BOUNDARY EXPANSIONS FOR LIOUVILLE'S EQUATION IN PLANAR SINGULAR DOMAINS

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Abstract. We study asymptotic behaviors near the boundary of complete metrics of constant curvature in planar singular domains and establish an optimal estimate of these metrics by the corresponding metrics in tangent cones near isolated singular points on boundary. The conformal structure plays an essential role. We also discuss asymptotic behaviors of complete Kähler-Einstein metrics on singular product domains.

1. Introduction

Assume $\Omega \subset \mathbb{R}^2$ is a domain. We consider the following problem:

$$\Delta u = e^{2u} \quad \text{in } \Omega, \tag{1.1}$$

$$u = \infty \quad \text{on } \partial\Omega. \tag{1.2}$$

The equation (1.1) is known as Liouville's equation. For a large class of domains Ω, (1.1) and (1.2) admit a solution $u \in C^\infty(\Omega)$. Geometrically, $e^{2u}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2)$ is a complete metric with constant Gauss curvature -1 on Ω. Our main concern in this paper is the asymptotic behavior of solutions u near isolated *singular* points on boundary.

The higher dimensional counterpart is given by, for $\Omega \subset \mathbb{R}^n$, $n \geq 3$,

$$\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in } \Omega. \tag{1.3}$$

More generally, we can study, for a function f,

$$\Delta u = f(u) \quad \text{in } \Omega. \tag{1.4}$$

The study of these problems has a rich history. To begin with, we assume Ω has an at least C^2-boundary. Bieberbach [2] studied the problem (1.1)-(1.2) and proved the existence of its solutions. If f is monotone, Keller [10] established the existence for (1.4) and (1.2). In a pioneering work, Loewner and Nirenberg [17] studied asymptotic behaviors of solutions of (1.3) and (1.2) and proved

$$u(x) = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}} d^{-\frac{n-2}{2}} + o(d^{-\frac{n-2}{2}}),$$

where d is the distance to $\partial\Omega$. This result has been generalized to more general f and up to higher order terms, for example, by Brandle and Marcus [3], Diaz and Letelier [6], and Kichenassamy [11]. Moreover, if Ω has a smooth boundary, an estimate up to an arbitrarily finite order was established by Andersson, Chruściel and Friedrich [1] and

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Mazzeo [19]. In fact, they proved that solutions of (1.3) and (1.2) are polyhomogeneous. All these results require $\partial\Omega$ to have some degree of regularity. The case where $\partial\Omega$ is singular was studied by del Pino and Letelier [5], and Marcus and Veron [18]. However, no explicit estimates are known in neighborhoods of singular boundary points.

Other problems with a similar feature include complete Kähler-Einstein metrics discussed by Cheng and Yau [4], Fefferman [7], and Lee and Melrose [14], the complete minimal graphs in the hyperbolic space by Han and Jiang [9], Lin [15] and Tonegawa [20] and a class of Monge-Ampère equations by Jian and Wang [12].

Now we return to (1.1)-(1.2). For bounded domains $\Omega \subset \mathbb{R}^2$, let d be the distance function to $\partial\Omega$. If $\partial\Omega$ is C^2, then d is a C^2-function near $\partial\Omega$. Under this condition, the solution u of (1.1)-(1.2) satisfies

$$|u + \log d| \leq Cd, \tag{1.5}$$

where C is a positive constant depending only on the geometry of $\partial\Omega$. This follows from a simple comparison of u and the corresponding solutions in the interior and exterior tangent balls.

In this paper, we study the asymptotic behavior of u near isolated singular points on $\partial\Omega$. Taking a boundary point, say the origin, we assume $\partial\Omega$ has a conic singularity at the origin in the following sense: $\partial\Omega$ in a neighborhood of the origin consists of two C^2-curves σ_1 and σ_2, intersecting at the origin with an angle $\mu\pi$ for some constant $\mu \in (0, 2)$. Here, the origin is an end point of the both curves σ_1 and σ_2. Let l_1 and l_2 be two rays starting from the origin and tangent to σ_1 and σ_2 there, respectively. Then, an infinite cone V_μ formed by l_1 and l_2 is considered as a tangent cone of Ω at the origin, with an opening angle $\mu\pi$. Solutions of (1.1)-(1.2) in V_μ can be written explicitly. In fact, using polar coordinates, we write

$$V_\mu = \{(r, \theta) : r \in (0, \infty), \theta \in (0, \mu\pi)\}.$$

Here, l_1 corresponds to $\theta = 0$ and l_2 to $\theta = \mu\pi$. Then, the solution v_μ of (1.1)-(1.2) in V_μ is given by

$$v_\mu = -\log\left(\mu r \sin\frac{\theta}{\mu}\right). \tag{1.6}$$

Intuitively, v_μ should provide a good approximation of u near the origin. However, there is a major problem. The symmetric difference $(\Omega \setminus V_\mu) \cup (V_\mu \setminus \Omega)$ may be nonempty near the origin. For example, some $x \in \Omega$ may not be in the tangent cone V_μ. For a remedy, we need to modify v_μ to get a function defined in Ω near the origin.

To describe our result, we let d, d_1 and d_2 be the distances to $\partial\Omega, \sigma_1$ and σ_2, respectively. For $\mu \in (0, 1]$, we define, for any $x \in \Omega$,

$$f_\mu(x) = -\log\left(\mu|x| \sin\frac{\arcsin\frac{d(x)}{|x|}}{\mu}\right). \tag{1.7}$$

We note that f_μ in (1.7) is well-defined for x sufficiently small and that $\{x \in \Omega : d_1(x) = d_2(x)\}$ is a curve from the origin for $\mu \in (0, 1]$ near the origin. In fact, we can write, for

x sufficiently small,

$$f_\mu(x) = \begin{cases} -\log(\mu|x|\sin\frac{\arcsin\frac{d_1(x)}{|x|}}{\mu}) & \text{if } d_1(x) \le d_2(x), \\ -\log(\mu|x|\sin\frac{\arcsin\frac{d_2(x)}{|x|}}{\mu}) & \text{if } d_1(x) > d_2(x). \end{cases}$$

For $\mu \in (1,2)$, we define, for any $x \in \Omega$,

$$f_\mu(x) = \begin{cases} -\log(\mu|x|\sin\frac{\arcsin\frac{d_1(x)}{|x|}}{\mu}) & \text{if } d_1(x) < d_2(x), \\ -\log(\mu|x|\sin\frac{\theta}{\mu}) & \text{if } d_1(x) = d_2(x), \\ -\log(\mu|x|\sin\frac{\arcsin\frac{d_2(x)}{|x|}}{\mu}) & \text{if } d_1(x) > d_2(x), \end{cases} \tag{1.8}$$

where θ is the angle anticlockwise from the ray l_1 to $\overrightarrow{Ox}$. We note that f_μ in (1.8) is well-defined for x sufficiently small and that $\{x \in \Omega : d_1(x) = d_2(x)\}$ has a nonempty interior for $\mu \in (1,2)$. It is easy to see that f_μ in (1.7) and (1.8) is v_μ in (1.6) if Ω is the cone V_μ.

We now state our main result in this paper.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ in a neighborhood of the origin consist of two C^2-curves σ_1 and σ_2 intersecting at the origin at an angle $\mu\pi$, for some constant $\mu \in (0,2)$. Suppose $u \in C^\infty(\Omega)$ is a solution of* (1.1)-(1.2). *Then, for any $x \in \Omega \cap B_\delta$,*

$$|u(x) - f_\mu(x)| \le Cd(x), \tag{1.9}$$

where f_μ is the function defined in (1.7) *for $\mu \in (0,1]$ and in* (1.8) *for $\mu \in (1,2)$, d is the distance to $\partial\Omega$, and δ and C are positive constants depending only on the geometry of $\partial\Omega$.*

The estimate (1.9) generalizes (1.5) to singular domains and is optimal. The power one of the distance function in the right-hand side cannot be improved without better regularity assumptions of the boundary. The proof of Theorem 1.1 is based on a combination of conformal transforms and the maximum principle. An appropriate conformal transform changes the tangent cone at the origin to the upper half plane. The new boundary has a better regularity at the origin for $\mu \in (1,2)$ and becomes worse for $\mu \in (0,1)$. Such a change in the regularity of the boundary requires us to discuss the asymptotic behavior of solutions near $C^{1,\alpha}$-boundary and near $C^{2,\alpha}$-boundary.

The paper is organized as follows. In Section 2, we prove the existence and the uniqueness of solutions of (1.1)-(1.2) in a large class of domains. In Section 3, we study the asymptotic expansions near $C^{1,\alpha}$-boundary and derive an optimal estimate. In Section 4, we study the asymptotic expansions near $C^{2,\alpha}$-boundary and derive the corresponding optimal estimate. In Section 5, we study the asymptotic expansions near isolated singular points and prove Theorem 1.1. In Section 6, we discuss the asymptotic behavior of complete Kähler-Einstein metrics on singular product domains.

2. The Existence and Uniqueness

In this section, we prove the existence and the uniqueness of solutions of (1.1)-(1.2) in a large class of domains.

First, we introduce some notations. Let $x_0 \in \mathbb{R}^2$ be a point and $r > 0$ be a constant. For $\Omega = B_r(x_0)$, denote by u_{r,x_0} the corresponding solution of (1.1)-(1.2). Then,

$$u_{r,x_0}(x) = \log \frac{2r}{r^2 - |x - x_0|^2}.$$

With $d(x) = r - |x - x_0|$, we have

$$u_{r,x_0} = -\log d - \log\left(1 - \frac{d}{2r}\right).$$

For $\Omega = \mathbb{R}^2 \setminus B_r(x_0)$, denote by v_{r,x_0} the corresponding solution of (1.1)-(1.2). Then,

$$v_{r,x_0}(x) = \log \frac{2r}{|x - x_0|^2 - r^2}.$$

With $d(x) = |x - x_0| - r$, we have

$$v_{r,x_0} = -\log d - \log\left(1 + \frac{d}{2r}\right).$$

These two solutions play an important role in this paper.

Now we prove a preliminary result for domains with singularity. We note that a finite cone is determined by its vertex, its axis, its height and its opening angle.

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ satisfying a uniform exterior cone condition. Suppose $u \in C^\infty(\Omega)$ is a solution of* (1.1)-(1.2)*. Then, for any $x \in \Omega$ with $d < \delta$,*

$$|u + \log d| \le C,$$

where δ and C are positive constants depending only on the uniform exterior cone.

Proof. For any $x \in \Omega$ with $d(x) = d$, we have $B_d(x) \subset \Omega$. We assume $d = |x-p|$ for some $p \in \partial\Omega$. Let $u_{d,x}$ be the solution of (1.1)-(1.2) in $B_d(x)$. By the maximum principle, we have

$$u(x) \le u_{d,x}(x) = -\log d - \log\left(1 - \frac{d}{2d}\right) = -\log d + \log 2.$$

Next, there exists a cone V, with vertex p, axis $\overrightarrow{e_p}$, height h and opening angle 2θ, such that $V \cap \Omega = \emptyset$. Here, we can assume h and θ do not depend on the choice of $p \in \partial\Omega$. Set $\widetilde{p} = p + \frac{1}{\sin\theta} d\overrightarrow{e_p}$. It is straightforward to check $B_d(\widetilde{p}) \subset V \subset \Omega^C$, if $d < \frac{h}{1+\frac{1}{\sin\theta}}$, and $\mathrm{dist}(x, \partial B_d(\widetilde{p})) \le \frac{d}{\sin\theta}$. Let $v_{d,\widetilde{p}}$ be the solution of (1.1)-(1.2) in $\mathbb{R}^2 \setminus B_d(\widetilde{p})$. Then, by the maximum principle, we have

$$u(x) \ge v_{d,\widetilde{p}}(x) \ge -\log\left(\frac{d}{\sin\theta}\right) - \log\left(1 + \frac{d}{2d\sin\theta}\right) = -\log d - \log\left(\frac{1 + 2\sin\theta}{2\sin^2\theta}\right).$$

We have the desired result. $\square$

Next, we prove the existence and the uniqueness of solutions in a large class of domains. Such a result is well known. We include it here for completeness.

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ satisfying a uniform exterior cone condition. Then, there exists a unique solution $u \in C^\infty(\Omega)$ of* (1.1)-(1.2).

Proof. The proof consists of two steps. In the first step, we prove the existence of solutions by a standard method; while in the second step, we prove the uniqueness with the help of Lemma 2.1.

Step 1. We first construct a solution. For each positive integer k, let $u_k \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ be the solution of

$$\Delta u_k = e^{2u_k} \quad \text{in } \Omega,$$
$$u_k = k \quad \text{on } \partial\Omega.$$

By the maximum principle, we have $u_k \leq u_{k+1}$. Moreover, by comparing with solutions in balls in Ω and by the standard estimates for elliptic equations, we obtain, for any $k \geq 1$ and any subdomain $\Omega' \subset\subset \Omega$,

$$|u_k|_{L^\infty(\Omega')} \leq C_1(\Omega'),$$

and then, for any integer $m \geq 1$ and any $\alpha \in (0,1)$,

$$|u_k|_{C^{m,\alpha}(\Omega')} \leq C_2(m, \alpha, \Omega').$$

Therefore, there exists a $u \in C^\infty(\Omega)$ such that, for any $m \geq 1$ and any $\Omega' \subset\subset \Omega$,

$$u_k \to u \quad \text{in } C^m(\Omega'),$$

and hence u is a solution of (1.1). Moreover, u satisfies (1.2) by $u \geq u_k$ in Ω and $u_k = k$ on $\partial\Omega$.

Step 2. Let u be the solution constructed in Step 1. Without loss of generality, we assume Ω contains the origin. Then, $u(\frac{x}{\varepsilon}) + \log \frac{1}{\varepsilon}$ is a solution in $\varepsilon\Omega := \{x : \frac{x}{\varepsilon} \in \Omega\}$, for $\varepsilon > 0$. Hence, we may assume $\Omega \subset B_{1/2}$. Since the solution $u_{1/2,0}$ of (1.1)-(1.2) in $B_{1/2}$ satisfies $u_{1/2,0} \geq \log 4$, we have, by the maximum principle,

$$u \geq \log 4 \quad \text{in } \Omega.$$

Suppose v is another solution of (1.1)-(1.2) in Ω. By the construction of u in Step 1 and the maximum principle, we have

$$u \leq v \quad \text{in } \Omega.$$

Set

$$w = \frac{v}{u}.$$

Then, $w \geq 1$ in Ω and, by Lemma 2.1, $w(x) \to 1$ uniformly as $x \to \partial\Omega$. By the equation (1.1) for u and v, we have

$$e^{2uw} = e^{2v} = \Delta v = (\Delta u)w + 2\nabla u \cdot \nabla w + (\Delta w)u,$$

and hence

$$\Delta w + 2\frac{\nabla u}{u} \cdot \nabla w = \frac{1}{u}\big[(e^{2u})^w - (e^{2u})w\big].$$

If w is not equal to 1 identically, w must assume its maximum $w(x_0) > 1$ at some point $x_0 \in \Omega$. Then at x_0, we have

$$\nabla w(x_0) = 0, \quad \Delta w(x_0) \leq 0.$$

Next, we set $f(s) = a^s - as$, for some constant $a > e$. Then, $f(1) = 0$ and

$$f'(s) = a^s \log a - a > 0 \quad \text{for any } s > 1.$$

Hence, $f(s) > 0$ for any $s > 1$. Therefore,

$$\frac{1}{u}\left[(e^{2u})^w - (e^{2u})w\right] > 0 \quad \text{at } x_0,$$

since $e^{2u} \geq 16$. This leads to a contradiction. Therefore, $u = v$ in Ω. □

3. Expansions near $C^{1,\alpha}$-boundary

In this section, we study the asymptotic behavior near $C^{1,\alpha}$-portions of $\partial\Omega$.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ be $C^{1,\alpha}$ near $x_0 \in \partial\Omega$ for some $\alpha \in (0, 1]$. Suppose $u \in C^\infty(\Omega)$ is a solution of (1.1)-(1.2). Then,*

$$|u + \log d| \leq Cd^\alpha \quad \text{in } \Omega \cap B_r(x_0),$$

where d is the distance to $\partial\Omega$, and r and C are positive constants depending only on α and the geometry of Ω.

Proof. We take $R > 0$ sufficiently small such that $\partial\Omega \cap B_R(x_0)$ is $C^{1,\alpha}$. We fix an $x \in \Omega \cap B_{R/4}(x_0)$ and take $p \in \partial\Omega$, also near x_0, such that $d(x) = |x - p|$. Then, $p \in \partial\Omega \cap B_{R/2}(x_0)$. By a translation and rotation, we assume $p = 0$ and the x_2-axis is the interior normal to $\partial\Omega$ at 0. Then, x is on the positive x_2-axis, with $d = d(x) = |x|$, and the x_1-axis is the tangent line of $\partial\Omega$ at 0. Moreover, a portion of $\partial\Omega$ near 0 can be expressed as a $C^{1,\alpha}$-function φ of $x_1 \in (-s_0, s_0)$, with $\varphi(0) = 0$, and

$$|\varphi(x_1)| \leq M|x_1|^{1+\alpha} \quad \text{for any } x_1 \in (-s_0, s_0). \tag{3.1}$$

Here, s_0 and M are positive constants chosen to be uniform, independent of x.

We first consider the case $\alpha = 1$. For any $r > 0$, the lower semi-circle of

$$x_1^2 + (x_2 - r)^2 = r^2$$

satisfies $x_2 \geq x_1^2/(2r)$. By fixing a constant r sufficiently small, (3.1) implies

$$B_r(re_2) \subset \Omega \text{ and } B_r(-re_2) \cap \Omega = \emptyset.$$

Let u_{r,re_2} and $v_{r,-re_2}$ be the solutions of (1.1)-(1.2) in $B_r(re_2)$ and $\mathbb{R}^2 \setminus B_r(-re_2)$, respectively. Then, by the maximum principle, we have

$$v_{r,-re_2} \leq u \leq u_{r,re_2} \quad \text{in } B_r(re_2).$$

For the x above in the positive x_2-axis with $|x| = d < r$, we obtain

$$-\log d - \log\left(1 + \frac{d}{2r}\right) \leq u \leq -\log d - \log\left(1 - \frac{d}{2r}\right).$$

This implies the desired result for $\alpha = 1$.

Next, we consider $\alpha \in (0,1)$. Recall that x is in the positive x_2-axis and $|x| = d$. We first note

$$|x_1|^{1+\alpha} \leq d^{1+\alpha} + \frac{1}{d^{1-\alpha}} x_1^2 \quad \text{for any } x_1 \in \mathbb{R}. \tag{3.2}$$

This follows from the Hölder inequality, or more easily, by considering $|x_1| \leq d$ and $|x_1| \geq d$ separately. Let $r = d^{1-\alpha}/(2M)$ and q be the point on the positive x_2-axis such that $|q| = Md^{1+\alpha} + r$. By taking d sufficiently small, (3.1) and (3.2) imply

$$B_r(q) \subset \Omega \text{ and } B_r(-q) \cap \Omega = \emptyset.$$

Let $u_{r,q}$ and $v_{r,-q}$ be the solutions of (1.1)-(1.2) in $B_r(q)$ and $\mathbb{R}^2 \setminus B_r(-q)$, respectively. Then, by the maximum principle, we have

$$v_{r,-q} \leq u \leq u_{r,q} \quad \text{in } B_r(q).$$

For the x above, $\text{dist}(x, \partial B_r(q)) = d - Md^{1+\alpha}$ and $\text{dist}(x, \partial B_r(-q)) = d + Md^{1+\alpha}$. Evaluating at such an x, we obtain

$$\begin{aligned} &- \log(d + Md^{1+\alpha}) - \log\left(1 + \frac{M}{d^{1-\alpha}}(d + Md^{1+\alpha})\right) \\ &\quad \leq u \leq - \log(d - Md^{1+\alpha}) - \log\left(1 - \frac{M}{d^{1-\alpha}}(d - Md^{1+\alpha})\right). \end{aligned}$$

This implies the desired result for $\alpha \in (0,1)$. □

4. Expansions near $C^{2,\alpha}$-boundary

In this section, we study the asymptotic behavior near $C^{2,\alpha}$-portions of $\partial\Omega$. It is straightforward to derive the upper bound and extra work is needed for lower bound. We also note that the curvature of the boundary is only C^α in the present case.

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ be $C^{2,\alpha}$ near $x_0 \in \partial\Omega$ for some $\alpha \in (0,1)$. Suppose $u \in C^\infty(\Omega)$ is a solution of (1.1)-(1.2). Then,*

$$\left| u + \log d - \frac{1}{2}\kappa d \right| \leq Cd^{1+\alpha} \quad \text{in } \Omega \cap B_r(x_0),$$

where d is the distance to $\partial\Omega$, κ is the curvature of $\partial\Omega$, and r and C are positive constants depending only on α and the geometry of Ω.

Proof. We take $R > 0$ sufficiently small such that $\partial\Omega \cap B_{2R}(x_0)$ is $C^{2,\alpha}$ and that d is $C^{2,\alpha}$ in $\Omega \cap B_{2R}(x_0)$. The proof consists of several steps.

Step 1. Set

$$u = v - \log d. \tag{4.1}$$

A straightforward calculation yields

$$S(v) = 0 \quad \text{in } \Omega, \tag{4.2}$$

where

$$S(v) = d\Delta v - \Delta d - \frac{1}{d}(e^{2v} - 1). \tag{4.3}$$

By Theorem 3.1 for $\alpha = 1$, we have

$$|v| \le C_0 d \quad \text{in } \Omega \cap B_R(x_0),$$

for some constant C_0 depending only on the geometry of Ω. In particular, $v = 0$ on $\partial\Omega \cap B_R(x_0)$.

To proceed, we denote by (x', d) the principal coordinates in $\bar{\Omega} \cap B_R(x_0)$. Then,

$$\Delta v = \frac{\partial^2 v}{\partial d^2} + G\frac{\partial^2 v}{\partial x'^2} + I_{x'}\frac{\partial v}{\partial x'} + I_d\frac{\partial v}{\partial d},$$

where G, $I_{x'}$ and I_d are at least continuous functions in $\bar{\Omega} \cap B_R(x_0)$. We note that G has a positive lower bound and I_d has the form

$$I_d = -\kappa + O(d^\alpha), \tag{4.4}$$

where κ is the curvature of $\partial\Omega$. Set, for any constant $r > 0$,

$$G_r = \{(x', d) : |x'| \le r, 0 < d < r\}.$$

Step 2. We now construct supersolutions and prove an upper bound of v. We set

$$w(x) = d(x'^2 + d^2)^{\frac{\alpha}{2}}, \tag{4.5}$$

and, for some positive constants A and B to be determined,

$$\overline{v} = \frac{1}{2}\kappa(0)d + Aw + Bd^{1+\alpha}.$$

We write

$$S(\overline{v}) = d\Delta\overline{v} - \Delta d - \frac{2}{d}\overline{v} - \frac{1}{d}(e^{2\overline{v}} - 1 - 2\overline{v}).$$

First, we note

$$e^{2\overline{v}} \ge 1 + 2\overline{v}.$$

Then,

$$S(\overline{v}) \le d\Delta\overline{v} - \Delta d - \frac{2}{d}\overline{v}.$$

Hence,

$$\begin{aligned} S(\overline{v}) \le &\frac{1}{2}\kappa(0)d\Delta d + Ad\Delta w + Bd\Delta d^{1+\alpha} \\ &- \Delta d - \kappa(0) - 2A(x'^2 + d^2)^{\frac{\alpha}{2}} - 2Bd^\alpha. \end{aligned}$$

Straightforward calculations yield

$$|d\Delta w| \le C(d^\alpha + w),$$

where C is a positive constant depending only on the geometry of Ω near x_0. Note

$$|\Delta d + \kappa(0)| \le K(|x'|^2 + d^2)^{\frac{\alpha}{2}},$$

for some positive constant K depending only on the geometry of Ω near x_0. Then,

$$\begin{aligned} S(\overline{v}) \le &CAd^\alpha + B[\alpha(\alpha+1) + (1+\alpha)dI_d - 2]d^\alpha \\ &+ (CAd - 2A)(x'^2 + d^2)^{\frac{\alpha}{2}} + K(|x'|^2 + d^2)^{\frac{\alpha}{2}} + Cd. \end{aligned}$$

Since $\alpha < 1$, we can take r sufficiently small such that

$$2 - \alpha(\alpha+1) - (1+\alpha)dI_d \geq c_0 \quad \text{in } G_r,$$

for some positive constant c_0. By taking r small further and choosing $A \geq K + C$, we have

$$S(\overline{v}) \leq CAd^\alpha - c_0 Bd^\alpha \quad \text{in } G_r.$$

We take A large further such that

$$C_0 d \leq \frac{1}{2}\kappa(0)d + Ad(x'^2 + d^2)^{\frac{\alpha}{2}} + Bd^{1+\alpha} \quad \text{on } \partial G_r.$$

Then, we take B large such that

$$c_0 B \geq CA.$$

Therefore,

$$S(\overline{v}) \leq S(v) \quad \text{in } G_r,$$
$$v \leq \overline{v} \quad \text{on } \partial G_r.$$

By the maximum principle, we have $v \leq \overline{v}$ in G_r.

Step 3. We now construct subsolutions and prove a lower bound of v. By taking the same w as in (4.5) and setting, for some positive constants A an B to be determined,

$$\underline{v} = \frac{1}{2}\kappa(0)d - Aw - Bd^{1+\alpha}.$$

We first assume

$$|\kappa(0)|r + A2^{\frac{\alpha}{2}}r^{1+\alpha} + Br^{1+\alpha} \leq \frac{2-\alpha(\alpha+1)}{16}. \tag{4.6}$$

Then,

$$\left|\frac{1}{d}(e^{2\underline{v}} - 1 - 2\underline{v})\right| \leq 2\kappa^2(0)d + \frac{1}{2}[2 - \alpha(\alpha+1)][A(x'^2 + d^2)^{\frac{\alpha}{2}} + Bd^\alpha].$$

Arguing as in Step 2, we obtain

$$S(\underline{v}) \geq -CAd^\alpha + B\left[1 - \frac{1}{2}\alpha(\alpha+1) - (1+\alpha)dI_d\right]d^\alpha$$
$$+ (A - CAd)(x'^2 + d^2)^{\frac{\alpha}{2}} - K(|x'|^2 + d^2)^{\frac{\alpha}{2}} - Cd.$$

We require

$$d \leq \frac{1}{2C}, \quad 1 - \frac{1}{2}\alpha(\alpha+1) - (1+\alpha)dI_d \geq c_0 \quad \text{in } G_r, \tag{4.7}$$

for some positive constant c_0. If $A \geq 2K + 2C$, we have

$$S(\underline{v}) \geq -CAd^\alpha + c_0 Bd^\alpha.$$

If

$$c_0 B \geq CA,$$

we have $S(\underline{v}) \geq 0$. In order to have $v \geq \underline{v}$ on ∂G_r, it is sufficient to require

$$|\kappa(0)| + C_0 \leq Ar^\alpha.$$

In summary, we first choose

$$A = \frac{|\kappa(0)| + C_0}{r^\alpha}, \quad B = \frac{AC}{c_0},$$

for some r small to be determined. Then, we choose r small satisfying (4.7) such that $A \geq 2K + 2C$ and (4.6) holds. Therefore, we have

$$\begin{aligned} S(\underline{v}) &\geq S(v) \quad \text{in } G_r, \\ v &\geq \underline{v} \quad \text{on } \partial G_r. \end{aligned}$$

By the maximum principle, we have $v \geq \underline{v}$ in G_r.

Step 4. Therefore, we obtain

$$\underline{v} \leq v \leq \overline{v} \quad \text{in } G_r.$$

By taking $x' = 0$, we obtain, for any $d \in (0, r)$,

$$\left| v(0, d) - \frac{1}{2}\kappa(0)d \right| \leq Cd^{1+\alpha}.$$

This is the desired estimate. □

We point out that the proof above can be adapted to yield a similar result as in Theorem 4.1 for the equation (1.3).

5. Expansions near Isolated Singular Boundary Points

In this section, we study the asymptotic behavior of u near isolated singular boundary points. Throughout this section, we will adopt notations from complex analysis and denote by $z = (x, y)$ points in the plane.

We fix a boundary point; in the following, we always assume this is the origin. We assume $\partial\Omega$ in a neighborhood of the origin consists of two C^2 curves σ_1 and σ_2. Here, the origin is an end of both σ_1 and σ_2. Suppose l_1 and l_2 are two rays from the origin such that σ_1 and σ_2 are tangent to l_1 and l_2 at the origin, respectively. The rays l_1 and l_2 divide $\mathbb{R}^2$ into two cones and one of the cones is naturally defined as the tangent cone of Ω at the origin. By a rotation, we assume the tangent cone V_μ is given by, for some positive constant $\mu \in (0, 2)$,

$$V_\mu = \{(r, \theta) \in \mathbb{R}^2 : 0 < r < \infty, 0 < \theta < \mu\pi\}. \tag{5.1}$$

Here, we used the polar coordinates in $\mathbb{R}^2$. In fact, the tangent cone V_μ can be characterized by the following: For any $\varepsilon > 0$, there exists an $r_0 > 0$ such that

$$\{(r, \theta) : r \in (0, r_0), \theta \in (\varepsilon, \mu\pi - \varepsilon)\} \subset \Omega \cap B_{r_0} \subset \{(r, \theta) : r \in (0, r_0), \theta \in (-\varepsilon, \mu\pi + \varepsilon)\}.$$

Our goal is to approximate solutions near an isolated singular boundary point by the corresponding solutions in tangent cones. To this end, we express explicitly the solutions in tangent cones. For any constant $\mu \in (0, 2)$, consider the unbounded cone V_μ defined by (5.1). Then, the solution of (1.1)-(1.2) in V_μ is given by

$$v_\mu = -\log\left(\mu r \sin\frac{\theta}{\mu}\right). \tag{5.2}$$

For $\mu \in (0,1)$ and $\theta \in (0, \mu\pi/2)$, we have $d = r\sin\theta$ and

$$v_\mu = -\log d - \log \frac{\mu \sin\frac{\theta}{\mu}}{\sin\theta}. \tag{5.3}$$

For $\mu \in (1,2)$, if $\theta \in (0,\pi/2)$, we have $d = r\sin\theta$ and the identity above; if $\theta \in (\pi/2, \mu\pi/2)$, we have $d = r$ and

$$v_\mu = -\log d - \log\left(\mu \sin\frac{\theta}{\mu}\right). \tag{5.4}$$

We note that the second terms in (5.3) and (5.4) are constant along the ray from the origin. This suggests that Lemma 2.1 cannot be improved in general if the boundary has a singularity.

Next, we modify the solution in (5.2) and construct super- and subsolutions. Define

$$\overline{u}_\mu = v_\mu + \log\left(1 + A|z|^{\frac{\sqrt{2}}{\mu}}\right), \tag{5.5}$$

and

$$\underline{u}_\mu = v_\mu - \log\left(1 + A|z|^{\frac{1}{\mu}}\right), \tag{5.6}$$

where v_μ is given by (5.2) and A is a positive constant.

Lemma 5.1. *Let V_μ be the cone defined in* (5.1)*, and $\overline{u}_\mu$ and $\underline{u}_\mu$ be defined by* (5.5) *and* (5.6)*, respectively. Then, $\overline{u}_\mu$ is a supersolution and $\underline{u}_\mu$ is a subsolution of* (1.1) *in V_μ, respectively.*

Proof. We calculate in polar coordinates. For functions of r only, we have

$$\Delta = \partial_{rr} + \frac{1}{r}\partial_r.$$

Note $r = |z|$. A straightforward calculation yields

$$\Delta\left(\log\left(1 + A|z|^{\frac{\sqrt{2}}{\mu}}\right)\right) = \frac{2}{\mu^2 r^2} \cdot \frac{Ar^{\frac{\sqrt{2}}{\mu}}}{1 + Ar^{\frac{\sqrt{2}}{\mu}}} - \frac{2}{\mu^2 r^2}\left(\frac{Ar^{\frac{\sqrt{2}}{\mu}}}{1 + Ar^{\frac{\sqrt{2}}{\mu}}}\right)^2.$$

Then,

$$\begin{aligned}
\Delta\overline{u}_\mu &= \frac{1}{\mu^2 r^2 \sin^2\frac{\theta}{\mu}} + \frac{2}{\mu^2 r^2} \cdot \frac{Ar^{\frac{\sqrt{2}}{\mu}}}{1 + Ar^{\frac{\sqrt{2}}{\mu}}} - \frac{2}{\mu^2 r^2}\left(\frac{Ar^{\frac{\sqrt{2}}{\mu}}}{1 + Ar^{\frac{\sqrt{2}}{\mu}}}\right)^2 \\
&= \frac{1}{\mu^2 r^2 \sin^2\frac{\theta}{\mu}}\left(1 + \frac{2Ar^{\frac{\sqrt{2}}{\mu}}}{1 + Ar^{\frac{\sqrt{2}}{\mu}}}\sin^2\frac{\theta}{\mu} - 2\left(\frac{Ar^{\frac{\sqrt{2}}{\mu}}}{1 + Ar^{\frac{\sqrt{2}}{\mu}}}\right)^2 \sin^2\frac{\theta}{\mu}\right) \\
&\le \frac{1}{\mu^2 r^2 \sin^2\frac{\theta}{\mu}}\left(1 + 2Ar^{\frac{\sqrt{2}}{\mu}}\right) \le \left(\frac{1}{\mu r \sin\frac{\theta}{\mu}}\right)^2 \left(1 + Ar^{\frac{\sqrt{2}}{\mu}}\right)^2 = e^{2\overline{u}_\mu}.
\end{aligned}$$

Hence, $\overline{u}_\mu$ is a supersolution in V_μ.

The proof for $\underline{u}_\mu$ is similar. In fact, we have

$$\begin{aligned}
\Delta\underline{u}_\mu &= \frac{1}{\mu^2 r^2\sin^2\frac{\theta}{\mu}} - \frac{1}{\mu^2r^2}\cdot\frac{Ar^{\frac{1}{\mu}}}{1+Ar^{\frac{1}{\mu}}} + \frac{1}{\mu^2r^2}\left(\frac{Ar^{\frac{1}{\mu}}}{1+Ar^{\frac{1}{\mu}}}\right)^2 \\
&= \frac{1}{\mu^2 r^2\sin^2\frac{\theta}{\mu}}\left(1-\frac{Ar^{\frac{1}{\mu}}}{1+Ar^{\frac{1}{\mu}}}\sin^2\frac{\theta}{\mu}+\left(\frac{Ar^{\frac{1}{\mu}}}{1+Ar^{\frac{1}{\mu}}}\right)^2\sin^2\frac{\theta}{\mu}\right) \\
&\geq \frac{1}{\mu^2 r^2\sin^2\frac{\theta}{\mu}}\left(1-\frac{Ar^{\frac{1}{\mu}}}{1+Ar^{\frac{1}{\mu}}}\right) \geq \left(\frac{1}{\mu r\sin\frac{\theta}{\mu}}\right)^2\left(1+Ar^{\frac{1}{\mu}}\right)^{-2} = e^{2\underline{u}_\mu}.
\end{aligned}$$

Hence, $\underline{u}_\mu$ is a subsolution in V_μ. □

Next, we describe how solutions of (1.1)-(1.2) change under one-to-one holomorphic mappings.

Lemma 5.2. *Let Ω_1 and Ω_2 be two domains in $\mathbb{R}^2$. Suppose $u_2 \in C^\infty(\Omega_2)$ is a solution of (1.1) in Ω_2 and f is a one-to-one holomorphic function from Ω_1 onto Ω_2. Then,*

$$u_1(z) = u_2(f(z)) + \log|f'(z)|$$

is a solution of (1.1) in Ω_1.

Proof. Note that $g_2 = e^{2u_2}(dx\otimes dx + dy\otimes dy)$ is a complete metric with constant Gauss curvature -1 on Ω_2. Since the Gauss curvature of the pull-back metric remains the same under the conformal mapping, then $g_1 = f^*g_2 = e^{2u_1}(dx\otimes dx+dy\otimes dy)$ is a complete metric with constant Gauss curvature -1 on Ω_1. Hence, u_1 solves (1.1) in Ω_1. □

Next, we prove that asymptotic expansions near singular boundary points are local properties.

Lemma 5.3. *Let Ω_1 and Ω_2 be two domains which coincide in a neighborhood of the origin and let V_μ be the tangent cone of Ω_1 and Ω_2 at the origin, for some $\mu \in (0,2)$. Suppose u_1 and u_2 are the solutions of (1.1)-(1.2) in Ω_1 and Ω_2, respectively. Then,*

$$u_1 = u_2 + O(|z|^{\frac{1}{\mu}}). \tag{5.7}$$

Proof. Taking $\widetilde{\mu}$ such that $\widetilde{\mu} > \mu$ and $\widetilde{\mu} < \min\{\sqrt{2}\mu, 2\}$ and set

$$\widetilde{V}_{\widetilde{\mu}} = \left\{(r,\theta)\in\mathbb{R}^2 :\ 0<r<\infty,\ -\frac{\widetilde{\mu}-\mu}{2}\pi<\theta<\frac{\widetilde{\mu}+\mu}{2}\pi\right\}. \tag{5.8}$$

For some constant $\delta_1 > 0$, we have

$$\Omega_1\cap B_{\delta_1}\subseteq \widetilde{V}_{\widetilde{\mu}}.$$

Set

$$\widetilde{\theta} = \theta + \frac{1}{2}(\widetilde{\mu}-\mu)\pi.$$

By Lemma 2.1, we have, for A_1 sufficiently large,

$$u_1(z) \geq -\log\left(\widetilde{\mu}|z|\sin\frac{\widetilde{\theta}}{\widetilde{\mu}}\right) - \log\left(1 + A_1|z|^{\frac{1}{\widetilde{\mu}}}\right) \quad \text{on } \Omega_1 \cap \partial B_{\delta_1}.$$

The estimate above obviously holds on $\partial\Omega_1 \cap B_{\delta_1}$. By Lemma 5.1 and the maximum principle, we have

$$u_1(z) \geq -\log\left(\widetilde{\mu}|z|\sin\frac{\widetilde{\theta}}{\widetilde{\mu}}\right) - \log\left(1 + A_1|z|^{\frac{1}{\widetilde{\mu}}}\right) \quad \text{in } \Omega_1 \cap B_{\delta_1}. \tag{5.9}$$

In particular, we can take $\delta_2 < \delta_1$ such that

$$e^{2u_1} \geq \frac{1}{2\mu^2|z|^2} \quad \text{in } \Omega_1 \cap B_{\delta_2}.$$

As in the proof of Lemma 5.1, we can verify that $u_1 - \log\left(1 + A|z|^{\frac{1}{\mu}}\right)$ is a subsolution of (1.1) in $\Omega_1 \bigcap B_{\delta_2}$. By Lemma 2.1 and the maximum principle, we have, for A sufficiently large,

$$u_1 \leq u_2 + \log\left(1 + A|z|^{\frac{1}{\mu}}\right) \quad \text{in } \Omega_1 \cap B_{\delta_2}.$$

Similarly, we have

$$u_2 \leq u_1 + \log\left(1 + A|z|^{\frac{1}{\mu}}\right) \quad \text{in } \Omega_1 \cap B_{\delta_2}.$$

This implies the desired result. □

Now we prove a simple calculus result.

Lemma 5.4. *Let σ be a curve defined by a function $y = \varphi(x) \in C^{1,\alpha}([0,\delta])$, for some constants $\alpha \in (0,1]$ and $\delta > 0$, satisfying $\varphi(0) = 0$ and*

$$|\varphi'(x)| \leq Mx^\alpha,$$

for some positive constant M. For any given point $z = (x, y)$ with $0 < x < \delta$ and $y > \varphi(x)$, let $p = (x', \varphi(x'))$ be the closest point to z on σ with the distance d. Then, for $|z|$ sufficient small,

$$x' \leq 2|z|.$$

Moreover, if $|y| \leq x/4$, then

$$|x - x'| \leq Cdx^\alpha,$$

where C is a positive constant depending only on M and α.

Proof. First, we note $d \leq |z|$ since d is the distance of z to σ. Then,

$$x' \leq |p| \leq |z| + |z - p| = |z| + d \leq 2|z|.$$

Next, for $x' \in (0, \delta)$, x' is characterized by

$$\frac{d}{dt}[(x-t)^2 + (y - \varphi(t))^2]|_{t=x'} = 0,$$

or

$$x - x' = (y - \varphi(x'))\varphi'(x').$$

If $|y| \le x/4$, then $|z| \le 5x/4$ and hence $x' \le 5x/2$. Moreover, $|y - \varphi(x')| \le d$. Then,

$$|x - x'| \le d|\varphi'(x')|. \tag{5.10}$$

This implies the desired result. □

We are ready to discuss the case when the opening angle of the tangent cone of Ω at the origin is less than π.

Theorem 5.5. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ in a neighborhood of the origin consist of two C^2-curves σ_1 and σ_2 intersecting at the origin with an angle $\mu\pi$, for some constant $\mu \in (0,1)$. Suppose $u \in C^\infty(\Omega)$ is a solution of* (1.1)-(1.2). *Then, for any $z \in \Omega \cap B_\delta$,*

$$|u(z) - f_\mu(z)| \le Cd(z), \tag{5.11}$$

where f_μ is given by

$$f_\mu(z) = -\log\left(\mu|z|\sin\frac{\arcsin\frac{d(z)}{|z|}}{\mu}\right), \tag{5.12}$$

d is the distance to $\partial\Omega$, and δ and C are positive constants depending only on the geometry of $\partial\Omega$.

Proof. We denote by d_1 and d_2 the distances to σ_1 and σ_2, respectively. We only consider the case $d_1 = d \le d_2$. We also denote by M the C^2-norm of σ_1 and σ_2. In the following, C and δ are positive constants depending only on the geometry of $\partial\Omega$. We will prove (5.11) with $d = d_1$.

Consider the conformal homeomorphism $T : z \mapsto z^{\frac{1}{\mu}}$. For

$$z = (x, y) = (|z|\cos\theta, |z|\sin\theta),$$

we write

$$T(z) = \widetilde{z} = (\widetilde{x}, \widetilde{y}) = \left(|z|^{\frac{1}{\mu}}\cos\frac{\theta}{\mu}, |z|^{\frac{1}{\mu}}\sin\frac{\theta}{\mu}\right).$$

By restricting to a small neighborhood of the origin, we assume σ_1 and σ_2 are curves over their tangent lines at the origin. Set $\widetilde{\sigma}_i = T(\sigma_i)$, $i = 1, 2$, and $\widetilde{\sigma} = \widetilde{\sigma}_1 \cup \widetilde{\sigma}_2$. We first study the regularity of $\widetilde{\sigma}$. By expressing $\widetilde{\sigma}$ by $\widetilde{y} = \widetilde{\varphi}(\widetilde{x})$, we claim

$$|\widetilde{\varphi}(\widetilde{x})| \le \widetilde{M}\widetilde{x}^{1+\mu}, \quad |\widetilde{\varphi}'(\widetilde{x})| \le \widetilde{M}\widetilde{x}^{\mu}, \quad |\widetilde{\varphi}''(\widetilde{x})| \le \widetilde{M}\widetilde{x}^{\mu-1}, \tag{5.13}$$

where $\widetilde{M}$ is a positive constant depending only on M and μ.

To prove (5.13), we assume σ_1 is given by the function $y = \varphi_1(x)$ satisfying $\varphi_1(0) = 0$, $\varphi_1'(0) = 0$ and

$$|\varphi_1''(x)| \le M.$$

Assume $\widetilde{\sigma}_1 = T(\sigma_1)$ is given by $\widetilde{y} = \widetilde{\varphi}_1(\widetilde{x})$. To prove the estimate of $\widetilde{\varphi}_1$, we note $|y| \le Cx^2$ on σ_1 and $|z| = O(\widetilde{x}^\mu)$ on $\widetilde{\sigma}_1$ for $|z|$ sufficiently small. Then,

$$|\widetilde{y}| = |z|^{\frac{1}{\mu}-1}\left||z|\sin\frac{\theta}{\mu}\right| \le C|z|^{\frac{1}{\mu}-1}|y| \le C|z|^{\frac{1}{\mu}-1}x^2 \le C|z|^{1+\frac{1}{\mu}} \le C\widetilde{x}^{1+\mu}.$$

This is the first estimate in (5.13). To prove estimates of derivatives of $\widetilde{\varphi}_1$, we first note that $(\widetilde{x},\widetilde{y})$ on $\widetilde{\sigma}_1$ is given by

$$\widetilde{x} = (x^2+\varphi_1(x)^2)^{\frac{1}{2\mu}} \cos \frac{\arcsin \frac{\varphi_1(x)}{((x^2+\varphi_1(x)^2)^{\frac{1}{2}}}}{\mu},$$

and

$$\widetilde{y} = (x^2+\varphi_1(x)^2)^{\frac{1}{2\mu}} \sin \frac{\arcsin \frac{\varphi_1(x)}{((x^2+\varphi_1(x)^2)^{\frac{1}{2}}}}{\mu}.$$

Straightforward calculations yield

$$\begin{aligned}\frac{d\widetilde{x}}{dx} &= \frac{1}{\mu} x^{\frac{1}{\mu}-1}(1+O(x)),\\ \frac{d^2\widetilde{x}}{dx^2} &= \frac{1}{\mu}\left(\frac{1}{\mu}-1\right) x^{\frac{1}{\mu}-2}(1+O(x)),\end{aligned}$$

and

$$\begin{aligned}\left|\frac{d\widetilde{y}}{dx}\right| &\le \frac{1}{\mu}\left(\frac{1}{\mu}+1\right) M x^{\frac{1}{\mu}}(1+O(x)),\\ \left|\frac{d^2\widetilde{y}}{dx^2}\right| &\le \frac{1}{\mu^2}\left(\frac{1}{\mu}+1\right) M x^{\frac{1}{\mu}-1}(1+O(x)).\end{aligned}$$

With $x = O(\widetilde{x}^{\mu})$, we get the second and third estimates in (5.13). This finishes the proof of (5.13) for $\widetilde{x} \ge 0$. A similar argument holds for $\widetilde{x} < 0$.

We now discuss three cases for $z \in \Omega \cap B_\delta$ with $d_1(z) \le d_2(z)$, for δ sufficiently small. For simplicity, we set

$$\begin{aligned}\Omega_1 &= \{z \in \Omega :\ d_1(z) > c_0|z|\},\\ \Omega_2 &= \{z \in \Omega :\ c_0|z| < d_1(z) < c_1|z|^2\},\\ \Omega_3 &= \{z \in \Omega :\ d_1(z) < c_1|z|^2\},\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\gamma_1 &= \{z \in \Omega :\ d_1(z) = c_0|z|\},\\ \gamma_2 &= \{z \in \Omega :\ d_1(z) = c_1|z|^2\},\end{aligned} \tag{5.15}$$

where c_0 and c_1 are appropriately chosen constants with $c_0 < \frac{1}{2}\mu \arctan \frac{1}{4}$.

Case 1. We consider $z \in \Omega_1 \cap B_\delta$.

Set

$$\Omega_+ = \Omega \cap B_\delta, \quad \Omega_- = \Omega \cup B_\delta^c.$$

Let u_+ and u_- be the solutions of (1.1)-(1.2) in Ω_+ and Ω_-, respectively. Then, we have

$$u_- \le u \le u_+ \quad \text{in } \Omega_+. \tag{5.16}$$

We take δ small so that T is one-to-one on Ω_+.

Set $\widetilde{\Omega}_+ = T(\Omega_+)$ and let $\widetilde{u}_+$ be the solution of (1.1)-(1.2) in $\widetilde{\Omega}_+$. By (5.13), the curve $\widetilde{\sigma}$ given by $\widetilde{y} = \widetilde{\varphi}(\widetilde{x})$ satisfies

$$-\widetilde{M}|\widetilde{x}|^{1+\mu} \leq |\widetilde{\varphi}(\widetilde{x})| \leq \widetilde{M}|\widetilde{x}|^{1+\mu}.$$

Theorem 3.1 implies, for $\widetilde{z}$ close to the origin,

$$\widetilde{u}_+(\widetilde{z}) \leq -\log \widetilde{d}_1 + O(\widetilde{d}_1^{\mu}), \tag{5.17}$$

and

$$\widetilde{u}_+(\widetilde{z}) \geq -\log \widetilde{d}_2 + O(\widetilde{d}_2^{\mu}), \tag{5.18}$$

where $\widetilde{d}_1$ and $\widetilde{d}_2$ are the distances from $\widetilde{z}$ to the curves $\widetilde{y} = \widetilde{M}|\widetilde{x}|^{1+\mu}$ and $\widetilde{y} = -\widetilde{M}|\widetilde{x}|^{1+\mu}$, respectively. Let $(\widetilde{x}', \widetilde{y}')$ be the point on $\widetilde{y} = \widetilde{M}|\widetilde{x}|^{1+\mu}$ realizing the distance from $\widetilde{z}$. Then,

$$\widetilde{y} - \widetilde{y}' \leq \widetilde{d}_1 \leq \widetilde{y},$$

and hence

$$|\widetilde{d}_1 - \widetilde{y}| \leq \widetilde{y}' = \widetilde{\varphi}(\widetilde{x}') \leq \widetilde{M}|\widetilde{x}'|^{1+\mu} \leq C\left(|z|^{\frac{1}{\mu}}\right)^{1+\mu} = C|z|^{1+\frac{1}{\mu}}.$$

By $d \geq c_0|z|$, we have $\theta \geq \theta_0$ for some positive constant θ_0, for $|z|$ sufficiently small, and then $|z| = O(y) = O(\widetilde{y}^{\mu})$. Hence, $|\widetilde{d}_1 - \widetilde{y}| \leq C\widetilde{y}|z|$ and then $\widetilde{d}_1 = \widetilde{y}(1 + O(|z|))$. Therefore,

$$\log \widetilde{d}_1 = \log \widetilde{y} + O(|z|) = \log \widetilde{y} + O(d).$$

Next, we note

$$\widetilde{d}_1^{\mu} \leq |\widetilde{z}|^{\mu} = |z| \leq Cd.$$

Similar estimates hold for $\widetilde{d}_2$. Then, (5.17) and (5.18) imply

$$\widetilde{u}_+(\widetilde{z}) = -\log \widetilde{y} + O(d). \tag{5.19}$$

Let V_μ be the tangent cone of Ω at the origin given by (5.1) and v be the corresponding solution in the tangent cone V_μ given by (5.2). Then, $T(V)$ is the upper half-plane and the solution $\widetilde{v}$ of (1.1)-(1.2) in $T(V)$ is given by

$$\widetilde{v}(\widetilde{z}) = -\log \widetilde{y}.$$

Hence, (5.19) implies

$$\widetilde{u}_+(\widetilde{z}) = \widetilde{v}(\widetilde{z}) + O(d).$$

By Lemma 5.2, we have

$$u_+(z) = \widetilde{u}_+(\widetilde{z}) + \log\left(\frac{1}{\mu}|z|^{\frac{1}{\mu}-1}\right),$$

and

$$v(z) = \widetilde{v}(\widetilde{z}) + \log\left(\frac{1}{\mu}|z|^{\frac{1}{\mu}-1}\right).$$

Therefore, we obtain

$$u_+(z) = v(z) + O(d). \tag{5.20}$$

Next, we fix a point $P \in \Omega_-^c$ and consider the conformal homeomorphism $\widehat{T} : z \mapsto \frac{1}{z-P}$. We assume that $\widehat{T}$ maps Ω_- to $\widehat{\Omega}_-$, V_μ to $\widehat{V}_\mu$, σ_i to $\widehat{\sigma}_i$, and l_i to $\widehat{l}_i$. Then, $\widehat{\sigma}_i$ and $\widehat{l}_i$ are C^2-curves with bounded C^2-norms in a small neighborhood of $\widehat{T}(0)$ since $\widehat{T}$ is smooth in $\overline{B}_{|0P|/2}$. The tangent cone of $\widehat{\Omega}_-$ at $\widehat{T}(0)$, denoted by $\underline{V}$, has an opening angle $\mu\pi$ since $\widehat{T}$ is conformal. Let $\widehat{u}_-$, $\widehat{v}$ and $\underline{v}$ be the solutions of (1.1)-(1.2) in $\widehat{\Omega}_-$, $\widehat{V}_\mu$ and $\underline{V}$, respectively. By Lemma 5.2, we have

$$u_-(z) = \widehat{u}_-(\widehat{z}) - 2\ln|z - P|,$$

and

$$v(z) = \widehat{v}(\widehat{z}) - 2\ln|z - P|.$$

By applying (5.20) (for u_+ in Ω_+) to $\widehat{u}_-$ and $\widehat{v}$ in $\widehat{\Omega}_-$ and $\widehat{V}_\mu$, respectively, we have

$$\widehat{u}_-(\widehat{z}) = \underline{v}(\widehat{z}) + O(\widehat{d}),$$

and

$$\widehat{v}(\widehat{z}) = \underline{v}(\widehat{z}) + O(\widehat{d}).$$

We note that the distance $\widehat{d}$ from $\widehat{z}$ to $\partial\widehat{\Omega}_-$ is comparable to that from $\widehat{z}$ to $\partial\widehat{V}_\mu$. Therefore,

$$\widehat{u}_-(\widehat{z}) = \widehat{v}(\widehat{z}) + O(\widehat{d}),$$

and hence

$$u_-(z) = v(z) + O(d). \tag{5.21}$$

By combining (5.16), (5.20) and (5.21), we have

$$u(z) = v(z) + O(d).$$

By the explicit expression of v in (5.2), it is straightforward to verify

$$v(z) = -\log\left(\mu|z|\sin\frac{\arcsin\frac{d}{|z|}}{\mu}\right) + O(d).$$

We hence have (5.11).

Case 2. We consider $z \in \Omega_2 \cap B_\delta$ and discuss in two cases.

Case 2.1. First, we assume T is one-to-one in Ω. Set $\widetilde{\Omega} = T(\Omega)$ and let $\widetilde{u}$ be the solution of (1.1)-(1.2) in $\widetilde{\Omega}$. Let $\widetilde{p} = (\widetilde{x}', \widetilde{y}')$ be the closest point to $\widetilde{z}$ on $\widetilde{\sigma}_1$ with the distance $\widetilde{d}$. If c_0 is small, then $|\widetilde{y}| \le c_*|\widetilde{x}|$ for some constant c_* small. By Lemma 5.4, we have

$$\widetilde{x}' = \widetilde{x} + O(\widetilde{x}'^{\mu}\widetilde{d}).$$

Note that $|z|$ is comparable with x and $|\widetilde{z}|$ is comparable with $\widetilde{x}$. With $\widetilde{x}'^{\mu} \le |z|$, we also have

$$|\widetilde{\varphi}'(\widetilde{x}')| \le \widetilde{M}\widetilde{x}'^{\mu} \le C|z|. \tag{5.22}$$

Similarly, we have

$$\frac{|\widetilde{\varphi}(\widetilde{x}')|}{|\widetilde{x}'|} \le C|z|. \tag{5.23}$$

Next, we claim, for any $\widehat{x}$ sufficient small,

$$|\widetilde{\varphi}(\widehat{x}) - \widetilde{\varphi}(\widetilde{x}') - \widetilde{\varphi}'(\widetilde{x}')(\widehat{x} - \widetilde{x}')| \leq K|z|^{1-\frac{1}{\mu}}(\widehat{x} - \widetilde{x}')^2, \tag{5.24}$$

where K is a positive constant depending only on M and μ. We prove (5.24) in three cases. If $\widehat{x} \geq |z|^{\frac{1}{\mu}}/3$, then, with $\mu \in (0, 1)$,

$$|\widetilde{\varphi}''(\widehat{x})| \leq \widetilde{M}\widehat{x}^{\mu-1} \leq C|z|^{1-\frac{1}{\mu}},$$

and (5.24) holds by the Taylor expansion. If $0 \leq \widehat{x} \leq |z|^{\frac{1}{\mu}}/3$, we have

$$|\widetilde{\varphi}(\widehat{x})| \leq \widetilde{M}\widehat{x}^{1+\mu} \leq C|z|^{1+\frac{1}{\mu}},$$
$$|\widetilde{\varphi}'(\widetilde{x}')(\widehat{x} - \widetilde{x}')| \leq C|z|^{1+\frac{1}{\mu}},$$

and

$$|z|^{1+\frac{1}{\mu}} = |z|^{1-\frac{1}{\mu}}(|z|^{\frac{1}{\mu}})^2 \leq C|z|^{1-\frac{1}{\mu}}(\widehat{x} - \widetilde{x}')^2.$$

Then, (5.24) follows. If $\widehat{x} \leq 0$, we have

$$|\widetilde{\varphi}(\widehat{x})| \leq \widetilde{M}|\widehat{x}|^{1+\mu} \leq Cr^{1-\frac{1}{\mu}}(\widehat{x} - \widetilde{x}')^2,$$

and

$$|\widetilde{\varphi}'(\widetilde{x}')(\widehat{x} - \widetilde{x}')| \leq Cr^{1-\frac{1}{\mu}}(\widehat{x} - \widetilde{x}')^2.$$

Then, (5.24) also holds.

By (5.24), it is easy to check, for some $R = C'|z|^{\frac{1}{\mu}-1}$,

$$B_R(\widetilde{p} + R\vec{n}) \subset \widetilde{\Omega} \quad \text{and} \quad B_R(\widetilde{p} - R\vec{n}) \cap \widetilde{\Omega} = \emptyset,$$

where $\vec{n}$ is the unit inward normal vector of $\widetilde{\sigma}_1$ at $\widetilde{p}$ and C' is some constant depending only on the geometry of $\partial\Omega$. Let $u_{R,\widetilde{p}+R\vec{n}}$ and $v_{R,\widetilde{p}-R\vec{n}}$ be the solutions of (1.1)-(1.2) in $B_R(\widetilde{p} + R\vec{n})$ and $\mathbb{R}^2 \setminus B_R(\widetilde{p} - R\vec{n})$, respectively. Then, by the maximum principle, we have

$$v_{R,\widetilde{p}-R\vec{n}} \leq \widetilde{u} \leq u_{R,\widetilde{p}+R\vec{n}} \quad \text{in } B_R(\widetilde{p} + R\vec{n}),$$

and hence, at $\widetilde{z}$,

$$-\log \widetilde{d} - \log\left(1 + \frac{\widetilde{d}}{2R}\right) \leq \widetilde{u} \leq -\log \widetilde{d} - \log\left(1 - \frac{\widetilde{d}}{2R}\right).$$

Therefore,

$$\widetilde{u}(\widetilde{z}) = -\log \widetilde{d} + O\left(\frac{\widetilde{d}}{|z|^{\frac{1}{\mu}-1}}\right). \tag{5.25}$$

For T: $z \mapsto z^{\frac{1}{\mu}}$, if $z_1, z_2 \in B_{|z|/3}(z)$, we have

$$|T(z_1) - T(z_2)| \leq \frac{1}{\mu} \max_{z' \in B_{|z|/3}(z)} \{|z'|^{\frac{1}{\mu}-1}\}|z_1 - z_2|.$$

Let q be the closest point to z on σ_1. By $d_1 \le c_0|z|$ for c_0 small, we have $q \in B_{|z|/3}(z)$ if $|z|$ is small. Therefore,

$$\widetilde{d} \le \mathrm{dist}(\widetilde{z}, \widetilde{q}) \le C|z|^{\frac{1}{\mu}-1}d.$$

With (5.25), we obtain

$$\widetilde{u}(\widetilde{z}) = -\log \widetilde{d} + O(d).$$

Let $\widetilde{l}$ be the tangent line of $\widetilde{\sigma}_1$ at $\widetilde{p}$ and $\widetilde{l}'$ be the line passing the origin and intersecting $\widetilde{\sigma}_1$ at the point $\widetilde{p}$. Then, the slopes of these two straight lines are bounded by $C|z|$ by (5.22) and (5.23). Therefore, the included angle $\widetilde{\theta}$ between $\widetilde{l}$ and $\widetilde{l}'$ is less than $C|z|$, and hence,

$$\mathrm{dist}(\widetilde{z}, \widetilde{l}') = \widetilde{d}\cos\widetilde{\theta} = \widetilde{d}(1 + O(\widetilde{\theta}^2)) = \widetilde{d}(1 + O(|z|^2)).$$

By $c_1|z|^2 \le d$, we obtain

$$\mathrm{dist}(\widetilde{z}, \widetilde{l}') = \widetilde{d}(1 + O(d)).$$

Let $\widetilde{V}'$ be the region above the line $\widetilde{l}'$ and $\widetilde{v}'(z)$ be the solution of (1.1)-(1.2) in $\widetilde{V}'$. Then,

$$\widetilde{v}'(\widetilde{z}) = -\log \widetilde{d} + O(d),$$

and hence

$$\widetilde{u}(\widetilde{z}) = \widetilde{v}'(\widetilde{z}) + O(d). \tag{5.26}$$

Let V' be the image of $\widetilde{V}'$ under the conformal homeomorphism $T^{-1} : \widetilde{z} \mapsto \widetilde{z}^{\mu}$, and l' be the image of $\widetilde{l}' \bigcap \{\widetilde{x} > 0\}$ under T^{-1}. For the solution v' of (1.1)-(1.2) in V', we have

$$v'(z) = \widetilde{v}'(\widetilde{z}) + \log\left(\frac{1}{\mu}|z|^{\frac{1}{\mu}-1}\right).$$

Combining with (5.26), we have

$$u(z) = v'(z) + O(d).$$

By Lemma 5.4, we have

$$|\widetilde{x}' - \widetilde{x}| \le C\widetilde{x}^{\mu}\widetilde{d},$$

and hence

$$\mathrm{dist}(\widetilde{p}, (\widetilde{x}, \widetilde{\varphi}_1(\widetilde{x})) \le C\widetilde{x}^{\mu}\widetilde{d}.$$

Under the conformal homeomorphism $T^{-1} : \widetilde{z} \mapsto \widetilde{z}^{\mu}$, we assume

$$\widetilde{p} \mapsto p, \qquad (\widetilde{x}, \widetilde{\varphi}_1(\widetilde{x})) \mapsto (x, \varphi_1(x)).$$

Then,

$$\begin{aligned}\mathrm{dist}(p, (x, \varphi_1(x))) &\le C(|z|^{\frac{1}{\mu}})^{\mu-1}\mathrm{dist}(\widetilde{p}, (\widetilde{x}, \widetilde{\varphi_1}(\widetilde{x}))) \le C(|z|^{\frac{1}{\mu}})^{\mu-1}\widetilde{x}^{\mu}\widetilde{d} \\ &\le C(|z|^{\frac{1}{\mu}})^{\mu-1}(|z|^{\frac{1}{\mu}})^{\mu}|z|^{\frac{1}{\mu}-1}d \le C|z|d.\end{aligned}$$

Recall that q is the closest point to z on σ_1. By Lemma 5.4, we have

$$\mathrm{dist}(q, (x, \varphi_1(x))) \le C|z|d.$$

By setting $p = (x', \varphi_1(x'))$ and $q = (\overline{x}, \varphi_1(\overline{x}))$, we have

$$|x' - \overline{x}| \le \mathrm{dist}(p, q) \le C|z|d.$$

Denote by l' and $\bar{l}$ the straight lines passing the origin and intersecting σ_1 at p and q, respectively. Then, the difference of their slopes can be estimated by

$$\left|\frac{\varphi_1(x')}{x'} - \frac{\varphi_1(\overline{x})}{\overline{x}}\right| \leq C|z|d,$$

and a similar estimate holds for the angle between l' and $\bar{l}$. With $c_1|z|^2 \leq d$, we have

$$|\operatorname{dist}(z, l') - \operatorname{dist}(z, \bar{l})| \leq |z| \cdot C|z|d = C|z|^2 d \leq Cd^2.$$

Denote by $\widehat{\theta}$ the angle between the line $\bar{l}$ and the tangent line of σ_1 at q. Then,

$$\operatorname{dist}(z, \bar{l}) = d\cos\widehat{\theta} = d(1 + O(|z|^2)) = d(1 + O(d)),$$

and hence

$$\operatorname{dist}(z, l') = d(1 + O(d)).$$

By the explicit expressions of v' in (5.2), it is straightforward to verify

$$v'(z) = -\log\left(\mu r \sin\frac{\arcsin\frac{d}{|z|}}{\mu}\right) + O(d).$$

We hence have (5.11).

Case 2.2. Now we consider the general case that the map $T : z \mapsto z^{\frac{1}{\mu}}$ is not necessary one-to-one on Ω. Take $R > 0$ sufficiently small such that T is one-to-one on $D = \Omega \cap B_R$. Let u_D be the solution of (1.1)-(1.2) in D. Then, the desired estimates for u_D holds in Ω_1 and Ω_2 by Case 1 and Case 2. In the following, we denote the given solution u in Ω by u_Ω. Then, (5.11) holds for u_Ω in Ω_1. We now prove (5.11) holds for u_Ω in Ω_2. Since D and Ω coincide in a neighborhood of the origin, we have, by (5.7),

$$u_\Omega(z) = -\log\left(\mu|z| \sin\frac{\arcsin\frac{d}{|z|}}{\mu}\right) + O(d) + O(|z|^{\frac{1}{\mu}}).$$

We need to estimate $|z|^{\frac{1}{\mu}}$.

If $\frac{1}{\mu} \geq 2$, we have $|z|^{\frac{1}{\mu}} \leq Cd$ and then (5.11) for u_Ω. For $\frac{1}{\mu} < 2$, we adopt notations in the proof of Lemma 5.3. We take $\widetilde{\mu} > \mu$ sufficiently close to μ and set

$$\widetilde{\theta} = \theta + \frac{1}{2}(\widetilde{\mu} - \mu)\pi.$$

By (5.9), we have

$$e^{2u_D} \geq \left(\frac{1}{\widetilde{\mu}|z| \sin\frac{\widetilde{\theta}}{\widetilde{\mu}}}\right)^2 \left(1 + A|z|^{\frac{1}{\widetilde{\mu}}}\right)^{-2} \quad \text{in } \Omega \cap B_\delta,$$

for δ sufficient small. Consider

$$\widehat{\Omega} = \Omega_2 \cup \gamma_2 \cup \Omega_3 = \{z \in \Omega : d_1(z) < c_0|z|\}.$$

For c_0 small, we have

$$e^{2u_D} \geq \frac{2}{|z|^2} \quad \text{in } \widehat{\Omega} \cap B_\delta,$$

if δ is smaller. Then, it is straightforward to verify that $u_D + \log(1 + A|z|^2)$ is a supersolution of (1.1) in $\widehat{\Omega} \cap B_\delta$. By Case 1, we have

$$u_\Omega \le u_D + Cd_1 \quad \text{on } \gamma_1 \cap B_\delta. \tag{5.27}$$

We set, for two constants a and b,

$$\phi(d_1) = ad_1 - bd_1^2.$$

Then,

$$\Delta\phi = -a\kappa - 2b + O(d).$$

We can take positive constants a and b depending only on M and μ such that

$$\phi(d_1) > 0, \quad \Delta\phi(d_1) < 0 \quad \text{in } \widehat{\Omega} \cap B_\delta,$$

and

$$u_\Omega \le u_D + \phi(d_1) \quad \text{on } \gamma_1 \cap B_\delta.$$

By Lemma 2.1 and the maximum principle, we have

$$u_\Omega \le u_D + \log(1 + A|z|^2) + \phi(d_1) \quad \text{in } \widehat{\Omega} \cap B_\delta.$$

Similarly, we obtain

$$u_D \le u_\Omega + \log(1 + A|z|^2) + \phi(d_1) \quad \text{in } \widehat{\Omega} \cap B_\delta,$$

and hence

$$u_\Omega = u_D + \log(1 + A|z|^2) + \phi(d_1) \quad \text{in } \widehat{\Omega} \cap B_\delta.$$

Note $c_1|z|^2 \le d_1$ in Ω_2, we get

$$u_\Omega = u_D + O(d_1) \quad \text{in } \Omega_2 \cap B_\delta, \tag{5.28}$$

and hence (5.11) for u_Ω.

We note that $\mu < 1$ is not used here. What we proved is the following statement: If

$$u_\Omega = u_D + O(d_1) \quad \text{in } \gamma_1 \cap B_\delta,$$

then (5.28) holds in $\Omega_2 \cap B_\delta$.

Case 3. We consider $z \in \Omega_3 \cap B_\delta$. We point out that we will not need the transform T in this case.

Let q be the closest point to z on σ_1 and set $B = B_{\frac{1}{20c_1}}(q + \frac{1}{20c_1}\vec{n})$, where $\vec{n}$ is the unit inward normal vector of σ_1 at q. Denote by Q one of the intersects of ∂B and the curve γ_2, with the larger distance to the origin. Then for $c_1 = c_1(M, \mu)$ large, we have $\operatorname{dist}(O, Q) < 3|z|$. With $d_1 \le c_1|z|^2$, we have

$$\mu|z| \sin \frac{\arcsin \frac{d_1}{|z|}}{\mu} = |z| \left[\frac{d_1}{|z|} + O\left(\left(\frac{d_1}{|z|} \right)^3 \right) \right] = d_1(1 + O(d_1)) \quad \text{in } \Omega_3. \tag{5.29}$$

By what we proved in Case 2, we have

$$u = -\log d_1 + O(d_1) \quad \text{on } \gamma_2.$$

For some positive constants a and b, set

$$\phi(d_1) = ad_1 - bd_1^2.$$

Then,

$$\Delta\phi = -a\kappa - 2b + O(d_1).$$

Let u_B be the solution of (1.1)-(1.2) in B. By taking a and b depending only on M and μ, we have

$$\phi(d_1) > 0, \quad \Delta\phi(d_1) < 0 \quad \text{in } \Omega_3 \cap B_\delta,$$

and

$$u \le u_B + \phi(d_1) \quad \text{on } \gamma_2 \cap B.$$

By the maximum principle, we obtain

$$u \le u_B + \phi(d_1) \quad \text{in } \Omega_3 \cap B.$$

With $u_B = -\log d_1 + O(d_1)$, we have, at the fixed z,

$$u \le -\log d_1 + Cd_1.$$

Since we can always put a ball outside Ω and tangent to $\partial\Omega$ at q due to $\mu < 1$, we get

$$u \ge -\log d_1 - Cd_1.$$

Therefore, we obtain

$$u(z) = -\log d_1 + O(d_1),$$

and hence (5.11) by (5.29).

By combining Cases 1-3, we finish the proof of (5.11). □

Now, we discuss the case when the opening angle of the tangent cone of Ω at the origin is larger than π. We first introduce the limit function. Let $\partial\Omega$ in a neighborhood of the origin consist of two C^2-curves σ_1 and σ_2 intersecting at the origin at an angle $\mu\pi$, for some constant $\mu \in (1, 2)$. Define, for any $z \in \Omega$,

$$f_\mu(z) = \begin{cases} -\log(\mu|z|\sin\dfrac{\arcsin\frac{d_1(z)}{|z|}}{\mu}) & \text{if } d_1(z) < d_2(z), \\ -\log(\mu|z|\sin\frac{\theta}{\mu}) & \text{if } d_1(z) = d_2(z), \\ -\log(\mu|z|\sin\dfrac{\arcsin\frac{d_2(z)}{|z|}}{\mu}) & \text{if } d_1(z) > d_2(z), \end{cases} \tag{5.30}$$

where d, d_1 and d_2 are the distances to $\partial\Omega, \sigma_1$ and σ_2, respectively, θ is the angle anticlockwise from the tangent line of σ_1 at the origin to $\overrightarrow{Oz}$. We note that $\{z \in \Omega : \ d_1(z) = d_2(z)\}$ has a nonempty interior for $\mu \in (1, 2)$ and that f_μ is well-defined for z sufficiently small. It is straightforward to verify that $\partial\{z \in \Omega : d_1(z) < (\text{or} >) d_2(z)\} \cap \Omega$ near the origin is a line segment perpendicular to the tangent line of σ_1 (or σ_2) at the origin. Hence, f_μ is continuous in $\Omega \cap B_\delta$ for δ sufficiently small.

Theorem 5.6. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ in a neighborhood of the origin consist of two C^2-curves σ_1 and σ_2 intersecting at the origin at an angle $\mu\pi$, for some constant $\mu \in (1, 2)$. Suppose $u \in C^\infty(\Omega)$ is a solution of (1.1)-(1.2). Then, for any $z \in \Omega \cap B_\delta$,*

$$|u(z) - f_\mu(z)| \le Cd(z),$$

where f_μ is the function defined by (5.30), and δ and C are positive constants depending only on the geometry of $\partial\Omega$.

Proof. We proceed similarly as in the proof of Theorem 5.5 and adopt the same notations. We denote by M the C^2-norms of σ_1 and σ_2, and define Ω_1, Ω_2, Ω_3 and γ_1, γ_2 by (5.14) and (5.15), respectively. Consider T: $z \mapsto z^{\frac{1}{\mu}}$.

We fix a point $z \in \Omega \cap B_\delta$ for some δ sufficiently small. Without loss of generality, we assume $d_1 = d_1(z) = d(z) \leq d_2 = d_2(z)$.

Case 1. We consider $z \in \Omega_1 \cap B_\delta$, where c_0 is some small constant such that $c_0 < \frac{1}{2}\arctan\frac{1}{4}$.

Set $\Omega_+ = \Omega \cap B_\delta$ and let u_+ be the solution of (1.1)-(1.2) in Ω_+. We take δ small so that T is one-to-one on Ω_+. Set $\widetilde{\Omega}_+ = T(\Omega_+)$ and let $\widetilde{u}_+$ be the solution of (1.1)-(1.2) in $\widetilde{\Omega}_+$. By (5.13), the curve $\widetilde{\sigma}$ given by $\widetilde{y} = \widetilde{\varphi}(\widetilde{x})$ satisfies

$$-\widetilde{M}|\widetilde{x}|^{1+\mu} \leq |\widetilde{\varphi}(\widetilde{x})| \leq \widetilde{M}|\widetilde{x}|^{1+\mu}.$$

We note here $1 + \mu > 2$. Theorem 4.1 implies, for $\widetilde{z}$ close to the origin,

$$\widetilde{u}_+(\widetilde{z}) \leq -\log \widetilde{d}_1 + \frac{1}{2}\kappa_1 \widetilde{d}_1 + O(\widetilde{d}_1^\mu), \tag{5.31}$$

and

$$\widetilde{u}_+(\widetilde{z}) \geq -\log \widetilde{d}_2 + \frac{1}{2}\kappa_2 \widetilde{d}_1 + O(\widetilde{d}_2^\mu), \tag{5.32}$$

where $\widetilde{d}_1$ and $\widetilde{d}_2$ are the distances from $\widetilde{z}$ to the curves $\widetilde{y} = \widetilde{M}|\widetilde{x}|^{1+\mu}$ and $\widetilde{y} = -\widetilde{M}|\widetilde{x}|^{1+\mu}$, respectively, and κ_1 and κ_2 are the curvatures of the curves $\widetilde{y} = \widetilde{M}|\widetilde{x}|^{1+\mu}$ and $\widetilde{y} = -\widetilde{M}|\widetilde{x}|^{1+\mu}$, respectively. Recall, for $c_0|z| < d$,

$$\log \widetilde{d}_i = \log \widetilde{y} + O(d),$$

and

$$\widetilde{d}_i^\mu \leq Cd.$$

Moreover,

$$|\kappa_i| \leq C|\widetilde{z}|^{\mu-1} = C|z|^{\frac{\mu-1}{\mu}} \leq Cd^{\frac{\mu-1}{\mu}}.$$

Therefore, (5.31) and (5.32) imply

$$\widetilde{u}_+(\widetilde{z}) = -\log \widetilde{y} + O(d).$$

This is (5.19). The rest of the proof for Case 1 is identical to that in the proof of Theorem 5.5.

Case 2. We consider $z \in \Omega_2 \cap B_\delta$.

Arguing similarly as in the proof of Theorem 5.5, we have

$$|\widetilde{\varphi}(\widehat{x}) - \widetilde{\varphi}(\widetilde{x}') - \widetilde{\varphi}'(\widetilde{x}')(\widehat{x} - \widetilde{x}')| \leq K(|z|^{1-\frac{1}{\mu}} + |\widehat{x} - \widetilde{x}'|^{\mu-1})(\widehat{x} - \widetilde{x}')^2. \tag{5.33}$$

This plays a similar role as (5.24). Then, we have

$$\widetilde{u}(\widetilde{z}) \leq -\log \widehat{d}_1 + \frac{1}{2}\kappa_1 \widehat{d}_1 + O(\widehat{d}_1^\mu),$$

and

$$\widetilde{u}(\widetilde{z}) \geq -\log \widehat{d}_2 + \frac{1}{2}\kappa_2 \widehat{d}_2 + O(\widehat{d}_2^\mu),$$

where $\widehat{d}_1$ is the distance from $\widetilde{z}$ to the curve

$$\widehat{y} = \widetilde{\varphi}(\widetilde{x}') + \widetilde{\varphi}'(\widetilde{x}')(\widehat{x} - \widetilde{x}') + K(|z|^{1-\frac{1}{\mu}} + |\widehat{x} - \widetilde{x}'|^{\mu-1})(\widehat{x} - \widetilde{x}')^2,$$

and $\widehat{d}_2$ is the distance from $\widetilde{z}$ to the curve

$$\widehat{y} = \widetilde{\varphi}(\widetilde{x}') + \widetilde{\varphi}'(\widetilde{x}')(\widehat{x} - \widetilde{x}') - K(|z|^{1-\frac{1}{\mu}} + |\widehat{x} - \widetilde{x}'|^{\mu-1})(\widehat{x} - \widetilde{x}')^2.$$

Then, we proceed similarly as in Case 2 in the proof of Theorem 5.5.

Case 3. We consider $z \in \Omega_3 \cap B_\delta$.

We take $q \in \sigma_1$ with the least distance to z, and denote by l the tangent line of σ_1 at p. We put q at the origin of the line l. A portion of σ_1 near q, including the part from the origin to q, can be expressed as a C^2-function φ in $(-s_0, s_0)$, with $\varphi(-s_0)$ corresponding to the origin in $\mathbb{R}^2$ and $\varphi(0)$ corresponding to q, i.e., $\varphi(0) = 0$. Then,

$$|\varphi(s)| \leq \frac{1}{2}M|s|^2 \quad \text{for any } s \in (-s_0, s_0). \tag{5.34}$$

In the present case, M is uniform, independent of z; however, s_0 depends on z. We should first estimate s_0 in terms of d_2. We note, for d_2 sufficiently small,

$$\frac{1}{2}|z| \sin\frac{(2-\mu)\pi}{2} \leq d_2 \leq |z|. \tag{5.35}$$

By the triangle inequality and (5.34), we have

$$s_0 \leq \frac{1}{2}Ms_0^2 + |z| + d_1,$$

and

$$s_0 \geq -\frac{1}{2}Ms_0^2 + |z| - d_1.$$

Then, $s_0/|z| \to 1$ as $|z| \to 0$. We take $|z|$ sufficiently small such that $s_0 \geq 2|z|/3$.

By taking $|z|$ sufficiently small, (5.35) implies

$$B_{r_1}(q - r_1\vec{n}) \cap \Omega = \emptyset,$$

where $\vec{n}$ is the unit inward normal vector of σ_1 at q and

$$r_1 = \frac{1}{2}|z| \sin\frac{(2-\mu)\pi}{8}.$$

By the maximum principle, we have

$$u(z) \geq v_{r_1, q-r_1\vec{n}} \quad \text{in } \Omega.$$

Hence,

$$u \geq -\log d - C|z| \quad \text{in } \Omega_3 \cap B_\delta. \tag{5.36}$$

By taking $R = R(M, \mu)$ small, we have

$$\text{dist}(z', \sigma_1) \leq \frac{1}{2}\text{dist}(z', \partial B_R(q - R\vec{n})).$$

By what we proved in Case 2, we get

$$u(z) = -\log d_1 + O(d_1) \quad \text{on } \gamma_2 \cap B_\delta.$$

Combining with (5.36), we have, for $|z|$ sufficient small,

$$u(z) \geq v_{R,q-R\vec{n}_L} \quad \text{in } \Omega_3 \cap \partial B_{3|z|}(z).$$

Set

$$\phi(d_1) = ad_1 - bd_1^2.$$

We can take two positive constants a and b depending only the geometry of Ω such that

$$\phi(d_1) > 0, \quad \Delta\phi(d_1) < 0 \quad \text{in } \Omega_3 \cap \partial B_\delta,$$

and

$$v_{R,q-R\vec{n}} \leq u + \phi(d_1) \quad \text{on } \gamma_2 \cap B_\delta.$$

By the maximum principle, we obtain

$$v_{R,q-R\vec{n}} \leq u + \phi(d_1) \quad \text{in } \Omega_3 \cap B_{3|z|}(z).$$

By

$$v_{R,q-R\vec{n}}(z) = -\log d_1 + O(d_1),$$

we have

$$u(z) \geq -\log d_1 - Cd_1.$$

Since we can always put a ball inside Ω and tangent to $\partial\Omega$ at q due to $\mu > 1$, we get

$$u(z) \leq -\log d_1 + Cd_1.$$

Therefore,

$$u(z) = -\log d_1 + O(d_1),$$

and hence

$$u(z) = -\log\left(\mu r \sin\frac{\arcsin\frac{d_1}{r}}{\mu}\right) + O(d_1).$$

This is the desired estimate. □

Remark 5.7. We point out the estimates in Theorem 5.5 and Theorem 5.6 are local; namely, they hold in Ω near the origin, independent of Ω away from the origin.

Remark 5.8. With a slightly more complicated argument, we can prove the following estimate: if σ_1 and σ_2 are $C^{1,\alpha}$-curves, for some $\alpha \in (0,1)$, then for any $z \in \Omega \cap B_\delta$,

$$|u(z) - f_\mu(z)| \leq Cd^\alpha(z),$$

where f_μ is given by (5.12) for $\mu \in (0,1]$ and by (5.30) for $\mu \in (1,2)$, and δ and C are positive constants depending only on the geometry of $\partial\Omega$. This estimate can be viewed as a generalization of Theorem 3.1.

6. Application to Kähler-Einstein metrics

Cheng and Yau [4] studied the following problem:

$$\det u_{i\bar{j}} = e^{(n+1)u} \quad \text{in } \Omega, \tag{6.1}$$

$$u = \infty \quad \text{on } \partial\Omega, \tag{6.2}$$

where $\Omega \subset \mathbb{C}^n$ is a smooth bounded strictly pseudoconvex domain, $n \geq 2$. They proved that (6.1)-(6.2) admits a smooth strictly plurisubharmonic solution. Geometrically, if u is a strictly plurisubharmonic solution to (6.1)-(6.2), then

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial z_{\bar{j}}} dz_i dz_{\bar{j}}$$

is a complete Kähler-Einstein metric on Ω. Lee and Melrose [14] discussed boundary expansions of u in smooth bounded strictly pseudoconvex domain.

In this section, we discuss the asymptotic behavior in singular product domains. We note that (6.1)-(6.2) reduces to (1.1)-(1.2) upon a rescaling, for $n = 1$.

Theorem 6.1. *Assume that $\Omega \subset \mathbb{C}^n$ has the form*

$$\Omega = \Omega_1^1 \times \ldots \times \Omega_k^1 \times \Omega_{k+1}^{n-k},$$

where $\Omega_i^1 \subset \mathbb{C}^1$ is a bounded Lipschitz domain bounded by finite many C^2-curves, $i = 1, \cdots, k$, for some $1 \leq k \leq n$, and $\Omega_{k+1}^{n-k} \subset \mathbb{C}^{n-k}$ is a smooth bounded strictly pseudoconvex domain. Then, (6.1)-(6.2) *admits a unique smooth strictly plurisubharmonic solution u in the form*

$$u(z_1, .., z_n) = u_1(z_1) + \cdots + u_k(z_k) + u_{k+1}(z_{k+1}, ..., z_n), \tag{6.3}$$

where u_i, $i = 1, ..., k$, is the unique solution of

$$\Delta u_i = 4e^{(n+1)u_i} \quad \text{in } \Omega_i^1, \tag{6.4}$$

$$u_i = \infty \quad \text{on } \partial\Omega_i^1, \tag{6.5}$$

and u_{k+1} is the unique strictly plurisubharmonic solution of

$$\det u_{k+1, i\bar{j}} = e^{(n+1)u_{k+1}} \quad \text{in } \Omega_{k+1}^{n-k}, \tag{6.6}$$

$$u_{k+1} = \infty \quad \text{on } \partial\Omega_{k+1}^{n-k}. \tag{6.7}$$

In (6.4), Δ is the standard Laplacian in $\mathbb{R}^2$.

Proof. Without loss of generality, we assume Ω contains the origin. Set, for $i = 1, \cdots, k$,

$$v_i(x) = \frac{n+1}{2} u_i\left(\sqrt{\frac{8}{n+1}} x\right).$$

Then, v_i satisfies (1.1)-(1.2) for $\Omega = \sqrt{\frac{n+1}{8}}\Omega_i^1$. By Lemma 2.1, we have,

$$\left| u_i(z_i) - \frac{2}{n+1} \log d_i(z_i) \right| \leq C \quad \text{in } \Omega_i^1,$$

where C is a positive constant depending only on the geometry of Ω_i^1, and $d_i(z_i)$ is the distance from z_i to $\partial\Omega_i^1$. Set

$$v_{k+1}(z_{k+1},\cdots,z_n)=\frac{n+1}{n-k+1}u_{k+1}\left(\left(\frac{n-k+1}{n+1}\right)^{\frac{1}{2(n-k)}}(z_{k+1},\cdots,z_n)\right),$$

and

$$\Omega^{n-k}=\left(\frac{n+1}{n-k+1}\right)^{\frac{1}{2(n-k)}}\Omega_{k+1}^{n-k}.$$

Then, v_{k+1} satisfies

$$\det v_{k+1,i\bar{j}}=e^{(n-k+1)v_{k+1}}\quad\text{in }\Omega^{n-k},\tag{6.8}$$

$$v_{k+1}=\infty\quad\text{on }\partial\Omega^{n-k}.\tag{6.9}$$

We can get the asymptotic behavior of v_{k+1} by applying the result in [14] and scaling back. Hence, $u(z_1,..,z_n)$ given by (6.3) satisfies

$$\begin{aligned}\Big|u(z)&+\frac{2}{n+1}(\log d_1(z_1)+\cdots+\log d_k(z_k))\\&+\frac{n-k+1}{n+1}\log d_{k+1}(z_{k+1},...,z_n)\Big|\le C,\end{aligned}\tag{6.10}$$

where $d_1,\cdots,d_k$ and d_{k+1} are distances to $\partial\Omega_1^1,\cdots,\partial\Omega_k^1$ and $\partial\Omega_{k+1}^{n-k}$, respectively. In the following, we set

$$c_1=\frac{2}{n+1},\quad c_2=\frac{n-k+1}{n+1}.$$

We now prove that u given by (6.3) is the only solution of (6.1)-(6.2). Let w be an arbitrary strictly plurisubharmonic solution of (6.1)-(6.2). Then,

$$u\left(\frac{z}{\varepsilon}\right)+\frac{2n}{n+1}\log\frac{1}{\varepsilon}$$

is a solution in $\varepsilon\Omega:=\{z:\frac{z}{\varepsilon}\in\Omega\}$, for $\varepsilon>0$. Hence, we may assume $\Omega\subset B_r\times\cdots\times B_r$. Since the solution u_r of (1.1)-(1.2) in $B_r\times\cdots\times B_r$ satisfies

$$u_r\ge\frac{2n}{n+1}\left(-\log r+\frac{1}{2}\log\frac{8}{n+1}\right),$$

we have, by the maximum principle,

$$u\ge\frac{2n}{n+1}\left(-\log r+\frac{1}{2}\log\frac{8}{n+1}\right).$$

Therefore, we can assume u is large enough since we can take r sufficiently small. This also holds for w.

Set

$$f(z)=\frac{w(z)}{u(z)}.$$

Then, it is easy to see

$$f(z)\ge 1\quad\text{in }\Omega.$$

We now claim

$$f(z) \to 1 \quad \text{as } z \to \partial\Omega. \tag{6.11}$$

To this end, we approximate $\Omega_1^1, \cdots, \Omega_k^1$ and Ω_{k+1}^{n-k} appropriately from their interiors by $\Omega_{1,m}^1, \cdots, \Omega_{k,m}^1$ and $\Omega_{k+1,m}^{n-k}$. Set

$$\Omega_m = \Omega_{1,m}^1 \times \cdots \times \Omega_{k,m}^1 \times \Omega_{k+1,m}^{n-k}.$$

Assume u_i^m, $i = 1, \cdots, k$, is the unique solution of

$$\begin{aligned} \Delta u_i^m &= 4e^{(n+1)u_i^m} \quad \text{in } \Omega_{i,m}^1, \\ u_i^m &= \infty \quad \text{on } \partial\Omega_{i,m}^1, \end{aligned}$$

and u_{k+1}^m is the unique strictly plurisubharmonic solution of

$$\begin{aligned} \det u_{k+1,\, i\bar{j}}^m &= e^{(n+1)u_{k+1}^m} \quad \text{in } \Omega_{k+1,m}^{n-k}, \\ u_{k+1}^m &= \infty \quad \text{on } \partial\Omega_{k+1,m}^{n-k}. \end{aligned}$$

Set

$$u_m(z_1, .., z_n) = u_{1,m}(z_1) + \cdots + u_{k,m}(z_k) + u_{k+1,m}(z_{k+1}, ..., z_n).$$

Fix a point $z = (z_1, .., z_n) \in \Omega$. Then for m large, we have $z \in \Omega_m$. By the maximum principle, we have $w(z) \le u_m(z)$, and hence by (6.10),

$$w(z) \le -c_1 \left(\log d_{1,m}(z_1) + \cdots + \log d_{k,m}(z_k)\right) - c_2 \log d_{k+1,m}(z_{k+1}, ..., z_n) + C_m,$$

where $d_{i,m}$ is the distance to $\partial\Omega_{i,m}^1$, $i = 1, ..., k$, $d_{k+1,m}$ is the distance to $\partial\Omega_{k+1,m}^{n-k}$, and C_m is a positive constant. By the geometry of Ω_i^1 and Ω_{k+1}^{n-k}, we can choose C_m independent of m. Letting $m \to \infty$, we have

$$w(z) \le -c_1 \left(\log d_1(z_1) + \cdots + \log d_k(z_k)\right) - c_2 \log d_{k+1}(z_{k+1}, ..., z_n) + C, \tag{6.12}$$

where d_i is the distance to $\partial\Omega_i^1$, $i = 1, ..., k$, d_{k+1} is the distance to $\partial\Omega_{k+1}^{n-k}$ and C is a positive constant depending only on the geometry of Ω. On the other hand, we have

$$w(z) \ge u(z) > -c_1 \left(\log d_1(z_1) + \cdots + \log d_k(z_k)\right) - c_2 \log d_{k+1}(z_{k+1}, ..., z_n) - C, \tag{6.13}$$

for some constant C depending only on the geometry of Ω. By combining (6.12) and (6.13), we obtain (6.11).

If f is not equal to 1 identically, f must assume its maximum $f(z_0) > 1$ at some $z_0 \in \Omega$. It is easy to check $\det w_{i\bar{j}} \le f^n \det u_{i\bar{j}}$ at z_0, and hence,

$$e^{(n+1)uf} \le f^n e^{(n+1)u} \quad \text{at } z_0.$$

Next, we set $h(s) = a^s - as^n$, for some constant a. Then, $h(1) = 0$ and $h(s) > 0$ for any $s > 1$ if a is large. This leads to a contradiction. Therefore, $f = 1$ and then $u = w$ in Ω. □

For the solution u in (6.3), we can apply Theorem 5.5 and Theorem 5.6 to get, near the singular point of $\partial\Omega_i^1$,

$$\left| u_i(z_i) - \frac{2}{n+1}\left(f_\mu(z_i) + \frac{1}{2}\log\frac{8}{n+1} \right) \right| \le C d_{\Omega_i^1}(z_i),$$

where f_μ is given by (5.12) for $\mu \in (0,1]$ and by (5.30) for $\mu \in (1,2)$, and $d_{\Omega_i^1}(z_i)$ is the distance from z_i to $\partial\Omega_i^1$. By applying the result in [14], we get an expansion for u_{k+1}. By putting these estimates together, we get an expansion for u.

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