\mathbf{M}} \frac{1}{(t-s)^{np/[2(p-1)]}} e^{-cp/(p-1) \frac{d^2(x,y)}{t-s}} dy \right)^{(p-1)/p} ds \|V\|_{L^p} \\
& \leq C_0 t^{1 - \frac{n}{2p}}.
\end{aligned} \quad (2.31)$$

In the following, we prove the theorem under the condition (b), so that (2.30) holds. The proof under condition (a) works verbatim after replacing (2.30) by (2.31).

From (2.29) and (2.30), we see that

$$Pw - w_0 \leq C_0(a-1)\sqrt{t}w_0.$$

If we choose

$$T_0 = [C_0(a-1)]^{-2} \eta^2 = [C_0(5\delta^{-1} - 1)]^{-2} \eta^2, \quad (2.32)$$

then

$$Pw - w_0 \leq \eta w_0. \quad (2.33)$$

Thus P maps X into X .

Next we show that P is a contraction mapping on X when T_0 is chosen as in (2.32). Let w_1 and w_2 be two elements in X . Then (2.28) implies

$$\begin{aligned}
& |Pw_2 - Pw_1|(x, t) = 2(a-1) \left| \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) (w_2 - w_1)(y, s) dy ds \right| \\
& \leq 2(a-1) \int_0^t \int_{\mathbf{M}} G(x, t; y, s) V(y) dy ds \|w_2 - w_1\|_{\infty} \\
& \leq C_0(a-1)\sqrt{t} \|w_2 - w_1\|_{\infty}.
\end{aligned}$$

By (2.32), we know that under condition (b) of the theorem, (2.33) holds and also

$$\|Pw_2 - Pw_1\|_\infty \leq \eta \|w_2 - w_1\|_\infty. \quad (2.34)$$

Hence P is a contraction map from X to X . The unique fixed point, named w , is a solution to (2.12) and (2.11). By the definition of X , we already know that on $\mathbf{M} \times [0, T_0]$,

$$1 \leq w \leq 1 + \eta.$$

From the relations (2.10), we know that

$$J = w^{\frac{1}{a-1}} = w^{-\frac{\delta}{5-\delta}}.$$

Hence

$$(1 + \eta)^{-\frac{\delta}{5-\delta}} \leq J \leq 1. \quad (2.35)$$

Let

$$\beta = \frac{n\alpha}{n + (1 - \alpha)^2} (1 + \eta)^{-\frac{\delta}{5-\delta}}. \quad (2.36)$$

Then, (2.26) can be rewritten as

$$\beta \frac{n + (1 - \alpha)^2 |\nabla u|^2}{n u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2\beta t},$$

which obviously implies (1.11).

Moreover, from (2.32), (2.24), and (2.36), we see that

$$T_0 = c(1 - \beta)^4 \quad (2.37)$$

under condition (b) of the theorem.

Similarly, under condition (a), one can get

$$T_0 = [C_0(5\delta - 1)]^{-2p/(2p-n)} \eta^{2p/(2p-n)}, \quad (2.38)$$

and hence

$$T_0 = c(1 - \beta)^{4p/(2p-n)}. \quad (2.39)$$

□

3. RICCI FLOW CASE

In this section, we consider the Li-Yau bound in the Ricci flow case and prove Theorem 1.4. The main tool is still the maximum principle applied on a differential inequality involving Li-Yau type quantity. However, due to the Ricci flow, extra terms involving the Ricci curvature and Hessian of the solution will come out. In order to proceed we need to create a new term with the scalar curvature in the denominator.

Before proving the theorem, we carry out some basic computations.

Lemma 3.1. *Let*

$$F = -\Delta u + \delta \frac{|\nabla u|^2}{u} - \alpha R u + \frac{\beta u}{R + C}, \quad (3.1)$$

and operator $\mathcal{L} = \Delta - \frac{\partial}{\partial t}$, where δ , α , and β are arbitrary constants and C is a constant so that $R + C > 0$. Then

$$\begin{aligned} \mathcal{L}F &= \frac{1}{u} \left| u_{ij} - \frac{u_i u_j}{u} + u R_{ij} \right|^2 + \frac{2\delta - 1}{u} \left| u_{ij} - \frac{u_i u_j}{u} \right|^2 + \frac{1}{(2\alpha - 1)u} \left| (2\alpha - 1)u R_{ij} + \frac{u_i u_j}{u} \right|^2 \\ &\quad - \frac{1}{2\alpha - 1} \frac{|\nabla u|^4}{u^3} + \frac{\alpha u}{R + C} \left| \nabla R - \frac{R + C}{u} \nabla u \right|^2 + \left(\frac{\beta}{(R + C)^3} - \frac{\alpha}{R + C} \right) u |\nabla R|^2 \\ &\quad - \frac{\alpha(R + C) |\nabla u|^2}{u} + \frac{2\beta u |R_{ij}|^2}{(R + C)^2} + \frac{\beta u}{R + C} \left| \frac{\nabla R}{R + C} - \frac{\nabla u}{u} \right|^2 - \frac{\beta |\nabla u|^2}{u(R + C)}. \end{aligned} \quad (3.2)$$

Proof. It follows from (1.6) that

$$\mathcal{L}(\Delta u) = -2R_{ij}u_{ij}, \quad (3.3)$$

and

$$\mathcal{L}(|\nabla u|^2) = 2|u_{ij}|^2. \quad (3.4)$$

Also, it is well known that under the Ricci flow we have

$$\mathcal{L}R = -2|R_{ij}|^2. \quad (3.5)$$

On the other hand, it is straightforward to check that for any smooth functions f and g , one has

$$\mathcal{L}\left(\frac{f}{g}\right) = \frac{1}{g}\mathcal{L}f - \frac{f}{g^2}\mathcal{L}g - \frac{2}{g}\nabla_i \frac{f}{g} \nabla_i g, \quad (3.6)$$

and

$$\mathcal{L}(fg) = f\mathcal{L}g + g\mathcal{L}f + 2\nabla_i f \nabla_i g. \quad (3.7)$$

It then follows from (1.6), (3.4), (3.5), (3.6) and (3.7) that

$$\begin{aligned} \mathcal{L}\left(\frac{|\nabla u|^2}{u}\right) &= \frac{1}{u}\mathcal{L}|\nabla u|^2 - \frac{2}{u}\nabla_i \frac{|\nabla u|^2}{u} \nabla_i u \\ &= \frac{2}{u}|u_{ij}|^2 - \frac{4}{u^2}u_{ij}u_i u_j + \frac{2|\nabla u|^4}{u^3} \\ &= \frac{2}{u}\left|u_{ij} - \frac{u_i u_j}{u}\right|^2, \end{aligned} \quad (3.8)$$

$$\mathcal{L}(Ru) = u\mathcal{L}R + 2\nabla_i R \nabla_i u = -2u|R_{ij}|^2 + 2\nabla_i R \nabla_i u, \quad (3.9)$$

and

$$\begin{aligned} \mathcal{L}\left(\frac{u}{R + C}\right) &= -\frac{u}{(R + C)^2}\mathcal{L}R - \frac{2}{R + C}\nabla_i \frac{u}{R + C} \nabla_i R \\ &= \frac{2u|R_{ij}|^2}{(R + C)^2} + \frac{2u|\nabla R|^2}{(R + C)^3} - \frac{2}{(R + C)^2}\nabla_i R \nabla_i u. \end{aligned} \quad (3.10)$$

Thus, by (3.1), (3.3), (3.8), (3.9) and (3.10), we have, after splitting zeros in four occasions, that

$$\begin{aligned}
 \mathcal{L}F &= 2R_{ij}u_{ij} + \frac{2\delta}{u} \left| u_{ij} - \frac{u_i u_j}{u} \right|^2 + 2\alpha u |R_{ij}|^2 - 2\alpha \nabla_i R \nabla_i u \\
 &\quad + \frac{2\beta u |R_{ij}|^2}{(R+C)^2} + \frac{2\beta u |\nabla R|^2}{(R+C)^3} - \frac{2\beta}{(R+C)^2} \nabla_i R \nabla_i u \\
 &= \frac{1}{u} \left| u_{ij} - \frac{u_i u_j}{u} + u R_{ij} \right|^2 + \frac{2\delta-1}{u} \left| u_{ij} - \frac{u_i u_j}{u} \right|^2 + (2\alpha-1)u |R_{ij}|^2 + 2R_{ij} \frac{u_i u_j}{u} \\
 &\quad + \frac{1}{2\alpha-1} \frac{|\nabla u|^4}{u^3} - \frac{1}{2\alpha-1} \frac{|\nabla u|^4}{u^3} - 2\alpha \nabla_i R \nabla_i u + \frac{\alpha u |\nabla R|^2}{R+C} + \frac{\alpha(R+C)|\nabla u|^2}{u} \\
 &\quad + \left(\frac{\beta}{(R+C)^3} - \frac{\alpha}{R+C} \right) u |\nabla R|^2 - \frac{\alpha(R+C)|\nabla u|^2}{u} + \frac{2\beta u |R_{ij}|^2}{(R+C)^2} - \frac{2\beta}{(R+C)^2} \nabla_i R \nabla_i u \\
 &\quad + \frac{\beta u |\nabla R|^2}{(R+C)^3} + \frac{\beta |\nabla u|^2}{u(R+C)} - \frac{\beta |\nabla u|^2}{u(R+C)}.
 \end{aligned}$$

Observe that the 3rd, 4th and 5th terms, 7th, 8th and 9th terms and 13th, 14th and 15th terms form complete squares, respectively. Hence we get (3.2). \square

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4: Assume that $1 > \delta \geq \frac{1}{2}$ and $\alpha > 1$. By choosing $C = 2$ and $\beta = \alpha(R_1 + 2)^2$ in the above lemma, we have

$$\begin{aligned}
 \mathcal{L}F &\geq \frac{1}{nu} \left| \Delta u - \frac{|\nabla u|^2}{u} + uR \right|^2 + \frac{2\delta-1}{nu} \left| \Delta u - \frac{|\nabla u|^2}{u} \right|^2 - \frac{1}{2\alpha-1} \frac{|\nabla u|^4}{u^3} \\
 &\quad - \frac{\alpha(R+2)|\nabla u|^2}{u} - \frac{\beta|\nabla u|^2}{u(R+2)}.
 \end{aligned} \tag{3.11}$$

Notice that

$$\Delta u - \frac{|\nabla u|^2}{u} = \left(\Delta u - \frac{|\nabla u|^2}{u} + uR \right) - uR.$$

We rewrite (3.11) as

$$\begin{aligned}
 \mathcal{L}F &\geq \frac{2\delta}{nu} \left| \Delta u - \frac{|\nabla u|^2}{u} + uR \right|^2 - \frac{2(2\delta-1)R}{n} \left(\Delta u - \frac{|\nabla u|^2}{u} + uR \right) + \frac{(2\delta-1)R^2 u}{n} \\
 &\quad - \frac{1}{2\alpha-1} \frac{|\nabla u|^4}{u^3} - \frac{\alpha(R+2)|\nabla u|^2}{u} - \frac{\beta|\nabla u|^2}{u(R+2)}.
 \end{aligned}$$

According to the definition of F in (3.1), the above inequality becomes

$$\begin{aligned}
 \mathcal{L}F &\geq \frac{2\delta}{nu} \left| F + (1-\delta) \frac{|\nabla u|^2}{u} + (\alpha-1)Ru - \frac{\beta u}{R+2} \right|^2 \\
 &\quad + \frac{2(2\delta-1)R}{n} \left(F + (1-\delta) \frac{|\nabla u|^2}{u} + (\alpha-1)Ru - \frac{\beta u}{R+2} \right) \\
 &\quad + \frac{(2\delta-1)R^2 u}{n} - \frac{1}{2\alpha-1} \frac{|\nabla u|^4}{u^3} - \left[\alpha(R+2) + \frac{\beta}{R+2} \right] \frac{|\nabla u|^2}{u}.
 \end{aligned}$$

Let $Q = tF - \theta u$. Then at $t = 0$, we have $Q < 0$. Suppose that at time $t_0 > 0$ and point $x_0 \in \mathbf{M}$, Q reaches 0 for the first time. Then at (x_0, t_0) , we have $t_0 F = \theta u$ and

$$\begin{aligned} 0 &\geq t_0 \mathcal{L}Q(x_0, t_0) \\ &\geq -\theta u + \frac{2\delta}{nu} \left| \theta u + (1-\delta) \frac{|\nabla u|^2}{u} t_0 + (\alpha-1) R u t_0 - \frac{\beta u t_0}{R+2} \right|^2 \\ &\quad + \frac{2(2\delta-1)Rt_0}{n} \left(\theta u + (1-\delta) \frac{|\nabla u|^2}{u} t_0 + (\alpha-1) R u t_0 - \frac{\beta u t_0}{R+2} \right) \\ &\quad + \frac{(2\delta-1)R^2 u}{n} t_0^2 - \frac{1}{2\alpha-1} \frac{|\nabla u|^4}{u^3} t_0^2 - \left[\alpha(R+2) + \frac{\beta}{R+2} \right] \frac{|\nabla u|^2}{u} t_0^2 \end{aligned}$$

After expanding the first square, we deduce

$$\begin{aligned} 0 &\geq -\theta u + \frac{2\delta}{n} \theta^2 u + \frac{2\delta(1-\delta)^2}{n} \frac{|\nabla u|^4}{u^3} t_0^2 + \frac{2\delta(\alpha-1)^2 R^2 u}{n} t_0^2 \\ &\quad + \frac{2\delta\beta^2 u}{n(R+2)^2} t_0^2 + \frac{4\delta(1-\delta)\theta}{n} \frac{|\nabla u|^2}{u} t_0 + \frac{4\delta(\alpha-1)\theta R u}{n} t_0 - \frac{4\delta\beta\theta u}{n(R+2)} t_0 \\ &\quad + \frac{4\delta(1-\delta)(\alpha-1)R}{n} \frac{|\nabla u|^2}{u} t_0^2 - \frac{4\delta(1-\delta)\beta}{n(R+2)} \frac{|\nabla u|^2}{u} t_0^2 - \frac{4\delta(\alpha-1)\beta R u}{n(R+2)} t_0^2 \\ &\quad + \frac{2(2\delta-1)\theta R u}{n} t_0 + \frac{2(2\delta-1)(1-\delta)R}{n} \frac{|\nabla u|^2}{u} t_0^2 + \frac{2(2\delta-1)(\alpha-1)R^2 u}{n} t_0^2 \\ &\quad - \frac{2(2\delta-1)\beta R u}{n(R+2)} t_0^2 + \frac{(2\delta-1)R^2 u}{n} t_0^2 - \frac{1}{2\alpha-1} \frac{|\nabla u|^4}{u^3} t_0^2 - \left[\alpha(R+2) + \frac{\beta}{R+2} \right] \frac{|\nabla u|^2}{u} t_0^2. \end{aligned}$$

This becomes, after combining similar terms,

$$\begin{aligned} 0 &\geq -\theta u + \frac{2\delta}{n} \theta^2 u - \frac{4\delta\beta\theta u}{n(R+2)} t_0 - \frac{(4\delta\alpha-2)\theta u}{n} t_0 - \frac{(4\delta\alpha-2)\beta R_1 u}{n(R+2)} t_0^2 \\ &\quad + \left(\frac{2\delta(1-\delta)^2}{n} - \frac{1}{2\alpha-1} \right) \frac{|\nabla u|^4}{u^3} t_0^2 \\ &\quad + \left(\frac{4\delta(1-\delta)\theta}{n} t_0 - \frac{4\delta(1-\delta)\beta}{n(R+2)} t_0^2 - \frac{2\beta}{(R+2)} t_0^2 - \frac{(4\delta\alpha-2)(1-\delta)}{n} t_0^2 \right) \frac{|\nabla u|^2}{u}. \end{aligned}$$

It is straightforward to check that by choosing

$$\alpha = \frac{n}{2\delta(1-\delta)^2}, \quad \text{and} \quad \theta = \frac{n}{2\delta} + \frac{4n\beta T}{\delta(1-\delta)}$$

one has

$$\frac{2\delta(1-\delta)^2}{n} - \frac{1}{2\alpha-1} > 0, \quad (3.12)$$

$$\frac{4\delta(1-\delta)\theta}{nT} \geq \left[\frac{4\delta(1-\delta)}{n} + 2 \right] \beta + \frac{(4\delta\alpha-2)(1-\delta)}{n}, \quad (3.13)$$

and

$$\frac{2\delta}{nT^2} \theta^2 - \left(\frac{1}{T^2} + \frac{4\delta\beta}{nT} + \frac{4\delta\alpha}{nT} \right) \theta - \frac{4\delta\alpha\beta R_1}{n} > 0. \quad (3.14)$$

Therefore, we have a contradiction. It follows that

$$-\Delta u + \delta \frac{|\nabla u|^2}{u} - \alpha R u + \frac{\beta u}{R+2} \leq \frac{\theta u}{t}$$

for any $t \in (0, T)$, which is (1.13). \square

In general, along the Ricci flow we have

$$-\sup_{\mathbf{M}} R(x, 0) \leq R(x, t) \leq \sup_{\mathbf{M} \times [0, T]} R(x, t).$$

Denote by $R_1 = \sup_{\mathbf{M} \times [0, T]} R(x, t)$ and

$$R_0 = \begin{cases} \sup_{\mathbf{M}} R^-(x, 0), & \text{if } \sup_{\mathbf{M}} R^-(x, 0) > 0 \\ \inf_{\mathbf{M}} R(x, 0), & \text{if } \sup_{\mathbf{M}} R^-(x, 0) = 0. \end{cases}$$

It is not hard to check that by choosing $C = 2R_0$ and $\beta = \alpha(R_1 + 2R_0)^2$ in Lemma 3.1 and repeating the proof of Theorem 1.4, we can get the following scaling invariant Li-Yau bounds.

Theorem 3.2. *Let \mathbf{M} be a compact n dimensional Riemannian manifold, and $g_{ij}(t)$, $t \in [0, T)$, a solution of the Ricci flow (1.5) on \mathbf{M} . Suppose that u is a positive solution of the heat equation (1.6). Then, for any $\delta \in [\frac{1}{2}, 1)$, when $\sup_{\mathbf{M}} R^-(x, 0) > 0$, we have*

$$\delta \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} - \alpha R + \frac{\beta}{R + 2R_0} \leq \frac{\theta}{t}, \quad (3.15)$$

and when $\sup_{\mathbf{M}} R^-(x, 0) = 0$, we have

$$\delta \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} - \alpha R + \frac{\beta}{R} \leq \frac{\theta}{t}, \quad (3.16)$$

for any $t \in (0, T)$, where $\alpha = \frac{n}{2\delta(1-\delta)^2}$, $\beta = \alpha(R_1 + 2R_0)^2$, and $\theta = \frac{n}{2\delta} + \frac{4n\beta T}{\delta(1-\delta)R_0}$.

Remark 3.3. *From (1.8), one can see that if $R > 0$, then there are both a forward inequality $u_t \geq -\frac{Ba}{t}$, and a backward inequality $u_t \leq \frac{Ba}{t}$.*

The Li-Yau bound (1.13) obtained above gives us a stronger forward Harnack inequality $\frac{u_t}{u} \geq -\frac{k}{t}$ when $R > 0$. However, it seems that a backward Harnack inequality of the form $\frac{u_t}{u} \leq \frac{k}{t}$ cannot be expected. Because if this were the case, then one would have $u(x, t_2) \leq u(x, t_1) \left(\frac{t_2}{t_1}\right)^k$. Now suppose that M is an Einstein manifold $R_{ij} = \rho g_{ij}$ with $\rho > 0$ and $u(x, t) = G(x, t; x_0, 0)$ the heat kernel under the Ricci flow. According to a result in [CZ], we have the Gaussian lower and upper bounds of G , i.e.,

$$Ct^{-\frac{n}{2}} e^{-\frac{cd_{t_1}^2(x, x_0)}{t}} \leq G(x, t; x_0, 0) \leq Ct^{-\frac{n}{2}} e^{-\frac{d_t^2(x, y)}{ct}}.$$

It then follows that

$$Ct_2^{-\frac{n}{2}} e^{-\frac{cd_{t_2}^2(x, x_0)}{t_2}} \leq G(x, t_2; x_0, 0) \leq G(x, t_1; x_0, 0) \left(\frac{t_2}{t_1}\right)^k \leq C \left(\frac{t_2}{t_1}\right)^k t_1^{-\frac{n}{2}} e^{-\frac{d_{t_1}^2(x, y)}{ct_1}},$$

i.e.,

$$e^{-\frac{(1-2\rho t_2)d_0^2(x, x_0)}{t_2} + \frac{(1-2\rho t_1)d_0^2(x, x_0)}{t_1}} \leq C \left(\frac{t_2}{t_1}\right)^{\frac{n}{2} + k}.$$

Obviously, when $t_2 = 2t_1$ and $x \neq x_0$, we get a contradiction for t_1 small enough.

4. APPLICATIONS ON EXTENDING COLDING-NABER RESULT

In this section, we mainly apply the Li-Yau bound (1.11) to extend parabolic approximations of distance functions of Colding-Naber [CoNa] to the case where $|Ric^-| \in L^p$ for some $p > \frac{n}{2}$. Meanwhile, some of the intermediate results can also be proved by replacing the condition that $|Ric^-| \in L^p$ by $|Ric^-| \in K^{2,n-2}$, where $K^{p,\lambda}$ denotes the Kato type space with the norm

$$\|w\|_{K^{p,\lambda}} = \left(\sup_{\mathbf{M}} \int_{\mathbf{M}} \frac{|w|^p}{d^\lambda(x,y)} dy \right)^{\frac{1}{p}}.$$

Let (\mathbf{M}, g_{ij}) be a compact n dimensional Riemannian manifold. Parts of the following three assumptions will be used in the results of this section.

A1: M is κ -noncollapsed for some constant κ , i.e.,

$$\text{Vol}(B_r(x)) \geq \kappa r^n, \quad \forall x \in \mathbf{M}, \text{ and } r \leq 1. \quad (4.1)$$

A2: $\|Ric^-\|_{L^p} \leq \Lambda$ for some $p > \frac{n}{2}$.

A3: $\|Ric^-\|_{K^{2,n-2}} \leq \Gamma$, the heat kernel of (1.1) has a Gaussian upper bound for $0 < t \leq 1$ as in (1.12), and $|B(x, r)| \leq Cr^n$.

We mention that conditions A3 holds on each time slice of the normalized Kähler-Ricci flow, which is the motivation for imposing such a condition ([TZq1] and [TZq2]).

Let $h_t^\pm(x)$ be the parabolic approximations of the local distance functions as defined in (4.25). The main result of this section is

Theorem 4.1. (Tian-Z. Zhang in [TZz]) *Assume that A1 and A2 are satisfied. Let O^+ and O^- be two fixed points in \mathbf{M} . Denote by $d_0 = d(O^+, O^-)$. Then for some fixed $\delta > 0$, there exist constants $C = C(n, p, \kappa, \Lambda, \delta)$ and $\bar{\epsilon} = \bar{\epsilon}(n, p, \delta)$, such that for any $0 < \epsilon \leq \bar{\epsilon}$,*

$$x \in M_{\delta,2} \equiv \{x \in \mathbf{M} \mid \delta d_0 < d(x, \{O^+, O^-\}) \leq 2d_0\}$$

with

$$e(x) \equiv d(O^-, x) + d(O^+, x) - d(O^+, O^-) \leq \epsilon^2 d_0,$$

and any ϵ -geodesic σ connecting O^+ and O^- , there exists $r \in [\frac{1}{2}, 2]$ satisfying

- (1) $\left| h_{r\epsilon^2 d_0^2}^\pm - d^\pm \right| \leq C d_0 (\epsilon^2 + \epsilon^{2-\frac{n}{2p}}).$
- (2) $\oint_{B_{\epsilon d_0}(x)} \left| |\nabla h_{r\epsilon^2 d_0^2}^\pm|^2 - 1 \right| \leq C (\epsilon + \epsilon^{1-\frac{n}{2p}}).$
- (3) $\oint_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{\epsilon d_0}(\sigma(s))} \left| |\nabla h_{r\epsilon^2 d_0^2}^\pm|^2 - 1 \right| \leq C (\epsilon^2 + \epsilon^{2-\frac{n}{p}}).$
- (4) $\oint_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{\epsilon d_0}(\sigma(s))} \left| \nabla^2 h_{r\epsilon^2 d_0^2}^\pm \right|^2 \leq \frac{C(1 + \epsilon^{-\frac{n}{p}})}{d_0^2}.$

More explanations of the notations in the above theorem can be found in the following context. The theorem was first obtained by Tian-Z. Zhang in [TZz]. Here by using the Li-Yau bound (1.11), following the original route in [CoNa] for the case where the Ricci curvature is bounded from below, we are able to exemplify some of the results. Alternatively, one may also derive a Gaussian estimate of $|\frac{\partial G}{\partial t}|$ which then can be used in place of the Li-Yau bound.

A very important tool that will be used repeatedly in this section is the following volume comparison theorem proved by Petersen-Wei.

Theorem 4.2. (Petersen-Wei [PW1]) *If A2 is satisfied, then there exists a constant $C = C(n, p)$ which is nondecreasing in R such that for all $r \leq R$ and $x \in \mathbf{M}$, we have*

$$\left(\frac{\text{Vol}(B_R(x))}{R^n}\right)^{1/2p} - \left(\frac{\text{Vol}(B_r(x))}{r^n}\right)^{1/2p} \leq C\Lambda^{1/2p}R^{1-\frac{n}{2p}}, \quad (4.2)$$

where $B_r(x)$ denotes the geodesic ball centered at x with radius r .

A very important corollary of the above theorem is the following volume doubling property (see [PW2] Theorem 2.1).

Theorem 4.3. (Petersen-Wei [PW2]) *Given $\alpha < 1$ and $p > n/2$. Assume that A1 and A2 are satisfied. Then there exists an $R = R(\alpha, p, n, \Lambda) > 0$ such that for any $0 < r_1 \leq r_2 \leq R$, we have*

$$\alpha \frac{r_1^n}{r_2^n} \leq \frac{\text{vol} B_{r_1}(x)}{\text{vol} B_{r_2}(x)}. \quad (4.3)$$

By using the above theorem, Petersen-Wei also obtained the following cut-off function, which was first observed by Cheeger-Colding in [ChCo] for manifolds with Ricci curvature bounded from below.

Lemma 4.4. (Petersen-Wei [PW2]) *Suppose that A1 and A2 are satisfied. There exist $r_0 = r_0(n, p, \kappa, \Lambda)$ and $C = C(n, p, \kappa, \Lambda)$ such that on any geodesic ball $B_r(x)$, $r \leq r_0$, there exists a function $\phi \in C_0^\infty(B_r(x))$ such that*

$$\phi \geq 0, \quad \phi = 1 \text{ in } B_{r/2}(x),$$

and

$$|\nabla\phi|^2 + |\Delta\phi| \leq Cr^{-2}.$$

Let E be a closed subset of M . Denote the r -tubular neighborhood of E by

$$T_r(E) = \{x \in \mathbf{M} \mid d(x, E) \leq r\}.$$

For $0 < r_1 < r_2$, define the annulus $A_{r_1, r_2}(E) = T_{r_2}(E) \setminus \overline{T_{r_1}(E)}$. Using the lemma above and a similar argument as in the proof of Lemma 2.6 in [CoNa], one has

Lemma 4.5. (Tian-Z. Zhang [TZz]) *Suppose that A1 and A2 are satisfied. For any $R > 0$, there exists $C = C(n, p, \kappa, \Lambda, R)$ such that the following holds. Let E be any closed subset and $0 < r_1 < 10r_2 < R$. There exists a function $\phi \in C^\infty(B_R(E))$ satisfying*

$$\phi \geq 0, \quad \phi = 1 \text{ in } A_{3r_1, r_2/3}(E), \quad \phi = 0 \text{ outside } A_{2r_1, r_2/2}(E),$$

$$|\nabla\phi|^2 + |\Delta\phi| \leq Cr_1^{-2} \text{ in } A_{2r_1, 3r_1}(E),$$

and

$$|\nabla\phi|^2 + |\Delta\phi| \leq Cr_2^{-2} \text{ in } A_{r_2/3, r_2/2}(E).$$

Let $G(y, t; x, 0) = G(x, t; y, 0)$ be the heat kernel on M . It can be showed that $G(y, t; x, 0)$ has both Gaussian upper and lower bounds as follows

Lemma 4.6. (*Tian-Z. Zhang [TZz]*) *Suppose that A1 and A2 are satisfied. There exist positive constants $C_i = C_i(n, p, \kappa, \Lambda)$, $i = 1, 2, 3, 4$, such that*

$$C_1 t^{-\frac{n}{2}} e^{-\frac{C_2 d^2(x,y)}{t}} \leq G(y, t; x, 0) \leq C_3 t^{-\frac{n}{2}} e^{-\frac{d^2(x,y)}{C_4 t}}, \quad \forall x, y \in \mathbf{M}, \text{ and } 0 < t \leq 1. \quad (4.4)$$

Actually, the Gaussian upper bound can be obtained by an L^1 mean value inequality for $G(y, t; x, 0)$ and Grigor'yan's method in [Gri]. Then the lower bound follows from the upper bound and an on-diagonal gradient bound for $G(y, t; x, 0)$.

By using Duhamel's principle, it is not hard to prove the following L^1 Harnack inequalities (see e.g. [TZz]).

Lemma 4.7. *Let $u(x, t)$ be a nonnegative function satisfying*

$$\frac{\partial u}{\partial t} \geq \Delta u - \xi, \quad (4.5)$$

where $\xi = \xi(x) \geq 0$ is a smooth function.

(i) *If A1 and A2 are satisfied, then for any $q > \frac{n}{2}$, there exists a constant $C = C(n, p, q, \kappa, \Lambda)$ such that*

$$\oint_{B_r(x)} u(y, 0) dy \leq C \left(u(x, r^2) + r^{2-\frac{n}{q}} \|\xi\|_{L^q} \right) \quad (4.6)$$

holds for any $x \in \mathbf{M}$ and $0 < r \leq 1$.

More generally, we have

$$\oint_{B_r(x)} u(y, 0) dy \leq C \left(\inf_{B_r(x)} u(\cdot, r^2) + r^{2-\frac{n}{q}} \|\xi\|_{L^q} \right). \quad (4.7)$$

(ii) *If A1 and A3 are satisfied, then for any $q > 0$ and $\lambda > n - 2q$, there exists a constant $C = C(n, q, \lambda, \kappa, \Gamma)$ such that*

$$\oint_{B_r(x)} u(y, 0) dy \leq C \left(u(x, r^2) + r^{2-\frac{n-\lambda}{q}} \|\xi\|_{K^{q,\lambda}} \right) \quad (4.8)$$

holds for any $x \in \mathbf{M}$ and $0 < r \leq 1$.

More generally, we have

$$\oint_{B_r(x)} u(y, 0) dy \leq C \left(\inf_{B_r(x)} u(\cdot, r^2) + r^{2-\frac{n-\lambda}{q}} \|\xi\|_{K^{q,\lambda}} \right). \quad (4.9)$$

Proof. By Duhamel's principle, we have

$$u(x, t) - \int_{\mathbf{M}} u(y, 0) G(y, t; x, 0) dy \geq - \int_0^t \int_{\mathbf{M}} \xi(y) G(y, t-s; x, 0) dy ds.$$

If $A1$ and $A2$ are satisfied, from Lemma 4.6, we have

$$\begin{aligned} \int_{B_r(x)} u(y, 0)G(y, t; x, 0)dy &\leq u(x, t) + \int_0^t \int_{\mathbf{M}} \xi(y)G(y, t-s; x, 0)dyds \\ &\leq u(x, t) + C \int_0^t \|\xi\|_{L^q} \left(\int_{\mathbf{M}} (t-s)^{-\frac{nq}{2(q-1)}} e^{-\frac{d^2(x,y)}{C(t-s)}} dy \right)^{\frac{q-1}{q}} ds \\ &\leq u(x, t) + Ct^{1-\frac{n}{2q}} \|\xi\|_{L^q}. \end{aligned}$$

By (4.1) and the lower bound of $G(y, t; x, 0)$ in (4.4), we have

$$\oint_{B_r(x)} u(y, 0)dy \leq C \int_{B_r(x)} u(y, 0)G(y, r^2; x, 0)dy,$$

from which (4.6) follows easily.

If $A1$ and $A3$ are satisfied, by a similar argument as above, we get

$$\begin{aligned} \int_{B_r(x)} u(y, 0)G(y, t; x, 0)dy &\leq u(x, t) + \int_0^t \int_{\mathbf{M}} \xi(y)G(y, t-s; x, 0)dyds \\ &\leq u(x, t) + \int_0^t \int_{\mathbf{M}} \frac{\xi(y)}{d^{\lambda/q}(x, y)} \frac{Cd^{\lambda/q}(x, y)}{(t-s)^{\frac{n}{2}}} e^{-\frac{d^2(x,y)}{C(t-s)}} dyds \\ &\leq u(x, t) + C\|\xi\|_{K^{q,\lambda}} \int_0^t \left(\int_{\mathbf{M}} \frac{d^{\lambda/(q-1)}(x, y)}{(t-s)^{\frac{nq}{2(q-1)}}} e^{-\frac{d^2(x,y)}{C(t-s)}} dy \right)^{\frac{q-1}{q}} ds \\ &\leq u(x, t) + C\|\xi\|_{K^{q,\lambda}} \int_0^t \frac{1}{(t-s)^{\frac{n-\lambda}{2q}}} ds \\ &= u(x, t) + Ct^{1-\frac{n-\lambda}{2q}} \|\xi\|_{K^{q,\lambda}}, \end{aligned}$$

and

$$\oint_{B_r(x)} u(y, 0)dy \leq C \int_{B_r(x)} u(y, 0)G(y, r^2; x, 0)dy.$$

Hence, (4.8) follows. □

When the function u does not depend on t , the above lemma becomes

Corollary 4.8. *Let $u(x)$ be a nonnegative function satisfying*

$$\Delta u \leq \xi(x), \tag{4.10}$$

where $\xi(x) \geq 0$ is a smooth function.

(i) *If $A1$ and $A2$ are satisfied, then for any $q > \frac{n}{2}$, there exists a constant $C = C(n, p, q, \kappa, \Lambda)$ such that*

$$\oint_{B_r(x)} u(y)dy \leq C \left(u(x) + r^{2-\frac{n}{q}} \|\xi\|_{L^q} \right) \tag{4.11}$$

holds for any $x \in M$ and $0 < r \leq 1$.

More generally, we have

$$\oint_{B_r(x)} u(y) dy \leq C \left(\inf_{B_r(x)} u(\cdot) + r^{2-\frac{n}{q}} \|\xi\|_{L^q} \right). \quad (4.12)$$

(ii) If A1 and A3 are satisfied, then for any $q > 0$ and $\lambda > n - 2q$, there exists a constant $C = C(n, q, \lambda, \kappa, \Gamma)$ such that

$$\oint_{B_r(x)} u(y) dy \leq C \left(u(x) + r^{2-\frac{n-\lambda}{q}} \|\xi\|_{K^{q,\lambda}} \right) \quad (4.13)$$

holds for any $x \in M$ and $0 < r \leq 1$.

More generally, we have

$$\oint_{B_r(x)} u(y) dy \leq C \left(\inf_{B_r(x)} u(\cdot) + r^{2-\frac{n-\lambda}{q}} \|\xi\|_{K^{q,\lambda}} \right). \quad (4.14)$$

Now let O^+ and O^- be two fixed points in \mathbf{M} . Following [CoNa], define

$$d^-(x) = d(O^-, x), \quad d^+(x) = d(O^+, O^-) - d(O^+, x), \quad (4.15)$$

and

$$e(x) = d^-(x) - d^+(x) = d(O^-, x) + d(O^+, x) - d(O^+, O^-). \quad (4.16)$$

First of all, we have in barrier sense that

$$\Delta d^-(x) \leq \frac{n-1}{d^-} + \psi^- \quad (4.17)$$

$$-\Delta d^+(x) \leq \frac{n-1}{d^+} + \psi^+, \quad (4.18)$$

where $\psi^- = \max\{\Delta d^-(x) - \frac{n-1}{d^-}, 0\}$, and $\psi^+ = \max\{-\Delta d^+(x) - \frac{n-1}{d^+}, 0\}$. Moreover, it follows from Lemma 2.2 in [PW1] that

$$\int_{B_r(x)} |\psi^\pm|^{2p}(y) dy \leq C(n, p) \int_{B_r(x)} |\text{Ric}^-|^p(y) dy. \quad (4.19)$$

Denote by

$$d_0 = d(O^+, O^-) \text{ and } M_{r_1, r_2} = A_{r_1 d_0, r_2 d_0}(\{O^+, O^-\}).$$

With out loss of generality, we may assume that $d_0 \leq 1$.

Lemma 4.9. *For some fixed $\delta > 0$,*

i) if A1 and A2 are satisfied, then there exist a small constant $\bar{\epsilon} = \bar{\epsilon}(n, p, \delta)$, and a constant $C = C(n, p, \kappa, \Lambda, \delta)$ such that for any $0 < \epsilon \leq \bar{\epsilon}$, we have

$$\oint_{B_{\epsilon d_0}(x)} e(y) dy \leq C \left[e(x) + \epsilon^2 d_0 + (\|\psi^+\|_{L^{2p}} + \|\psi^-\|_{L^{2p}}) \epsilon^{2-\frac{n}{2p}} d_0 \right] \leq C(e(x) + \epsilon^{2-\frac{n}{2p}} d_0), \quad (4.20)$$

for all $x \in M_{\frac{\delta}{4}, 16}$.

In particular, this implies the excess estimate of Abresch-Gromoll [AbGr], i.e.,

$$e(y) \leq C \epsilon^{1+\alpha(n,p)} d_0, \quad \forall y \in B_{\frac{1}{2}\epsilon d_0}(x) \quad (4.21)$$

whenever $e(x) \leq \epsilon^{2-\frac{n}{2p}} d_0$, where $\alpha(n, p) = \frac{1}{n+1} \left(1 - \frac{n}{2p}\right)$.

ii) if A1 and A3 are satisfied, then for any $q > 0$ and $\lambda > n - 2q$, there exist a small constant $\bar{\epsilon} = \bar{\epsilon}(n, \delta)$, and a constant $C = C(n, q, \lambda, \kappa, \Gamma, \delta)$ such that for any $0 < \epsilon \leq \bar{\epsilon}$, we have

$$\oint_{B_{\epsilon d_0}(x)} e(y) dy \leq C \left[e(x) + \epsilon^2 d_0 + (\|\psi^+\|_{K^{q,\lambda}} + \|\psi^-\|_{K^{q,\lambda}}) \epsilon^{2-\frac{n-\lambda}{q}} d_0^{2-\frac{n-\lambda}{q}} \right], \quad (4.22)$$

for all $x \in M_{\frac{\delta}{4}, 16}$.

Proof. Inequalities (4.20) and (4.22) follow directly from Corollary 4.8, since

$$\Delta e(x) = \Delta d^- - \Delta d^+ \leq \frac{C}{d_0} + \psi^- + \psi^+.$$

To see that (4.20) implies (4.21), notice that for some $q > 1$ satisfying $2\epsilon^q \leq \epsilon$, and any $y \in B_{(\epsilon-\epsilon^q)d_0}(x)$, we have

$$\int_{B_{\epsilon^q d_0}(y)} e(z) dz \leq \int_{B_{\epsilon d_0}(x)} e(z) dz \leq C(e(x) + \epsilon^{2-\frac{n}{2p}} d_0) \text{Vol}(B_{\epsilon d_0}(x)) \leq C\epsilon^{2-\frac{n}{2p}} d_0 (\epsilon d_0)^n.$$

Thus,

$$\int_{B_{\epsilon^q d_0}(y)} e(z) dz \leq C\epsilon^{2-\frac{n}{2p}} d_0 \frac{(\epsilon d_0)^n}{(\epsilon^q d_0)^n} = C\epsilon^{2-\frac{n}{2p}+n-nq} d_0. \quad (4.23)$$

This means that there exists a point $y' \in B_{\epsilon^q}(y)$ such that

$$e(y') \leq C\epsilon^{2-\frac{n}{2p}+n-nq} d_0.$$

Hence,

$$e(y) \leq e(y') + 2d(y, y') \leq C(\epsilon^{2-\frac{n}{2p}+n-nq} + \epsilon^q) d_0.$$

By choosing $q = 1 + \alpha(n, p) = 1 + \frac{1}{n+1}(1 - \frac{n}{2p})$, one has

$$e(y) \leq C\epsilon^{1+\alpha(n,p)} d_0,$$

for any $y \in B_{\frac{1}{2}\epsilon d_0}(x) \subset B_{(\epsilon-\epsilon^q)d_0}(x)$. □

Under the assumptions A1 and A2, according to Lemma 4.5, we can construct a cut-off function $\phi \geq 0$ such that

$$\phi = 1 \text{ on } M_{\frac{\delta}{4}, 8}, \text{ supp}(\phi) \subset M_{\frac{\delta}{16}, 16}, \text{ and } |\Delta\phi| + |\nabla\phi|^2 \leq \frac{C}{d_0^2}. \quad (4.24)$$

Define $h_0^\pm(x) = \phi d^\pm(x)$, and $e_0(x) = \phi e(x)$. Also, denote by $h_t^\pm(x)$ and $e_t(x) = h_t^- - h_t^+$ the solutions of the equations

$$\begin{cases} (\frac{\partial}{\partial t} - \Delta)h^\pm(x, t) = 0 \\ h^\pm(x, 0) = h_0^\pm(x), \end{cases} \quad (4.25)$$

and

$$\begin{cases} (\frac{\partial}{\partial t} - \Delta)e(x, t) = 0 \\ e(x, 0) = e_0(x) \end{cases} \quad (4.26)$$

If only $A1$ and $A3$ are satisfied, assuming further that there exists a cut-off function as in (4.24), we can still construct $h_t^\pm(x)$ and $e_t(x)$ as above.

In the following, we derive estimates of $h_t^\pm(x)$ and $e_t(x)$. We will use the notation

$$\|\psi^\pm\|_{K^{q,\lambda}} := \|\psi^-\|_{K^{q,\lambda}} + \|\psi^+\|_{K^{q,\lambda}}.$$

The following lemmas that we obtained are under the assumptions that either $A1$ and $A2$ are satisfied, or $A1$ and $A3$ are satisfied, and there exists a cut-off function as in (4.24). But we will only present the proofs for the former case since the proofs for the latter case follow in a similar way.

Lemma 4.10. *i) If $A1$ and $A2$ are satisfied, then there exists a constant $C = C(n, p, \kappa, \Lambda, \delta)$ such that*

$$\Delta h_t^-, -\Delta h_t^+, \Delta e_t \leq C \left(\frac{1}{d_0} + (\|\psi^+\|_{L^{2p}} + \|\psi^-\|_{L^{2p}}) t^{-\frac{n}{4p}} \right) \leq C \left(\frac{1}{d_0} + t^{-\frac{n}{4p}} \right)$$

in $M_{\frac{\delta}{16}, 16}$.

ii) If $A1$ and $A3$ are satisfied, and there exists a cut-off function as in (4.24), then for any $q > 0$ and $\lambda > 0$, there exists a constant $C = C(n, q, \lambda, \kappa, \Gamma, \delta)$ such that

$$\Delta h_t^-, -\Delta h_t^+, \Delta e_t \leq C \left(\frac{1}{d_0} + \|\psi^\pm\|_{K^{q,\lambda}} t^{\frac{n-\lambda}{2q}} \right).$$

in $M_{\frac{\delta}{16}, 16}$.

Proof. Following Tian-Zhang [TZz], here we only prove the estimate for Δe_t . The proofs of the other two estimates are similar. First, notice that for $x \in M_{\frac{\delta}{16}, 16}$, we have

$$\begin{aligned} \Delta e_0(x) &= \Delta \phi e(x) + 2\nabla \phi \cdot \nabla e(x) + \phi \Delta e(x) \\ &\leq \frac{C}{d_0} + \psi^- + \psi^+. \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta e_t(x) &= \int_{M_{\frac{\delta}{16}, 16}} \Delta_x G(y, t; x, 0) e_0(y) dy \\ &= \int_{M_{\frac{\delta}{16}, 16}} \Delta_y G(y, t; x, 0) e_0(y) dy \\ &= \int_{M_{\frac{\delta}{16}, 16}} G(y, t; x, 0) \Delta e_0(y) dy \\ &\leq \int_{M_{\frac{\delta}{16}, 16}} G(y, t; x, 0) \left(\frac{C}{d_0} + \psi^-(y) + \psi^+(y) \right) dy \\ &\leq \frac{C}{d_0} + (\|\psi^-\|_{L^{2p}} + \|\psi^+\|_{L^{2p}}) \|G(y, t; x, 0)\|_{L^{\frac{2p}{2p-1}}}. \end{aligned}$$

In the last step above, we have used the Gaussian upper bound of $G(y, t; x, 0)$ in Lemma 4.4. This is part (i). Part (ii) can then be proven by the argument at the end of the proof of Lemma 4.7. \square

Lemma 4.11. *i) If A1 and A2 are satisfied, there exists a constant $C = C(n, p, \kappa, \Lambda, \delta)$, such that for any $x \in M_{\frac{\delta}{2}, 4}$ and $0 < t \leq \bar{\epsilon}^2 d_0^2$, the following estimates hold for $y \in B_{\sqrt{t}}(x)$,*

- (1) $|e_t(y)| \leq C \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right).$
- (2) $|\nabla e_t|(y) \leq \frac{C}{\sqrt{t}} \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right).$
- (3) $\left| \frac{\partial}{\partial t} e_t(y) \right| = |\Delta e_t(y)| \leq \frac{C}{t} \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right).$
- (4) $\oint_{B_{\sqrt{t}}(y)} |\nabla^2 e_t|^2 \leq \frac{C}{t^2} \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right)^2.$

ii) If A1 and A3 are satisfied, and there exists a cut-off function as in (4.24), then for any $q > 0$ and $\lambda > n - 2q$, there exists a constant $C = C(n, q, \lambda, \Gamma, \delta)$ such that

- (1') $|e_t(y)| \leq C \left(e(x) + td_0^{-1} + \|\psi^\pm\|_{K^{q,\lambda}} t^{1-\frac{n-\lambda}{2q}} \right).$
- (2') $|\nabla e_t|(y) \leq \frac{C}{\sqrt{t}} \left(e(x) + td_0^{-1} + \|\psi^\pm\|_{K^{q,\lambda}} t^{1-\frac{n-\lambda}{2q}} \right).$
- (3') $\left| \frac{\partial}{\partial t} e_t(y) \right| = |\Delta e_t(y)| \leq \frac{C}{t} \left(e(x) + td_0^{-1} + \|\psi^\pm\|_{K^{q,\lambda}} t^{1-\frac{n-\lambda}{2q}} \right).$
- (4') $\oint_{B_{\sqrt{t}}(y)} |\nabla^2 e_t|^2 \leq \frac{C}{t^2} \left(e(x) + td_0^{-1} + \|\psi^\pm\|_{K^{q,\lambda}} t^{1-\frac{n-\lambda}{2q}} \right)^2.$

Here $\bar{\epsilon}$ is the constant in Lemma 4.9.

Proof. Again we will only prove (i). The main idea of the proof is from [CoNa]. Here we use our Li-Yau bound in section 2 instead of the original Li-Yau bound used in [CoNa].

For any $x \in M$, Lemma 4.10 implies that

$$e_t(x) = e_0(x) + \int_0^t \Delta e_s(x) ds \leq e(x) + Ctd_0^{-1} + Ct^{1-\frac{n}{4p}}, \quad (4.27)$$

which is (1).

It is not hard to check that the Li-Yau bound (1.11) gives the following Harnack inequality

$$u(x_1, t_1) \leq u(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{\frac{n}{k}} e^{\frac{d^2(x_1, x_2)}{C(t_2 - t_1)}}. \quad (4.28)$$

for any positive solution $u(x, t)$ for the heat equation, where $0 < t_1 \leq t_2$, and $k = k(n)$ is a constant. By applying (4.28) to e_t , one has

$$e_t(y) \leq Ce_{2t}(x) \leq C \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right).$$

Also, from (1.11), we have

$$\frac{\partial}{\partial t} e_t(y) \geq -\frac{n}{kt} e_t(y) \geq -\frac{C}{t} \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right).$$

Therefore, we have

$$\left| \frac{\partial}{\partial t} e_t(y) \right| = |\Delta e_t(y)| \leq \frac{C}{t} \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right), \quad (4.29)$$

which is (3).

Again, (1.11) implies that

$$\begin{aligned} |\nabla e_t|^2(y) &\leq C \left(\frac{1}{t} + \frac{\frac{\partial}{\partial t} e_t(y)}{e_t(y)} \right) e_t^2(y) \\ &\leq \frac{C}{t} \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right)^2 + C \left(e(x) + t^{-1}d_0 + t^{1-\frac{n}{4p}} \right) \left(\frac{1}{d_0} + t^{-\frac{n}{4p}} \right) \\ &\leq \frac{C}{t} \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right)^2. \end{aligned}$$

Thus, (2) follows immediately.

For the last inequality, assume that $\eta \geq 0$ is a cut-off function so that

$$\eta = 1 \text{ in } B_{\sqrt{t}}(y), \quad \eta = 0 \text{ outside of } B_{2\sqrt{t}}(y),$$

and

$$|\Delta \eta| + |\nabla \eta|^2 \leq \frac{C}{t}.$$

Then

$$\begin{aligned} \oint_{B_{\sqrt{t}}(y)} |\nabla^2 e_t|^2 &\leq \frac{1}{\text{Vol}(B_{\sqrt{t}}(y))} \int_M \eta |\nabla^2 e_t|^2 \\ &= \frac{1}{\text{Vol}(B_{\sqrt{t}}(y))} \int_M \frac{1}{2} \eta \Delta |\nabla e_t|^2 - \eta \nabla(\Delta e_t) \cdot \nabla e_t - \eta R_{ij} \nabla_i e_t \nabla_j e_t \\ &\leq \frac{1}{\text{Vol}(B_{\sqrt{t}}(y))} \int_{B_{2\sqrt{t}}(y)} \Delta \eta |\nabla e_t|^2 + \Delta e_t \nabla e_t \cdot \nabla \eta + \eta (\Delta e_t)^2 + \eta |\text{Ric}^-| |\nabla e_t|^2 \\ &\leq \frac{C}{t^2} \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right)^2. \end{aligned}$$

In the last step above, we have used (2) and (3) to bound $|\nabla e_t|^2$ and $|\Delta e_t|^2$, separately. \square

Alternatively, one may also use the Gaussian estimate of $|\frac{\partial G}{\partial t}|$ to obtain the estimates in the above lemma.

From the lemma above, one gets

Lemma 4.12. *i) If A1 and A2 are satisfied, then for any $x \in M_{\frac{\delta}{2}, 4}$, we have*

$$|h_t^\pm(x) - d^\pm(x)| \leq C(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}}).$$

ii) If A1 and A3 are satisfied, and there exists a cut-off function as in (4.24), then for any $q > 0$, $\lambda > n - 2q$ and $x \in M_{\frac{\delta}{2}, 4}$, we have

$$|h_t^\pm(x) - d^\pm(x)| \leq C(e(x) + td_0^{-1} + \|\psi^\pm\|_{K^{q, \lambda}} t^{1-\frac{n-\lambda}{2q}}).$$

Proof. As before we will only prove (i). Here we only present the proof for $h_t^-(x)$. The proof for $h_t^+(x)$ follows similarly. Firstly, we have

$$\begin{aligned} h_t^-(x) &= h_0^-(x) + \int_0^t \Delta h_s^-(x) ds \\ &\leq d^-(x) + Ctd_0^{-1} + Ct^{1-\frac{n}{4p}}. \end{aligned}$$

Thus,

$$h_t^-(x) - d^-(x) \leq C(td_0^{-1} + t^{1-\frac{n}{4p}}).$$

Similarly, we have

$$h_t^+(x) \geq d^+(x) - Ctd_0^{-1} - Ct^{1-\frac{n}{4p}}.$$

Since

$$h_t^-(x) - d^-(x) = h_t^+(x) - d^+(x) + e_t(x) - e(x),$$

and

$$|e_t(x)| \leq C \left(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}} \right),$$

we have

$$h_t^-(x) - d^-(x) \geq -C(e(x) + td_0^{-1} + t^{1-\frac{n}{4p}}).$$

□

Recall from [CoNa] that an ϵ -geodesic connecting O^+ and O^- is a unit speed curve σ such that $|\sigma| - d_0 \leq \epsilon^2 d_0$. Moreover, one has

Lemma 4.13. (Colding-Naber [CoNa])

- 1) Let σ be an ϵ -geodesic connecting O^+ and O^- . Then for any $z \in \sigma$, we have $e(z) \leq \epsilon^2 d_0$.
- 2) Let $x \in M$ such that $e(x) \leq \epsilon^2 d_0$. Then there exists an ϵ -geodesic σ such that $x \in \sigma$.

From Lemma 4.12 and Lemma 4.13, we immediately have

Corollary 4.14. For any ϵ -geodesic σ connecting O^+ and O^- , any $x \in \sigma \cap M_{\delta/2,4}$, and $0 < \epsilon \leq \bar{\epsilon}$, we have

i) when A1 and A2 are satisfied,

$$\left| h_{d_\epsilon^\pm}^\pm - d^\pm \right| \leq C(\epsilon^2 d_0 + \epsilon^{2-\frac{n}{2p}} d_0^{2-\frac{n}{2p}}).$$

ii) when A1 and A3 are satisfied, and there exists a cut-off function as in (4.24), for $q > 0$ and $\lambda > n - 2q$ we have

$$\left| h_{d_\epsilon^\pm}^\pm - d^\pm \right| \leq C(\epsilon^2 d_0 + \|\psi^\pm\|_{K^{q,\lambda}} \epsilon^{2-\frac{n-\lambda}{q}} d_0^{2-\frac{n-\lambda}{q}}).$$

Here $\bar{\epsilon}$ is the constant in Lemma 4.9.

To prove that h_t^\pm are L^1 close to d^\pm , we first need the following lemma.

Lemma 4.15. For any $x \in M_{\frac{\delta}{2},4}$, we have

$$\int_{M_{\frac{\delta}{16},16} \setminus M_{\frac{\delta}{4},8}} G(y, t; x, 0) dy \leq \frac{C}{d_0^2} t.$$

Proof. Let k be a positive integer, and $\phi_k \geq 0$ a cut-off function such that $\phi_k = 1$ in $M_{\frac{\delta}{k}, 2k}$, $\phi_k = 0$ outside of $M_{\frac{\delta}{2k}, 4k}$, and $|\Delta\phi_k| + |\nabla\phi_k|^2 \leq \frac{C}{d_0^2}$. Then, for any x , we have

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbf{M}} \phi_k(y) G(y, t; x, 0) dy \right| &= \left| \int_{\mathbf{M}} \phi_k(y) \Delta_y G(y, t; x, 0) dy \right| \\ &= \left| \int_{\mathbf{M}} \Delta\phi_k(y) G(y, t; x, 0) dy \right| \\ &\leq \frac{C}{d_0^2}. \end{aligned}$$

Thus, we have

$$-\frac{C}{d_0^2}t \leq -\phi_k(x) + \int_M \phi_k(y) G(y, t; x, 0) dy \leq \frac{C}{d_0^2}t.$$

It follows that for $x \in M_{\frac{\delta}{2}, 4}$,

$$\int_{M_{\frac{\delta}{16}, 32}} G(y, t; x, 0) dy \leq 1 + \frac{C}{d_0^2}t,$$

and

$$\int_{M_{\frac{\delta}{4}, 8}} G(y, t; x, 0) dy \geq 1 - \frac{C}{d_0^2}t.$$

Therefore,

$$\int_{M_{\frac{\delta}{16}, 16} \setminus M_{\frac{\delta}{4}, 8}} G(y, t; x, 0) dy \leq \frac{C}{d_0^2}t.$$

□

By using the above lemma, we can get

Lemma 4.16. *For any $x \in M_{\frac{\delta}{2}, 4}$, and $0 < t < \bar{\epsilon}^2 d_0^2$*

i) if A1 and A2 are satisfied, then we have

$$|\nabla h_t^\pm|^2(x) \leq 1 + \frac{C}{d_0^2}t + Ct^{1-\frac{n}{2p}}.$$

ii) if A1 and A3 are satisfied, and there exists a cut-off function as in (4.24), then

$$|\nabla h_t^\pm|^2(x) \leq 1 + \frac{C}{d_0^2}t + C\sqrt{t}.$$

Here $\bar{\epsilon}$ is the constant in Lemma 4.9.

Proof. Again we will only prove (i). Notice that

$$\frac{\partial}{\partial t} |\nabla h_t^\pm|^2 \leq \Delta |\nabla h_t^\pm|^2 + |\text{Ric}^-| |\nabla h_t^\pm|^2.$$

Moreover,

$$\begin{aligned} |\nabla h_0^\pm| &= 1, \text{ in } M_{\delta/4, 8}, \\ |\nabla h_0^\pm| &\leq C, \text{ in } M_{\delta/16, 16} \setminus M_{\delta/4, 8}, \end{aligned}$$

and

$$|\nabla h_0^\pm| = 0, \text{ outside of } M_{\delta/16,16}.$$

By Duhamel's principle, we have

$$\begin{aligned} |\nabla h_t^\pm|^2(x) &\leq \int_{M_{\delta/16,16}} |\nabla h_0^\pm|^2(y) G(y, t; x, 0) dy + \int_0^t \int_{\mathbf{M}} |\text{Ric}^-| |\nabla h_s^\pm|^2 G(y, t-s; x, 0) dy ds \\ &\leq C + \sup_{M \times [0,t]} |\nabla h_s^\pm|^2 \int_0^t \|\text{Ric}^-\|_{L^p} \left(\int_{\mathbf{M}} G^{\frac{p-1}{p}} dy \right)^{\frac{p-1}{p}} ds. \end{aligned} \quad (4.30)$$

Then the Gaussian upper bound of $G(y, t; x, 0)$ in (4.4) gives

$$|\nabla h_t^\pm|^2(x) \leq C + \sup_{M \times [0,t]} |\nabla h_s^\pm|^2 \cdot Ct^{1-\frac{n}{2p}}. \quad (4.31)$$

Since $Ct^{1-\frac{n}{2p}} \leq C\epsilon^{-2-\frac{n}{p}} \leq \frac{1}{2}$, it follows that

$$\sup_{M \times [0,t]} |\nabla h_s^\pm|^2 \leq C. \quad (4.32)$$

By plugging (4.32) in (4.30) and using the Gaussian upper bound of G again, we get

$$\begin{aligned} |\nabla h_t^\pm|^2(x) &\leq \int_{M_{\delta/16,16}} |\nabla h_0^\pm|^2(y) G(y, t; x, 0) dy + \int_0^t \int_{\mathbf{M}} |\text{Ric}^-| |\nabla h_s^\pm|^2 G(y, t-s; x, 0) dy ds \\ &\leq \int_{M_{\delta/4,8}} |\nabla h_0^\pm|^2(y) G(y, t; x, 0) dy + \int_{M_{\delta/16,16} \setminus M_{\delta/4,8}} |\nabla h_0^\pm|^2(y) G(y, t; x, 0) dy \\ &\quad + \int_0^t \int_{\mathbf{M}} |\text{Ric}^-| |\nabla h_s^\pm|^2 G(y, t-s; x, 0) dy ds \\ &\leq 1 + \frac{C}{d_0^2} t + C \int_0^t \|\text{Ric}^-\|_{L^p} \left(\int_{\mathbf{M}} G^{\frac{p-1}{p}} dy \right)^{\frac{p-1}{p}} ds \\ &\leq 1 + \frac{C}{d_0^2} t + Ct^{1-\frac{n}{2p}}. \end{aligned}$$

□

The above C^1 bound of h_t^\pm can be applied to show that

Lemma 4.17. *i) If A1 and A2 are satisfied, then there exists a constant $C = C(n, p, \kappa, \Lambda, \delta)$, such that for any $0 < \epsilon \leq \bar{\epsilon}$ and $0 < \sqrt{t} < \epsilon^2 d_0^2$, we have*

(1) *If $x \in M_{\delta,2}$, and $e(x) \leq \epsilon^2 d_0$, then $\oint_{B_{10\sqrt{t}}(x)} \left| |\nabla h_t^\pm|^2 - 1 \right| \leq Ct^{-\frac{1}{2}} (\epsilon^2 d_0 + td_0^{-1} + t^{1-\frac{n}{4p}})$.*

(2) *If σ is an ϵ -geodesic connecting O^+ and O^- , then*

$$\oint_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{10\sqrt{t}}(\sigma(s))} \left| |\nabla h_t^\pm|^2 - 1 \right| \leq C(\epsilon^2 d_0 + td_0^{-1} + t^{1-\frac{n}{2p}} d_0^{\frac{n}{2p}}).$$

ii) If A1 and A3 are satisfied, and there exists a cut-off function as in (4.24), then for any $q > 0$ and $\lambda > n - 2q$, there exists a constant $C = C(n, q, \lambda, \kappa, \Gamma, \delta)$, such that for any

$0 < \epsilon \leq \bar{\epsilon}$ and $0 < \sqrt{t} < \epsilon^2 d_0^2$, we have
 (1') If $x \in M_{\delta,2}$, and $e(x) \leq \epsilon^2 d_0$, then

$$\oint_{B_{10\sqrt{t}}(x)} \left| |\nabla h_t^\pm|^2 - 1 \right| \leq C t^{-\frac{1}{2}} (\epsilon^2 d_0 + t d_0^{-1} + \|\psi^\pm\|_{K^{q,\lambda}} t^{1-\frac{n-\lambda}{2q}}).$$

(2') If σ is an ϵ -geodesic connecting O^+ and O^- , then

$$\int_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{10\sqrt{t}}(\sigma(s))} \left| |\nabla h_t^\pm|^2 - 1 \right| \leq C (\epsilon^2 d_0 + t d_0^{-1} + \sqrt{t} d_0 + \|\psi^\pm\|_{K^{q,\lambda}} t^{1-\frac{n-\lambda}{2q}}).$$

Here $\bar{\epsilon}$ is the constant in Lemma 4.9.

Proof. Let us just prove (i). Let

$$w_t = 1 + \frac{C}{d_0^2} t + C t^{1-\frac{n}{2p}} - |\nabla h_t^-|^2,$$

and $\phi \geq 0$ a cut-off function such that $\phi = 1$ in $M_{\delta,2}$, $\phi = 0$ outside of $M_{\delta/2,4}$, and $|\Delta\phi| + |\nabla\phi|^2 \leq \frac{C}{d_0^2}$.

By direct computation and Lemma 4.16, one has

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) [\phi^2 w_t] &= \phi^2 \left(\frac{C}{d_0^2} + C t^{-\frac{n}{2p}} + 2|\nabla^2 h_t^-|^2 + 2R_{ij} \nabla_i h_t^- \nabla_j h_t^- \right) \\ &\quad - 2\phi \Delta\phi w_t - 2|\nabla\phi|^2 w_t + 4\phi \nabla\phi \cdot \nabla |\nabla h_t^-|^2 \\ &\geq 2\phi^2 |\nabla^2 h_t^-|^2 - 2\phi^2 |\text{Ric}^-| |\nabla h_t^-|^2 - \frac{C}{d_0^2} \left(1 + \frac{C}{d_0^2} t + C t^{1-\frac{n}{2p}} \right) \\ &\quad - \frac{C}{d_0} \left(1 + \frac{C}{d_0^2} t + C t^{1-\frac{n}{2p}} \right)^{\frac{1}{2}} \phi |\nabla^2 h_t^-| \\ &\geq -\frac{C}{d_0^2} \left(1 + \frac{C}{d_0^2} t + C t^{1-\frac{n}{2p}} \right) - |\text{Ric}^-| \left(1 + \frac{C}{d_0^2} t + C t^{1-\frac{n}{2p}} \right). \end{aligned}$$

It follows from Lemma 4.7 that

$$\oint_{B_{10\sqrt{t}}(x)} w_t(y) dy \leq C \left(\inf_{B_{10\sqrt{t}}(x)} w_{2t} + \frac{t}{d_0^2} \left(1 + \frac{C}{d_0^2} t + C t^{1-\frac{n}{2p}} \right) + \left(1 + \frac{C}{d_0^2} t + C t^{1-\frac{n}{2p}} \right) t^{1-\frac{n}{2p}} \right). \quad (4.33)$$

Assume that σ is a ϵ -geodesic connecting O^+ and O^- . For (1), since $e(x) \leq \epsilon^2 d_0$, we may further choose σ to be a piecewise geodesic passing through x . Then, Lemma 4.13 implies that $e(z) \leq \epsilon^2 d_0$ along σ . Moreover, we can get the following estimate from Lemma 4.12.

$$\begin{aligned} &\left| h_{2t}^-(x) - h_{2t}^-(\sigma(d^-(x) - 10\sqrt{t})) - 10\sqrt{t} \right| \\ &\leq \left| d^-(x) - d^-(\sigma(d^-(x) - 10\sqrt{t})) - 10\sqrt{t} \right| + C(\epsilon^2 d_0 + t d_0^{-1} + t^{1-\frac{n}{4p}}) \\ &= C(\epsilon^2 d_0 + t d_0^{-1} + t^{1-\frac{n}{4p}}). \end{aligned}$$

Thus,

$$\begin{aligned}
& \int_{d^-(x)-10\sqrt{t}}^{d^-(x)} w_{2t}(\sigma(s)) ds \\
&= \int_{d^-(x)-10\sqrt{t}}^{d^-(x)} \left[1 + C \frac{t}{d_0^2} + Ct^{1-\frac{n}{2p}} - |\nabla h_{2t}^-|^2 \right] ds \\
&\leq 10\sqrt{t} + C \frac{t^{3/2}}{d_0^2} + Ct^{\frac{3}{2}-\frac{n}{2p}} - \frac{1}{\sqrt{t}} \left(\int_{d^-(x)-10\sqrt{t}}^{d^-(x)} \nabla_{\dot{\sigma}(s)} h_{2t}^- ds \right)^2 \\
&\leq 10\sqrt{t} + C \frac{t^{3/2}}{d_0^2} + Ct^{\frac{3}{2}-\frac{n}{2p}} - \frac{1}{10\sqrt{t}} \left(h_{2t}^-(x) - h_{2t}^-(\sigma(d^-(x) - 10\sqrt{t})) \right)^2 \\
&\leq C(\epsilon^2 d_0 + td_0^{-1} + t^{1-\frac{n}{4p}}).
\end{aligned}$$

This means that there exists a point $y \in \sigma$ such that

$$w_{2t}(y) \leq Ct^{-\frac{1}{2}}(\epsilon^2 d_0 + td_0^{-1} + t^{1-\frac{n}{4p}}).$$

Substituting the inequality above in (4.33), one finishes the proof of (1) for h_t^- .

To prove statement (2) for h_t^- , by Lemma 4.12, for any s with $\delta d_0 < s < (1 - \delta)d_0$, we have

$$|h_{2t}^-(\sigma(s)) - h_{2t}^-(\sigma(\delta d_0)) - (s - \delta d_0)| \leq C(\epsilon^2 d_0 + td_0^{-1} + t^{1-\frac{n}{4p}}).$$

Thus,

$$\begin{aligned}
& \int_{\delta d_0}^{(1-\delta)d_0} w_{2t}(\sigma(s)) ds \\
&\leq (1 - 2\delta)d_0 + \frac{C}{d_0}t + Ct^{1-\frac{n}{2p}}d_0 - \frac{1}{(1 - 2\delta)d_0} \left(\int_{\delta d_0}^{(1-\delta)d_0} \nabla_{\dot{\sigma}(s)} h_{2t}^- ds \right)^2 \\
&\leq C(\epsilon^2 d_0 + td_0^{-1} + t^{1-\frac{n}{2p}}d_0^{\frac{n}{2p}}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{10\sqrt{t}}(\sigma(s))} \left| |\nabla h_t^-|^2 - 1 \right| \\
&\leq \int_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{10\sqrt{t}}(\sigma(s))} \left(w_t + Ctd_0^{-2} + Ct^{1-\frac{n}{2p}} \right) \\
&\leq C \int_{\delta d_0}^{(1-\delta)d_0} \inf_{B_{10\sqrt{t}}(\sigma(s))} w_{2t} + Ctd_0^{-1} + Ct^{1-\frac{n}{2p}}d_0 \\
&\leq C(\epsilon^2 d_0 + td_0^{-1} + t^{1-\frac{n}{2p}}d_0^{\frac{n}{2p}}).
\end{aligned}$$

The proofs of both (1) and (2) for h_t^+ can be carried out similarly. \square

Now we are ready to Theorem 4.1.

Proof of Theorem 4.1: Estimates (1), (2), and (3) are contained in Lemmas 4.12 and 4.17, respectively. In the following, we prove (4).

For any $\sigma(s)$, let $\eta(x) \geq 0$ be the cut-off function satisfying $\eta = 1$ in $B_{d_\epsilon}(\sigma(s))$, $\eta = 0$ outside of $B_{3d_\epsilon}(\sigma(s))$, and $|\Delta\eta| + |\nabla\eta|^2 \leq \frac{C}{d_\epsilon^2}$, where $d_\epsilon = \epsilon d_0$.

Let $a(t)$ be a smooth function in time such that $0 \leq a(t) \leq 1$, $a(t) = 1$ for $t \in [\frac{1}{2}d_\epsilon^2, 2d_\epsilon^2]$, $a(t) = 0$ for $t \notin [\frac{1}{4}d_\epsilon^2, 4d_\epsilon^2]$, and $|a'(t)| \leq \frac{C}{d_\epsilon^2}$.

Recall that

$$\left(\frac{\partial}{\partial t} - \Delta\right) (|\nabla h_t^\pm|^2 - 1) = -2|\nabla^2 h_t^\pm|^2 - 2Ric_{ij}\nabla_i h_t^\pm \nabla_j h_t^\pm.$$

Hence, we have

$$\begin{aligned} & 2 \int_{\mathbf{M}} a(t)\eta |\nabla^2 h_t^\pm|^2 \\ &= \int_{\mathbf{M}} a(t)\eta \Delta (|\nabla h_t^\pm|^2 - 1) - 2 \int_{\mathbf{M}} a(t)\eta Ric_{ij}\nabla_i h_t^\pm \nabla_j h_t^\pm - \int_{\mathbf{M}} a(t)\eta \frac{\partial}{\partial t} (|\nabla h_t^\pm|^2 - 1) \\ &\leq \frac{C}{d_\epsilon^2} \int_{B_{3d_\epsilon}(\sigma(s))} a(t) \left| |\nabla h_t^\pm|^2 - 1 \right| + 2 \int_{B_{3d_\epsilon}(\sigma(s))} a(t) Ric^- \left| |\nabla h_t^\pm|^2 \right| - \int_{\mathbf{M}} a(t)\eta \frac{\partial}{\partial t} (|\nabla h_t^\pm|^2 - 1). \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{\text{Vol}(B_{3d_\epsilon}(\sigma(s)))} \int_{\frac{1}{2}d_\epsilon^2}^{2d_\epsilon^2} \int_{B_{d_\epsilon}(\sigma(s))} |\nabla^2 h_t^\pm|^2 dy dt \\ &\leq \frac{C}{d_\epsilon^2} \int_{\frac{1}{4}d_\epsilon^2}^{4d_\epsilon^2} \oint_{B_{3d_\epsilon}(\sigma(s))} \left| |\nabla h_t^\pm|^2 - 1 \right| dy dt + C \int_{\frac{1}{4}d_\epsilon^2}^{4d_\epsilon^2} \oint_{B_{3d_\epsilon}(\sigma(s))} Ric^- \left(1 + \frac{C}{d_0^2}t + Ct^{1-\frac{n}{2p}}\right) dy dt \end{aligned}$$

It follows immediately from (4.1) and Theorem 4.2 that

$$\begin{aligned} & \int_{\frac{1}{2}d_\epsilon^2}^{2d_\epsilon^2} \int_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{d_\epsilon}(\sigma(s))} |\nabla^2 h_t^\pm|^2 dy ds dt \\ &\leq \frac{C}{d_\epsilon^2} \int_{\frac{1}{4}d_\epsilon^2}^{4d_\epsilon^2} \left(\epsilon^2 d_0 + \frac{t}{d_0} + t^{1-\frac{n}{2p}} d_0^{\frac{n}{2p}}\right) dt + C d_0 d_\epsilon^{-\frac{n}{p}} \int_{\frac{1}{4}d_\epsilon^2}^{4d_\epsilon^2} \left(1 + \frac{C}{d_0^2}t + Ct^{1-\frac{n}{2p}}\right) dt \\ &\leq C d_0 (\epsilon^2 + \epsilon^{2-\frac{n}{p}}). \end{aligned}$$

Therefore, there exists an $r \in [\frac{1}{2}, 2]$ such that

$$\int_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{d_\epsilon}(\sigma(s))} |\nabla^2 h_{rd_\epsilon^2}^\pm|^2 dy ds \leq \frac{C(1 + \epsilon^{-\frac{n}{p}})}{d_0}. \quad (4.34)$$

□

Similarly, we can prove

Theorem 4.18. *Assume that A1 and A3 are satisfied, and there is a cut-off function as in (4.24). Then for some fixed $\delta > 0$, any $q > 0$ and $\lambda > n - 2q$, there exist constants $C = C(n, q, \lambda, \kappa, \Gamma, \delta)$ and $\bar{\epsilon} = \bar{\epsilon}(n, \delta)$, such that for any $0 < \epsilon \leq \bar{\epsilon}$, $x \in M_{\delta, 2}$ with $e(x) \leq \epsilon^2 d_0$ and any ϵ -geodesic σ connecting O^+ and O^- , there exists $r \in [\frac{1}{2}, 2]$ satisfying*

$$(1) \left| h_{r\epsilon^2 d_0}^\pm - d^\pm \right| \leq C d_0 (\epsilon^2 + \|\psi^\pm\|_{K^{q, \lambda}} \epsilon^{2-\frac{n-\lambda}{q}} d_0^{2-\frac{n-\lambda}{q}}).$$

$$\begin{aligned}
(2) \quad & \oint_{B_{\epsilon d_0}(x)} \left| |\nabla h_{r\epsilon^2 d_0^\pm}^\pm|^2 - 1 \right| \leq C(\epsilon + \|\psi^\pm\|_{K^{q,\lambda}} \epsilon^{1-\frac{n-\lambda}{q}} d_0^{1-\frac{n-\lambda}{q}}). \\
(3) \quad & \oint_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{\epsilon d_0}(\sigma(s))} \left| |\nabla h_{r\epsilon^2 d_0^\pm}^\pm|^2 - 1 \right| \leq C(\epsilon^2 d_0 + \epsilon d_0^2 + \|\psi^\pm\|_{K^{q,\lambda}} \epsilon^{2-\frac{n-\lambda}{q}} d_0^{2-\frac{n-\lambda}{q}}). \\
(4) \quad & \oint_{\delta d_0}^{(1-\delta)d_0} \oint_{B_{\epsilon d_0}(\sigma(s))} \left| \nabla^2 h_{r\epsilon^2 d_0^\pm}^\pm \right|^2 \leq \frac{C(1 + \epsilon^{-1} + \|\psi^\pm\|_{K^{q,\lambda}} \epsilon^{-\frac{n-\lambda}{q}})}{d_0^2}.
\end{aligned}$$

Moreover, from Lemma 4.12 and Lemma 4.17, we can also get

Lemma 4.19. *Assume that A1 and A2 are satisfied. Let $x \in M_{\delta,2}$ with σ a unit speed minimizing geodesic from O^- to x . Then for any $\delta \leq \tau_1 < \tau_2 \leq d^-(x)$, the following estimates hold:*

$$\begin{aligned}
1) \quad & \int_{\delta}^{d^-(x)} \left| |\nabla h_t^-|^2 - 1 \right| \leq \frac{C}{d_0} (e(x) + t + t^{1-\frac{n}{4p}}). \\
2) \quad & \int_{\delta}^{d^-(x)} \left| \langle \nabla h_t^-, \nabla d^- \rangle - 1 \right| \leq \frac{C}{d_0} (e(x) + t + t^{1-\frac{n}{4p}}). \\
3) \quad & \int_{\tau_1}^{\tau_2} \left| \nabla h_t^- - \nabla d^- \right| \leq \frac{C\sqrt{\tau_2 - \tau_1}}{\sqrt{d_0}} (\sqrt{e(x)} + \sqrt{t} + t^{\frac{1}{2} - \frac{n}{8p}}).
\end{aligned}$$

ACKNOWLEDGEMENTS

The paper was originally intended to be a joint one with Richard Bamler who made an important contribution. However, Richard has withdrawn from authorship to concentrate on other projects. We sincerely thank him for his generosity.

M. Zhu would like to express his great appreciation to Prof. Huai-Dong Cao for constant encouragement and support. We are grateful to Professors H.-D. Cao, X.Z. Dai, H.Z. Li, G.F. Wei for their interest and comments on the result.

Q.S.Z. gratefully acknowledges the support of Simons' Foundation and Siyuan Foundation through Nanjing University.

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