FACTORIZATION OF FRACTIONAL-ORDER PSEUDODIFFERENTIAL OPERATORS, INTEGRATION BY PARTS, AND A POHOZAEV IDENTITY

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Abstract. Consider a classical elliptic pseudodifferential operator P on $\mathbb{R}^n$ of order $2a$ ($0<a<1$) with even symbol; the fractional Laplacian $(-\Delta)^a$ is a typical example. For solutions of the Dirichlet problem on a bounded smooth subset $\Omega\subset\mathbb{R}^n$, we show an integration-by-parts formula with a boundary term involving $\gamma_0(d^{-a}u)$, where $d(x)=\operatorname{dist}(x,\partial\Omega)$. This generalizes recent results of Ros-Oton, Serra and Valdinoci, to operators that are x-dependent, nonsymmetric and have lower-order parts. We also generalize their formula of Pohozaev-type, that can be used to prove unique continuation properties, and nonexistence of nontrivial solutions of semilinear problems. Applications are given with $P=(-\Delta+m^2)^a$. The basic step in our analysis is a factorization of the symbol of P, where we set up a calculus for the generalized pseudodifferential operators that come out of the construction.

1. Introduction.

A prominent example of a fractional-order pseudodifferential operator (ψdo) is the fractional Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, $0<a<1$;

$$(-\Delta)^a u=\operatorname{Op}(|\xi|^{2a})u=\mathcal{F}^{-1}(|\xi|^{2a}\hat u(\xi)),\quad \hat u(\xi)=\mathcal{F}u=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}u(x)\,dx. \tag{1.1}$$

It is currently of great interest in probability, finance, mathematical physics and differential geometry. It can also be described as a singular integral operator

$$(-\Delta)^a u(x)=c_{n,a}PV\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2a}}\,dy=c_{n,a}PV\int_{\mathbb{R}^n}\frac{u(x)-u(x+y)}{|y|^{n+2a}}\,dy, \tag{1.2}$$

with convolution kernel $c_{n,a}|y|^{-n-2a}=\mathcal{F}^{-1}|\xi|^{2a}$.

Both descriptions allow generalizations. In (1.1), one can replace the symbol $|\xi|^{2a}$ by a more general nonvanishing function $p_0(x,\xi)\in C^\infty(\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\}))$ that is homogeneous in ξ of degree $2a$, and add terms of lower order, to get a classical ψdo symbol $p(x,\xi)$; the operator is then no longer translation-invariant nor symmetric. Such operators and their

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boundary value problems on C^∞-subsets Ω of $\mathbb{R}^n$ have been studied recently in Grubb [G14,G15], obtaining regularity estimates of solutions divided by d^a ($d(x)=\operatorname{dist}(x,\partial\Omega)$) in full scales of function spaces.

In (1.2) one can replace the function $c_{n,a}|y|^{-n-2a}$ by other positive functions $K(y)$ that are homogeneous in y of degree $-n-2a$ and possibly less smooth. Such cases and further generalizations have recently been studied in probability and nonlinear analysis, see e.g. Caffarelli and Silvestre [CS09], Ros-Oton and Serra [RS14a,RS15b], and their references. Boundary integral formulas on subsets Ω are obtained in [RS14b,RSV15] (partly jointly with Valdinoci); these formulas rely on regularity estimates of solutions divided by d^a, in Hölder spaces C^t with t close to a, obtained in [RS14a,RS15b] under limited smoothness of Ω, e.g. $C^{1,1}$.

In the generalizations of (1.1) and (1.2), the fact that $|\xi|^{2a}$ is *even* (takes the same value at ξ and $-\xi$) is kept as a hypothesis, that p_0 is even in ξ, resp. that K is even in y.

The methods used in the pseudodifferential theory are complex, and differ radically from the real methods currently used for the singular integral formulations.

There are numerous other studies of boundary problems for these operators; let us mention e.g. [BG59,L72,HJ96,K97,CS98,J02,S07,MN14,SV14,BBS15,FG15,FKV15,BSV15].

In the present paper we shall develop the pseudodifferential treatment of the operators on domains still further. It was shown in [G15] how the a-transmission property together with the existence of a factorization of the principal symbol led to solvability and regularity properties of boundary value problems; here an important role was played by a reduction to a case where results from the 0-transmission theory (initiated by Boutet de Monvel [B71] and further developed e.g. in [G90,G96]) could be applied.

We now carry the factorization idea further, showing how it gives rise to generalized pseudodifferential operators with symbols in spaces that were used for Poisson and trace operators in the calculus of [B71,G96]. Here we show that the full operator P can be factorized as a product P^-P^+, modulo certain smoothing operators.

As an application of this analysis of products, we show a generalization to x-dependent classical elliptic even ψdo's of order $2a$ ($0<a<1$), of the integration-by-parts formulas shown in [RS14b] for $(-\Delta)^a$ and generalized in [RSV15] to a class of translation-invariant singular integral operators P.

Our main results are: When P is a classical elliptic x-dependent ψdo of order $2a$ on $\mathbb{R}^n$ with even symbol, and Ω is a smooth bounded subset of $\mathbb{R}^n$, then the solutions u of the Dirichlet problem

$$r^+Pu=f \text{ on } \Omega,\quad \operatorname{supp} u\subset\overline{\Omega}, \tag{1.3}$$

satisfy

$$\begin{aligned}\int_\Omega (Pu\,\partial_j\bar u' + \partial_j u\,\overline{P^*u'})\,dx &= \Gamma(a+1)^2\int_{\partial\Omega}\nu_j s_0\gamma_0(d^{-a}u)\,\gamma_0(d^{-a}\bar u')\,d\sigma\\ &\quad + \int_\Omega [P,\partial_j]u\,\bar u'\,dx,\quad j=1,\dots,n,\end{aligned} \tag{1.4}$$

$$\begin{aligned}\int_\Omega (Pu\,(x\cdot\nabla\bar u') + (x\cdot\nabla u)\,\overline{P^*u'})\,dx &= \Gamma(a+1)^2\int_{\partial\Omega}(x\cdot\nu)s_0\gamma_0(d^{-a}u)\,\gamma_0(d^{-a}\bar u')\,d\sigma\\ &\quad -n\int_\Omega Pu\,\bar u'\,dx + \int_\Omega \operatorname{Op}(\xi\cdot\nabla_\xi p)u\,\bar u'\,dx - \int_\Omega \operatorname{Op}(x\cdot\nabla_x p)u\,\bar u'\,dx.\end{aligned} \tag{1.5}$$

Here $\nu=(\nu_1,\dots,\nu_n)$ is the interior normal vector field to $\partial\Omega$, and $s_0(x)$ is the principal symbol of P at $(x,\nu(x))$.

The formulas hold when u and u' are solutions of problems (1.3) with $f\in \overline{H}^{1-a}(\Omega)$, or $f\in C^{1-a+\varepsilon}(\overline{\Omega})$ for some $\varepsilon>0$. They also hold when $f\in \overline{H}^{\frac12-a+\varepsilon}(\Omega)$, with some of the integrals replaced by Sobolev space dualities.

The formulas shown in [RS14b,RSV15] that (1.4)–(1.5) extend, are concerned with real solutions of (1.3) with $f\in C^{0,1}(\overline{\Omega})$; here Ω is a bounded $C^{1,1}$-domain, and P is translation-invariant with even nonnegative homogeneous kernel, with possibly less smoothness than in our C^∞-case. In comparison, our method allows nonselfadjointness, x-dependence, and nonhomogeneity (lower-order terms).

In the formulas (1.4) and (1.5), the terms with $[P,\partial_j]$ resp. $\operatorname{Op}(x\cdot\nabla_x p)$ represent the effect of the x-dependence of P.

The version of (1.5) shown in [RS14b,RSV15] is important in the study of existence questions for nonlinear problems where f is replaced by $f(u)$ in (1.3), since it leads to a Pohozaev-type identity for the possible solutions. We have the following extended Pohozaev identity, for translation-invariant selfadjoint operators P with lower-order terms:

$$
\begin{aligned}
-2n\int_\Omega F(u)\,dx+n\int_\Omega u\,f(u)\,dx&=\Gamma(1+a)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\gamma_0(d^{-a}u)^2\,d\sigma,\\
&\quad+\int_\Omega \operatorname{Op}(\xi\cdot\nabla_\xi p(\xi))u\,u\,dx,
\end{aligned}
\tag{1.6}
$$

for real solutions u; here $F(t)=\int_0^t f(s)\,ds$. For example, the formula applies to $P=(-\Delta+m^2)^a$, $m>0$, where we show a unique continuation principle, and nonexistence of bounded nontrivial solutions to (1.3) with f replaced by $f(u)=\operatorname{sign}u\,|u|^r$ when $r\ge\frac{n+2a}{n-2a}$.

Plan of the paper: The Appendix contains the notation for function spaces, and collects some facts on pseudodifferential operators that are known from the general theory and from preceding works such as [G15,G14]. Section 2 shows the factorization of symbols having the a-transmission property, and describes the symbol spaces and mapping properties of the generalized ψdo's that arise from the construction. In Section 3 we establish the formula (1.4) in the case where Ω is replaced by $\mathbb{R}^n_+$, for $j=n$. Finally in Section 4, we treat the problem for arbitrary smooth domains Ω, showing the formulas (1.4)–(1.6) in general and drawing some consequences.

2. Factorization of homogeneous symbols

2.1 Some notation.

The function $\langle\xi\rangle$ stands for $(1+|\xi|^2)^{\frac12}$, and we denote by $[\xi]$ a positive C^∞-function equal to $|\xi|$ for $|\xi|\ge1$ and $\ge\frac12$ for all ξ. Multi-index notation is used for differentiation (and polynomials): $\partial=(\partial_1,\dots,\partial_n)$, and $\partial^\alpha=\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$ for $\alpha\in\mathbb{N}_0^n$, with $|\alpha|=\alpha_1+\dots+\alpha_n$, $\alpha!=\alpha_1!\dots\alpha_n!$. $D=(D_1,\dots,D_n)$ with $D_j=-i\partial_j$.

Operators are considered acting on functions or distributions on $\mathbb{R}^n$, and on subsets $\mathbb{R}^n_\pm=\{x\in\mathbb{R}^n\mid x_n\gtrless0\}$ (where $(x_1,\dots,x_{n-1})=x'$), and bounded C^∞-subsets Ω with boundary $\partial\Omega$, and their complements.

Restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$ (or from $\mathbb{R}^n$ to Ω resp. $\complement\overline{\Omega}$) is denoted $r^\pm$, extension by zero from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$ (or from Ω resp. $\complement\overline{\Omega}$ to $\mathbb{R}^n$) is denoted $e^\pm$. Restriction from $\overline{\mathbb{R}}^n_+$ or $\overline{\Omega}$ to $\partial\mathbb{R}^n_+$ resp. $\partial\Omega$ is denoted γ_0.

The reader is encouraged to consult the Appendix for further notation, as it becomes relevant.

2.2 The factorization question.

Let there be given a function $p(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, homogeneous of degree $2a$ with $0 < a < 1$, even and elliptic, i.e.,

$$p(-\xi) = p(\xi) \text{ and } p(\xi) \neq 0 \text{ for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.1}$$

Consider the points in $\mathbb{R}^n$ as $\xi = (\xi', \xi_n)$, where $\xi' \in \mathbb{R}^{n-1}$, $\xi_n \in \mathbb{R}$. According to Vishik and Eskin, see Eskin [E81] Ch. 6, p can be written as a product of two factors $p_+(\xi', \xi_n)$ and $p_-(\xi', \xi_n)$ that extend analytically in ξ_n to $\mathbb{C}_-$ resp. $\mathbb{C}_+$; here $\mathbb{C}_\pm = \{\xi_n \in \mathbb{C} \mid \operatorname{Im} \xi_n \gtrless 0\}$.

Since the sign convention for the Fourier transform in [E81] is the opposite of the standard choice in Western literature, with consequences for other $\pm$-conventions, it is hard to avoid confusion when quoting the book directly. Therefore we shall show a detailed version of the factorization, where we moreover relate it to the symbol estimates and points of view that play a role in [H65], [B71] and later works such as [G96], [G09], [G15].

Assume for simplicity that $p(0, 1) = 1$ (this just amounts to a normalization).

We define (for $\xi \neq 0$)

$$q(\xi) = p(\xi)|\xi|^{-2a}, \quad \psi(\xi) = \log q(\xi), \tag{2.2}$$

they are both homogeneous of degree 0 and even. Actually, it suffices for the following considerations that q is "even in the ξ_n-direction", more precisely, has the 0-transmission property with respect to the surface $\{x_n = 0\}$:

$$\partial_\xi^\alpha q(0, -\xi_n) = (-1)^{|\alpha|} \partial_\xi^\alpha q(0, \xi_n), \text{ all } \alpha \in \mathbb{N}_0^n, \tag{2.3}$$

which clearly holds for even symbols of order 0.

When $\xi' \neq 0$,

$$\lim_{\xi_n \to \pm\infty} q(\xi', \xi_n) = \lim_{\xi_n \to \pm\infty} q(\xi'/\xi_n, 1) = 1, \quad \lim_{\xi_n \to \pm\infty} \psi(\xi', \xi_n) = 0. \tag{2.4}$$

To factorize q we shall decompose ψ into a sum of two terms that extend holomorphically into $\mathbb{C}_\pm$, respectively. This can be formulated in terms of Cauchy integral formulas.

Let us recall some facts about Cauchy integral decompositions. When $f(t)$ is $O(\langle t \rangle^{-1})$ on $\mathbb{R}$ with a continuous derivative $f'(t)$ that is $O(\langle t \rangle^{-2})$ on $\mathbb{R}$, one can define

$$\begin{aligned} f_+(t) &= \frac{i}{2\pi} \int_{\mathbb{R}} \frac{f(\sigma)}{\sigma - t} \,đ\sigma \text{ for } \operatorname{Im} t < 0, \\ f_-(t) &= \frac{-i}{2\pi} \int_{\mathbb{R}} \frac{f(\sigma)}{\sigma - t} \,đ\sigma \text{ for } \operatorname{Im} t > 0; \end{aligned} \tag{2.5}$$

they are holomorphic for $t \in \mathbb{C}_-$ resp. $\mathbb{C}_+$, and extend by continuity to $\overline{\mathbb{C}}_-$ resp. $\overline{\mathbb{C}}_+$ The values on $\mathbb{R}$ (the limits for $\operatorname{Im} t \to 0$ from $\mathbb{C}_=$ resp. $\mathbb{C}_+$) satisfy

$$f_+(t) + f_-(t) = f(t). \tag{2.6}$$

Moreover, for the functions on $\mathbb{R}$, the inverse Fourier transforms satisfy

$$\mathcal{F}^{-1}f_+ = e^+r^+\mathcal{F}^{-1}f, \quad \mathcal{F}^{-1}f_- = e^-r^-\mathcal{F}^{-1}f. \tag{2.7}$$

(they are in $L_2(\mathbb{R})$); here $r^\pm$ denotes restriction from functions on $\mathbb{R}$ to functions on $\mathbb{R}_\pm$, and $e^\pm$ denotes extension of functions on $\mathbb{R}_\pm$ to functions on $\mathbb{R}$ by zero on $\mathbb{R}_\mp$. These facts are well-known; proofs can be found e.g. in [E81] Lemma 6.1, Th. 5.1. ([H65] refers for the decomposition to Beurling's contribution to the Helsingfors congress 1938.)

As in [B71,G96,G09] we shall denote the mappings by $h^\pm\colon f \to f_\pm$; note that they are complementing projections, satisfying $h^+ + h^- = I$. (The mappings $h^\pm$ correspond to the mappings $\Pi^\pm$ in [E81], except that the holomorphy regions are exchanged because of a different convention for the Fourier transform.) The mappings are applied to special spaces of C^∞-functions in the calculus of [B71]; there are detailed accounts e.g. in [G96] Sect. 2.2 or [G09] Ch. 10, which serve our purposes here (and will be taken up below in Section 2.3). The projection properties are summed up e.g. in [G09] Th. 10.15.

Recall some ingredients: With $d \in \mathbb{Z}$, $\mathcal{H}_d$ denotes the space of C^∞-functions $f(t)$ on $\mathbb{R}$ such that $k(\tau) = \tau^d f(\tau^{-1})$ coincides with a C^∞-function for $-1 < \tau < 1$ (this means that the derivatives of f match in a good way for $t \to \pm\infty$). Here one can show that that

$$\mathcal{H}_{-1} = \mathcal{F}(e^-\mathcal{S}_- \oplus e^+\mathcal{S}_+), \quad \mathcal{H}_d = \mathcal{H}_{-1} \oplus \mathbb{C}_d[t] \text{ for } d \geq 0, \tag{2.8}$$

where $\mathcal{S}_\pm = r^\pm\mathcal{S}(\mathbb{R}) = \mathcal{S}(\overline{\mathbb{R}}_\pm)$ (defined from the Schwartz space $\mathcal{S}(\mathbb{R})$), and $\mathbb{C}_d[t]$ stands for the space of polynomials of degree $\leq d$ in t. Setting (with a slight asymmetry)

$$\mathcal{H}^+ = \mathcal{F}(e^+\mathcal{S}_+), \quad \mathcal{H}^-_d = \mathcal{F}(e^-\mathcal{S}_-) \oplus \mathbb{C}_d[t], \tag{2.9}$$

one defines the mappings $h^\pm$ on $\mathcal{H}_d$, consistently with their definition given above for $d \leq -1$, such that they are projections with ranges

$$h^+\mathcal{H}_d = \mathcal{H}^+, \quad h^-\mathcal{H}_d = \mathcal{H}^-_d, \quad \text{for } d \geq -1. \tag{2.10}$$

The symbol h_{-1} denotes the projection from $\mathcal{H}_d$ to $\mathcal{H}_{-1}$ that removes the polynomial part. The space $\mathcal{H}^-_{-1}$ equals the space of conjugates of functions in $\mathcal{H}^+$ ([G09] (10.55)). $\mathcal{H}^+$ can also be denoted $\mathcal{H}^+_{-1}$ when convenient.

In the case we want to work on, we are looking for a *factorization*, not a sum decomposition.

This was not treated in [B71, G09]. It involves taking the logarithm of q, decomposing $\log q$ into a sum by Cauchy integrals, and then deriving a factorization of q itself by exponentiating. The method is described in [E81] with a few estimates, but it has not been worked out what happens in terms of $\mathcal{H}^\pm$ spaces, so a new analysis is needed for our purposes. Here we moreover find a special structure of the factors, that in our application later will allow an integration by parts formula.

We first introduce some generalized symbol spaces and ψdo's.

2.3 Symbol spaces for generalized ψdo's.

Homogeneous functions of ξ are usually singular at $\xi = 0$. We use in general the convention that a symbol $p(x, \xi)$ is assumed to be C^∞ for all ξ, then in the homogeneous case, homogeneity is assumed only for $|\xi|\ge 1$, or $|\xi|\ge\delta$ for a suitable $\delta>0$ (if needed, the associated fully homogeneous function is then called the *strictly* homogeneous symbol).

Classical (also called polyhomogeneous) ψdo symbols of order m are C^∞-functions having series expansions $p(x,\xi)\sim\sum_{j\in\mathbb N_0}p_j(x,\xi)$, where the p_j are homogeneous of degree $m-j$ in ξ for $|\xi|\ge 1$ and $\partial_x^\beta\partial_\xi^\alpha(p(x,\xi)-\sum_{j<J}p_j(x,\xi))$ is $O(\langle\xi\rangle^{m-J-|\alpha|})$ for all α,β,J. The replacement of a strictly homogeneous function by a function that is smooth near $\xi=0$ is often achieved by multiplication by an *excision function* $\eta(\xi)$ satisfying:

$$\eta(\xi)=\eta(|\xi|)\in C^\infty(\mathbb R^n,[0,1])\text{ with }\eta(\xi)=0\text{ for }|\xi|\le\tfrac12,\ \eta(\xi)=1\text{ for }|\xi|\ge 1. \tag{2.11}$$

It is a basic fact in the Boutet de Monvel calculus (cf. e.g. [G09] Th. 10.21) that when $q(x,\xi)$ is a ψdo symbol of order $d\in\mathbb Z$ having the 0-transmission property with respect to the hyperplane $\{x_n=c\}$, then the symbol $q(x',c,\xi',\xi_n)$ is in $\mathcal H_d$ as a function of ξ_n, and

$$h^+q(x',c,\xi',\xi_n)\in S^d_{1,0}(\mathbb R^{n-1},\mathbb R^{n-1},\mathcal H^+), \tag{2.12}$$

where $h^+\colon f\mapsto f_+$ is the projection defined in (2.5)ff. (the space $S^d_{1,0}(\mathbb R^{n-1},\mathbb R^{n-1},\mathcal H^+)$ will be recalled in a moment). The function h^+q is not quite a ψdo symbol in ξ (although it is so in ξ' for each ξ_n), but we can still use the Op-definition (as in (A.1)), and we call such symbols generalized ψdo symbols.

The symbol spaces are explained e.g. in [G09], Section 10.3. With m denoting a positive integer, $S^d_{1,0}(\mathbb R^m,\mathbb R^{n-1},\mathcal H_+)$ consists of the following C^∞-functions:

$$\begin{gathered}f(X,\xi',\xi_n)\in S^d_{1,0}(\mathbb R^m,\mathbb R^{n-1},\mathcal H_+),\ \text{ when } f(X,\xi',\xi_n)\text{ is in }\mathcal H^+\text{ w.r.t. }\xi_n,\text{ and}\\ \|D_X^\beta D_{\xi'}^\alpha D_{\xi_n}^k h_{-1}(\xi_n^{k'}f(X,\xi',\xi_n))\|_{L_2(\mathbb R)}\le C_{\alpha,\beta,k,k'}\langle\xi'\rangle^{d+\frac12-k+k'-|\alpha|},\end{gathered} \tag{2.13}$$

for all indices $\alpha\in\mathbb N_0^{n-1},\beta\in\mathbb N_0^m,k,k'\in\mathbb N_0$, with constants $C_{\alpha,\beta,k,k'}$. m is usually taken equal to n or $n-1$. (The definition in [G09] has h^+ instead of h_{-1}; the projections h^+ and h_{-1} have the same effect of removing the polynomial terms arising from the multiplication of an $\mathcal H^+$-function by $\xi_n^{k'}$.)

The L_2-norms are useful when Fourier transforms are involved. In fact, the system of seminorms (2.13) is *equivalent with* the following system, applied to the inverse Fourier transforms $\tilde f=\mathcal F^{-1}_{\xi_n\to x_n}f$ restricted to $\{x_n>0\}$:

$$\|D_X^\beta D_{\xi'}^\alpha x_n^k D_{x_n}^{k'}r^+\tilde f(X,x_n,\xi')\|_{L_2(\mathbb R_+)}\le C_{\alpha,\beta,k,k'}\langle\xi'\rangle^{d+\frac12-k+k'-|\alpha|}, \tag{2.14}$$

the space of such functions $r^+\tilde f$ is denoted $S^d_{1,0}(\mathbb R^m,\mathbb R^{n-1},\mathcal S_+)$. Here $\tilde f$ is in $e^+\mathcal S_+$ as a function of x_n. The effect of h_{-1} is here replaced by that of r^+, which removes possible linear combinations of $D^j_{x_n}\delta_{x_n}$ (supported at $\{x_n=0\}$) that arise from differentiating $\tilde f\in e^+\mathcal S_+$.

It will be useful to observe that one can replace $L_2(\mathbb R_+)$-norms by $L_\infty(\mathbb R_+)$-norms or $L_1(\mathbb R_+)$-norms (as remarked for L_∞-norms around (10.17) in [G09], and used sporadically in the literature):

Lemma 2.1. *The family of estimates* (2.14) *is equivalent with the family of estimates:*

$$\|D_X^\beta D_{\xi'}^\alpha x_n^k D_{x_n}^{k'} r^+ \tilde f(X,x_n,\xi')))\|_{L_\infty(\mathbb{R}_+)} \le C_{\alpha,\beta,k,k'} \langle \xi'\rangle^{d+1-k+k'-|\alpha|}, \tag{2.15}$$

as well as with the family of estimates

$$\|D_X^\beta D_{\xi'}^\alpha x_n^k D_{x_n}^{k'} r^+ \tilde f(X,x_n,\xi')\|_{L_1(\mathbb{R}_+)} \le C_{\alpha,\beta,k,k'} \langle \xi'\rangle^{d-k+k'-|\alpha|}. \tag{2.16}$$

Proof. We have the elementary inequalities for functions $u(t)\in\mathcal{S}_+$, $\sigma>0$:

$$\begin{aligned}
\sup_{t\ge 0}|u(t)|^2 &\le \sup_{t\ge 0}\int_t^\infty |\partial_s(u(s)\bar u(s))|\,ds \le 2\|u\|_{L_2(\mathbb{R}_+)}\|\partial_t u\|_{L_2(\mathbb{R}_+)},\\
\sup_{t\ge 0}|u(t)| &\le \sup_{t\ge 0}\int_t^\infty |\partial_s u(s)|\,ds \le \|\partial_t u\|_{L_1(\mathbb{R}_+)},\\
\|u\|_{L_2} &\le \Big\|\frac{1+\sigma t}{1+\sigma t}u\Big\|_{L_2} \le c\sigma^{-\frac12}\|(1+\sigma t)u\|_{L_\infty},\\
\|u\|_{L_1} &= \int_0^\infty \frac{1+\sigma t}{1+\sigma t}|u(t)|\,dt \le c\sigma^{-\frac12}\|(1+\sigma t)u\|_{L_2},
\end{aligned} \tag{2.17}$$

where $\|(1+\sigma t)^{-1}\|_{L_2} = c\sigma^{-\frac12}$.

Thus when u satisfies

$$\|t^k D_t^{k'} u(t)\|_{L_2(\mathbb{R}_+)} \le C_{k,k'}\sigma^{d+\frac12-k+k'},\ \text{all } k,k'\in\mathbb{N}_0,$$

then we have from the first line:

$$\|u(t)\|_{L_\infty(\mathbb{R}_+)} \le (2C_{0,0}\,\sigma^{d+\frac12}\,C_{1,0}\,\sigma^{d+\frac32})^{\frac12} = c'\sigma^{d+1},$$

with a similar treatment of derived functions $t^k D_t^{k'} u$. The variables X,ξ' are easily included, to see with $\sigma=\langle\xi'\rangle$ that the system of estimates (2.14) implies (2.15). For the opposite direction, the basic step is that when inequalities

$$\|t^k D_t^{k'} u(t)\|_{L_\infty(\mathbb{R}_+)} \le C_{k,k'}\sigma^{d+1+k-k'}$$

hold, then we have from the third line in (2.17) that

$$\|u(t)\|_{L_2(\mathbb{R}_+)} \le c\sigma^{-\frac12}(C_{0,0}\,\sigma^{d+1} + \sigma\, C_{0,1}\,\sigma^d) = c''\sigma^{d+\frac12},$$

with a similar treatment of derived functions.

For L_1-norms, we moreover use the other lines in (2.17). □

Instead of the above estimates that are global in X, we can work with the constants $C_{\alpha,\dots}$ replaced by continuous, hence locally bounded, coefficients $C_{\alpha,\dots}(X)$; they can be applied in localized situations, and are more general than the above. Global estimates were considered in [G96,G09], and are useful when one considers operators defined over unbounded domains such as $\mathbb{R}^n$, $\mathbb{R}^n_+$ (more generally: "admissible manifolds", as defined in [G96]).

We also need a notation for the spaces where the functions are in $\mathcal{H}^-_{-1}$ or in $\mathcal{H}_{-1}$ as functions of ξ_n:

$$
\begin{aligned}
&f(X,\xi',\xi_n)\in S^d_{1,0}(\mathbb{R}^m,\mathbb{R}^{n-1},\mathcal{H}^-_{-1}), \text{ when } f\in\mathcal{H}^-_{-1} \text{ w.r.t. } \xi_n \text{ and}\\
&\|D^\beta_X D^\alpha_{\xi'} D^k_{\xi_n} h_{-1}(\xi_n^{k'} f(X,\xi',\xi_n))\|_{L_2(\mathbb{R})}\le C_{\alpha,\beta,k,k'}\langle\xi'\rangle^{d+\frac12-k+k'-|\alpha|},\\
&f(X,\xi',\xi_n)\in S^d_{1,0}(\mathbb{R}^m,\mathbb{R}^{n-1},\mathcal{H}_{-1}), \text{ when } f\in\mathcal{H}_{-1} \text{ w.r.t. } \xi_n \text{ and}\\
&\|D^\beta_X D^\alpha_{\xi'} D^k_{\xi_n} h_{-1}(\xi_n^{k'} f(X,\xi',\xi_n))\|_{L_2(\mathbb{R})}\le C_{\alpha,\beta,k,k'}\langle\xi'\rangle^{d+\frac12-k+k'-|\alpha|},
\end{aligned}
\tag{2.18}
$$

for all indices. Again, the estimates are equivalent with families of estimates of the inverse Fourier transforms in ξ_n as described above for $\mathcal{H}^+$. Note here that the inverse Fourier transform of $\mathcal{H}_{-1}=\mathcal{H}^-_{-1}\oplus\mathcal{H}^+$ is $e^-\mathcal{S}_-\oplus e^+\mathcal{S}_+$, so that in fact, the second system of estimates is equivalent with the system

$$
\|D^\beta_X D^\alpha_{\xi'} x_n^k D^{k'}_{x_n}\tilde f(X,x_n,\xi')|_{\mathbb{R}_-\cup\mathbb{R}_+}\|_{L_2(\mathbb{R}_-)\oplus L_2(\mathbb{R}_+)}\le C_{\alpha,\beta,k,k'}\langle\xi'\rangle^{d+\frac12-k+k'-|\alpha|}.
\tag{2.19}
$$

There are also versions of these spaces with local estimates in X (i.e., with the constants $C_{\alpha,\dots}$ replaced by continuous functions of X).

The symbols in $S^d_{1,0}(\mathbb{R}^m,\mathbb{R}^{n-1},\mathcal{H}_+)$ were used in [G96,G09] to define Poisson and trace operators (maps between the boundary and the interior of $\mathbb{R}^n_+$). We shall here use them to define ψdo's on $\mathbb{R}^n$. Since they do not satisfy all the estimates usually required of ψdo symbols, we view them as *generalized* ψdo symbols, and the operators resulting from applying the Op-definition in (A.1) as *generalized* ψdo's. To find their mapping properties, it is important to derive relevant sup-norm estimates from (2.13) (and here it is a point to avoid having to involve the projection h_{-1}).

Lemma 2.2. *Let $f\in S^d_{1,0}(\mathbb{R}^m,\mathbb{R}^{n-1},\mathcal{H}_{-1})$.*

1° *Then also $\xi_n^k D^k_{\xi_n} f$ is in the space for all $k\in\mathbb{N}_0$, and*

$$
|D^\beta_X D^\alpha_{\xi'}\xi_n^k D^k_{\xi_n} f(X,\xi',\xi_n)|\le C_{\alpha,\beta,k}\langle\xi'\rangle^{d-|\alpha|},
\tag{2.20}
$$

for all α,β,k.

2° *Moreover, $(\langle\xi'\rangle\pm i\xi_n)D_{\xi_n}f$ belongs to $S^d_{1,0}(\mathbb{R}^m,\mathbb{R}^{n-1},\mathcal{H}_{-1})$.*

Proof. When $\varphi(\xi_n)\in\mathcal{H}_{-1}$, then so are $D_{\xi_n}\varphi$ and $\xi_n D_{\xi_n}\varphi$; without going deeply into the definition of $\mathcal{H}_{-1}$ and h_{-1} we can see this by observing that the inverse Fourier transforms $-x_n\tilde\varphi(x_n)$ and $-D_{x_n}x_n\tilde\varphi(x_n)$ are in $e^-\mathcal{S}_-\oplus e^+\mathcal{S}_+$ without distribution terms supported at $x_n=0$.

For 1° we iterate these considerations, seeing that also $\xi_n^k D^k_{\xi_n}\varphi$ and $D_{\xi_n}\xi_n^k D^k_{\xi_n}\varphi$ are in $\mathcal{H}^+$. The estimates in (2.18) then show that when $f\in S^d_{1,0}(\mathbb{R}^m,\mathbb{R}^{n-1},\mathcal{H}_{-1})$, then

$$
\begin{aligned}
\|D^\beta_X D^\alpha_{\xi'}\xi_n^k D^k_{\xi_n} f(X,\xi',\xi_n))\|_{L_2(\mathbb{R})}&\le C_{\alpha,\beta,k}\langle\xi'\rangle^{d+\frac12-|\alpha|},\\
\|D^\beta_X D^\alpha_{\xi'}D_{\xi_n}\xi_n^k D^k_{\xi_n} f(X,\xi',\xi_n))\|_{L_2(\mathbb{R})}&\le C'_{\alpha,\beta,k}\langle\xi'\rangle^{d-\frac12-|\alpha|}.
\end{aligned}
$$

This implies (2.20) by the first line in (2.17), extended to functions on $\mathbb{R}$.

The other estimates needed for the space $S^d_{1,0}(\mathbb{R}^m,\mathbb{R}^{n-1},\mathcal{H}_{-1})$ follow easily by carrying the inspection a little further. This shows 1°, and 2° follows by adding a similar inspection of $\langle\xi'\rangle D_{\xi_n}f$. □

We now investigate the mapping properties of the generalized ψdo's defined from these symbols. Here it will be convenient to refer to not only the H^s_p-spaces recalled in the Appendix, but also spaces with a different differentiability degree in the x_n-direction (used e.g. in [H63,G96,G09] for $p=2$):

$$H^{s,t}_p(\mathbb{R}^n)=\{u\in\mathcal{S}'(\mathbb{R}^n\mid\mathcal{F}^{-1}(\langle\xi\rangle^s\langle\xi'\rangle^t\hat u)\in L_p(\mathbb{R}^n\}=\Xi^{-s}\Xi'^{-t}L_p(\mathbb{R}^n),$$

where $\Xi^t=\operatorname{Op}(\langle\xi\rangle^t)$, $\Xi'^t=\operatorname{Op}(\langle\xi'\rangle^t)$.

To simplify the notation, we in the following abbreviate $S^d_{1,0}(\mathbb{R}^n,\mathbb{R}^{n-1},\mathcal{H}^+)$ to $S^d(\mathcal{H}^+)$, and similarly with $\mathcal{H}^-_{-1}$ and $\mathcal{H}_{-1}$.

Proposition 2.3. *Let $f(x,\xi',\xi_n)\in S^d(\mathcal{H}_{-1})$ for some $d\in\mathbb{R}$. Then $F=\operatorname{Op}(f)$ is continuous*

$$F\colon H^{s,t}_p(\mathbb{R}^n)\to H^{s,t-d}_p(\mathbb{R}^n)\text{ for all } s,t\in\mathbb{R}. \tag{2.21}$$

Proof. Consider first the case $d=0$. By Lemma 2.2, we have that

$$\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}\xi_n^kD^k_{\xi_n}D^\beta_xf\text{ is bounded for all }\alpha\in\mathbb{N}_0^{n-1},\beta\in\mathbb{N}_0^n,k\in\mathbb{N}_0.$$

Then Lizorkin's criterion assures that $F\colon L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)$ is bounded; this shows (2.21) for $s=t=0$. The use of Lizorkin's criterion is explained e.g. in [GK93] around Th. 1.6, with references.

Next, observe that

$$\|Fu\|_{H^1_p}\le c(\sum_{j=1}^n\|D_jFu\|_{L_p}+\|Fu\|_{L_p}),$$

where $D_jFu=\operatorname{Op}(\xi_jf+D_{x_j}f)u=FD_ju+\operatorname{Op}(D_{x_j}f)u$. Here

$$\|FD_ju\|_{L_p}\le c\|D_ju\|_{L_p}\le c'\|u\|_{H^1_p}$$

by the preceding result, and since $D_{x_j}f$ is also in $S^d(\mathcal{H}_{-1})$,

$$\|\operatorname{Op}(D_{x_j}f)u\|_{L_p}\le c\|u\|_{L_p}\le c'\|u\|_{H^1_p},$$

implying altogether that

$$F\colon H^1_p\to H^1_p$$

is bounded. The argument extends easily to higher derivatives, implying boundedness of

$$F\colon H^s_p\to H^s_p \tag{2.22}$$

for all $s\in\mathbb{N}_0$. By interpolation, the result extends to $s\in\overline{\mathbb{R}}_+$.

Since the Lizorkin criterion also holds when the operator is in y-form (cf. [GK93]), we likewise find that the adjoint operator F^* satisfies (2.22) for $s\ge 0$. Her we can replace p by p', and hence conclude by duality that (2.22) holds for F, all $s\in\mathbb{R}$.

To extend the result to $H_p^{s,t}$-spaces, a first step is to observe that

$$\|Fu\|_{H_p^{s,1}}\le c(\sum_{j=1}^{n-1}\|D_jFu\|_{H_p^s}+\|Fu\|_{H_p^s})\le c'(\sum_{j=1}^{n-1}\|D_ju\|_{H_p^s}+\|u\|_{H_p^s})\le c''\|u\|_{H_p^{s,1}}.$$

Generalizing this to higher derivatives, we find (2.21) for $s\in\mathbb{R}$, $t\in\mathbb{N}_0$, $d=0$, and interpolation and a similar treatment of the adjoint leads to (2.21) for all $s,t\in\mathbb{R}$ when $d=0$.

For general $d\in\mathbb{R}$, we observe that $F\Xi'^{-d}=\operatorname{Op}(f\langle\xi'\rangle^{-d})$, where $f\langle\xi'\rangle^{-d}\in S^0(\mathcal{H}_{-1})$, hence satisfies (2.21) with $d=0$. Then since obviously $\Xi'^d\colon H_p^{s,t}(\mathbb{R}^n)\overset{\sim}{\to}H_p^{s,t-d}(\mathbb{R}^n)$; (2.21) follows for $F=F\Xi'^{-d}\Xi'^d$. $\square$

Theorem 2.4. *Let $f(x,\xi)\in S^d(\mathcal{H}_{-1})$ for some $d\in\mathbb{Z}$. Then $F=\operatorname{Op}(f)$ is continuous for all s,t:*

$$F\colon H_p^{s,t}(\mathbb{R}^n)\to H_p^{s-d,t}(\mathbb{R}^n)\ \text{if } d\ge -1. \tag{2.23}$$

The mapping property extends to $d=-k-1$, $k\in\mathbb{N}$, if $f(x,\xi)([\xi']+i\xi_n)^k\in S^{-1}(\mathcal{H}_{-1})$ (or $f(x,\xi)([\xi']-i\xi_n)^k\in S^{-1}(\mathcal{H}_{-1})$).

Proof. For $d\ge 0$, the result follows immediately from Proposition 2.3 since $H_p^{s,t-d}(\mathbb{R}^n)\subset H_p^{s-d,t}(\mathbb{R}^n)$.

Now let $d=-1$. Observe that

$$F=F\,\Xi_+^1\,\Xi_+^{-1},\ \text{where } F\,\Xi_+^1=\operatorname{Op}(f(x,\xi)([\xi']+i\xi_n)).$$

This symbol is in $\mathcal{H}_0$ as a function of ξ_n, and can be decomposed as

$$f(x,\xi)([\xi']+i\xi_n)=f(x,\xi)[\xi']+h_{-1}(if\xi_n)+(1-h_{-1})(if\xi_n).$$

The first two terms are in $S^0(\mathcal{H}_{-1})$, hence the corresponding operators act as in (2.23) for $d=0$. The third term is of the form $s(x,\xi')$, constant in ξ_n and with estimates $D_x^\beta D_\xi^\alpha s=O(\langle\xi'\rangle^{-|\alpha|})$ (it is the zero'th term in the expansion of $if\xi_n$ in powers ξ_n^{-j}, $j\in\mathbb{N}_0$, cf. e.g. [G09], Def. 10.12). It likewise defines a bounded operator in $H_p^{s,t}(\mathbb{R}^n)$. Since $\Xi_+^{-1}\colon H_p^{s,t}(\mathbb{R}^n)\overset{\sim}{\to}H_p^{s+1,t}(\mathbb{R}^n)$, we conclude (2.23) for $d=-1$. Note that we could just as well have used compositions to the right with $\Xi_-^{\pm1}=\operatorname{Op}(([\xi']-i\xi_n)^{\pm1})$.

For the lower values of d we apply the case $d=-1$ to the symbol $f(x,\xi)([\xi']+i\xi_n)^k$ (resp. $f(x,\xi)([\xi']-i\xi_n)^k$). $\square$

The most general symbols in $S^{-k-1}(\mathcal{H}_{-1})$, $k\in\mathbb{N}$, only have the mapping property

$$F\colon H_p^{s,t}(\mathbb{R}^n)\to H_p^{s+1,t+k}(\mathbb{R}^n),$$

(since they may only be $O(\xi_n^{-1})$ for $\xi_n\to\pm\infty$); this is shown by combining (2.23) for $d=-1$ with Proposition 2.3. Fortunately, our applications in this paper will mainly be in the cases $d=0$ and $d=-1$. Therefore we shall not burden the exposition with additional terminology for symbol classes.

2.4 The basic factorization theorem.

With these preparations, we shall etablish the factorization theorem for homogeneous symbols.

Theorem 2.5. *Let $q(x,\xi',\xi_n)$ be an elliptic pseudodifferential symbol of order* 0*, homogeneous in ξ of degree* 0 *for $|\xi|\ge 1$ and having the* 0*-transmission property at all hyperplanes $\{x_n=c\}$:*

$$\partial_x^\beta\partial_\xi^\alpha q(x,0,-\xi_n)=(-1)^{|\alpha|}\partial_x^\beta\partial_\xi^\alpha q(x,0,\xi_n),\ \textit{all}\ \alpha,\beta\in\mathbb{N}_0^n, |\xi|\ge 1. \tag{2.24}$$

1° *Assume that $q\ne 0$ also for $|\xi|<1$. Denote $q(x,0,1)=s_0(x)$. Then q has a factorization*

$$q(x,\xi)=s_0(x)q^-(x,\xi)q^+(x,\xi), \tag{2.25}$$

where $q^\pm(x,\xi',\xi_n)$ are invertible, and extend holomorphically into $\mathbb{C}_\mp$, respectively, as functions of ξ_n. Moreover,

$$q^+(x,\xi)=1+f(x,\xi)\ \textit{with}\ f(x,\xi)\in S^0_{1,0}(\mathbb{R}^n,\mathbb{R}^{n-1},\mathcal{H}^+), \tag{2.26}$$

homogeneous of degree 0 *in ξ for $|\xi'|\ge 1$, and $\overline{q^-}$ has these properties too.*

2° *Let $q^h(\xi)$ denote the strictly homogeneous function coinciding with $q(\xi)$ for $\xi\ne 0$. Without assuming that q^h can be modified smoothly as a nonvanishing function for $|\xi|<1$, we can for any $0<\delta<1$ find functions $q^\pm(\xi)$ that have the properties listed after* (2.25)*, such that $q(\xi)=\eta(\xi/\delta)q^h(\xi)$ satifies* (2.25) *for $|\xi|\ge\delta$.*

Proof. By division by $s_0(x)$ we can obtain that $q(x,0,1)=1$, which will be assumed from now on. Fix $x=(x',c)$. We shall suppress the explicit mention of x, since the estimates of derivatives in x follow in a standard way when the claims have been shown with respect to ξ at each x.

1°. Define $\psi(\xi)=\log q(\xi)$ (to take real values when $q(\xi)$ is positive). The function ψ is likewise homogeneous of degree 0 for $|\xi|\ge 1$, hence is a ψdo symbol; moreover, it again has the 0-transmission property. Since $q(0,1)=1$, we have (2.4) for all ξ'. In view of the 0-transmission property, ψ is in $\mathcal{H}_{-1}$ as a function of ξ_n for each ξ', and by [G09] Th. 10.21,

$$\psi_+=h^+\psi\in S^0_{1,0}(\mathbb{R}^{n-1},\mathbb{R}^{n-1},\mathcal{H}^+), \tag{2.27}$$

and is homogeneous in ξ of degree 0 when $|\xi'|\ge 1$; it extends holomorphically into $\mathbb{C}_-$ as a function of ξ_n. Moreover, we can define

$$\psi_-=h^-\psi=\overline{\psi_+};\ \text{then}\ \psi=\psi_++\psi_-, \tag{2.28}$$

and $\overline{\psi_-}$ is similar to ψ_+.

We now form

$$q^+(\xi)=\exp(\psi_+(\xi))=1+\psi_+(\xi)+\tfrac12\psi_+(\xi)^2+\dots, \tag{2.29}$$

and $q^- = \exp(\psi_-)$, then $q = q^- q^+$. We have to show the estimates claimed in (2.26). Let

$$f = q^+ - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} \psi_+^k. \tag{2.30}$$

Instead of considering f directly, consider the inverse Fourier transform from ξ_n to z_n (restricted to $z_n \in \mathbb{R}_+$),

$$\tilde{f} = \sum_{k=1}^{\infty} \frac{1}{k!} \tilde{\psi}_+^{*k}, \quad \tilde{\psi}_+^{*k} = \tilde{\psi}_+ * \cdots * \tilde{\psi}_+, \text{ (k factors).} \tag{2.31}$$

Observe that

$$\|\tilde{\psi}_+ * \tilde{\psi}_+\|_{L_\infty} \le \|\tilde{\psi}_+\|_{L_1} \|\tilde{\psi}_+\|_{L_\infty}, \ \dots\ , \|\tilde{\psi}_+^{*k}\|_{L_\infty} \le \|\tilde{\psi}_+\|_{L_1}^{k-1} \|\tilde{\psi}_+\|_{L_\infty}, \tag{2.32}$$

so that $\sum_{k=1}^{\infty} \frac{1}{k!} \|\tilde{\psi}_+\|_{L_1}^{k-1} \|\tilde{\psi}_+\|_{L_\infty}$ is a majorising series for the series in (2.31). Hence it converges in L_∞-norm, and the limit satisfies the estimate

$$\|\tilde{f}\|_{L_\infty} \le \sum_{k=1}^{\infty} \frac{1}{k!} \|\tilde{\psi}_+\|_{L_1}^{k-1} \|\tilde{\psi}_+\|_{L_\infty} = \|\tilde{\psi}_+\|_{L_1}^{-1} \|(\exp(\|\tilde{\psi}_+\|_{L_1} - 1) \|\tilde{\psi}_+\|_{L_\infty}. \tag{2.33}$$

Since $\tilde{\psi}_+$ satisfies the estimates in (2.14), (2.15) and (2.16) with $d = 0$ (in particular, $\|\tilde{\psi}_+\|_{L_1}$ is bounded in ξ'), this shows that $\tilde{f}$ satisfies the first estimate in (2.15), with $d = 0$.

The estimates of z_n-derivatives follow in the same way, when we note that D_{z_n} just hits one factor; the one on which we impose the L_∞-norm. Multiplication by a power z_n^l of z_n corresponds for the Fourier transform f to a derivative $D_{\xi_n}^l$, for which there is a Leibniz formula. We carry this over to the terms in $\tilde{f}$, seeing that it produces an expression where it hits at most l factors in the product $\tilde{\psi}_+^{*k}$. When $k \to \infty$, the estimates give $k - l$ factors $\|\tilde{\psi}_+\|_{L_1}$ besides at most l factors where specific estimates of functions derived from $\tilde{\psi}_+$ are needed. This allows a majorising sequence, leading to the desired estimate for $\tilde{f}$. Derivatives with respect to ξ' and x are straightforward to include.

There is a similar analysis of q^-.

2°. The property that q^h extends into $|\xi| \le 1$ as a smooth nonvanishing function can be assured e.g. when at each x, the values of $q^h(x, \xi/|\xi|)$ avoid some ray. Without imposing this, we proceed as follows:

Let $\delta' = \frac{1}{2}\delta$. The function $\log q^h(\xi)$ is defined for $\xi \ne 0$, and we multiply it by the excision function $\eta(\xi/\delta')$, defining $\psi(\xi) = \eta(\xi/\delta') \log q^h(\xi)$ for all ξ. From this we construct $q^\pm = \exp \psi^\pm$ as above, having the asserted properties (they are invertible since the exponential function is so). Here $q^-(\xi) q^+(\xi)$ equals $q(\xi) = \eta(\xi/\delta) q^h(\xi)$ for $|\xi| \ge \delta$. $\square$

It is important in Theorem 2.5 that q (after division by s_0) is not just factored into q^+ and q^- with the mentioned estimates, but that the first term in each of the two factors is 1, besides a term with a decrease in ξ_n. This will be very useful in the applications.

2.5 Factorization of full symbols.

For a general polyhomogeneous symbol that is elliptic and of type 0, the above can be extended to a factorization (in the sense of operator composition or Leibniz products) respecting also lower-order terms. Recall the composition rule for ψdo's:

$$\operatorname{Op}(a)\operatorname{Op}(b) \sim \operatorname{Op}(a\#b), \text{ where } a\#b = \sum_{\alpha\in\mathbb{N}_0^n} D_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi)/\alpha!. \tag{2.34}$$

The last expression is often called the Leibniz product of a and b.

We now show that it is possible to refine the factorization from Theorem 2.5, taking lower-order terms into account.

Theorem 2.6. *Let Q be a classical ψdo on $\mathbb{R}^n$ of order 0, with symbol $q \sim \sum_{j\in\mathbb{N}_0} q_j$ (where $q_j(x,\xi)$ is homogeneous of degree $-j$ in ξ for $|\xi|\ge 1$), elliptic (i.e., $q_0(x,\xi)\neq 0$ for $|\xi|\ge 1$) and having the 0-transmission property with respect to all hyperplanes $\{x_n = c\}$, i.e.,*

$$\partial_x^\beta\partial_\xi^\alpha q_j(x,0,-\xi_n) = (-1)^{j-|\alpha|}\partial_x^\beta\partial_\xi^\alpha q_j(x,0,\xi_n) \text{ for } j\in\mathbb{N}_0, \alpha,\beta\in\mathbb{N}_0^n,\ |\xi|\ge 1. \tag{2.35}$$

Denote $q_0(x,0,1) = s_0(x)$.

There exist two generalized pseudodifferential symbols $q^\pm(x,\xi) \sim \sum_{j\in\mathbb{N}_0} q_j^\pm(x,\xi)$, with $q_j^\pm(x,\xi)$ homogeneous of degree $-j$ in ξ for $|\xi'|\ge 1$, such that

$$q_0^+(x,\xi) = 1 + f(x,\xi) \text{ with } f(x,\xi)\in S^0_{1,0}(\mathbb{R}^n,\mathbb{R}^{n-1},\mathcal{H}^+);\quad q_j^+\in S^{-j}_{1,0}(\mathcal{H}^\pm_{-1}),\ \text{for } j>0. \tag{2.36}$$

and $\overline{q^-}(x,\xi)$ has a similar form, and

$$q \sim s_0\, q^-\# q^+, \tag{2.37}$$

in the sense that for all K, the difference between $s_0^{-1}q$ and the expression formed of the terms in q^+ and q^- down to order $-K$, composed by the Leibniz formula applied for $|\alpha|\le K$, is in $S^{-K-1}_{1,0}(\mathbb{R}^n,\mathbb{R}^{n-1},\mathcal{H}_{-1})$.

From the symbols $q^\pm$ we can define generalized ψdo's $Q^\pm$, respectively; then $Q - s_0Q^-Q^+$ has symbol in $S^{-\infty}(\mathcal{H}_{-1}) = \bigcap_d S^d(\mathcal{H}_{-1})$.

The operator $Q - s_0Q^-Q^+$ is smoothing in the sense that it maps $H^{s,t}(\mathbb{R}^n)$ to $H^{s+1,\infty}(\mathbb{R}^n) = \bigcap_t H^{s+1,t}(\mathbb{R}^n)$, for all s.

Proof. By multiplication by s_0^{-1} we can assume that $q_0(x,0,1) = 1$. The principal parts $q_0^\pm$ of $q^\pm$ are defined by application of Theorem 2.5 to q_0. Now we have to construct the lower-order symbols. This goes inductively as follows:

Collecting the terms of order -1 in (2.37) (cf. (2.34)), we find that $q_1^\pm$ should satisfy:

$$q_1 = q_0^- q_1^+ + q_1^- q_0^+ + \sum_{k\le n}\partial_{\xi_k} q_0^- D_{x_k} q_0^+.$$

Dividing by q_0 and using that $q_0 = q_0^- q_0^+$, we can rewrite this as

$$\frac{q_1^+}{q_0^+} + \frac{q_1^-}{q_0^-} = \frac{q_1}{q_0} - \frac{1}{q_0}\sum_{k\le n}\partial_{\xi_k} q_0^- D_{x_k} q_0^+, \tag{2.38}$$

where the right-hand side is already known. By Theorem 2.5, the function q_0^+ is 1 plus a function in $\mathcal{H}^+$ at each (x,ξ'), and since it is nonvanishing, the inverse is likewise of the form in (2.26). The same holds for $\overline{q_0^-}$. Moreover, q_1 being of order -1 and having the 0-transmission property implies that it is in $\mathcal{H}_{-1}$ as a function of ξ_n. Thus the right-hand side of (2.38) is in $\mathcal{H}_{-1}$, and the left-hand side expresses a decomposition in its $\mathcal{H}^+$-part and $\mathcal{H}^-_{-1}$-part, for each (x,ξ'). The decomposition is unique, and one checks that the two terms satisfy the appropriate estimates.

This shows the first step, and in the general step, one similarly determines the two terms q_k^+/q_0^+ and q_k^-/q_0^- as the components in $\mathcal{H}^+$ and $\mathcal{H}^-_{-1}$ of an expression formed of the preceding symbol terms of the relevant homogeneity:

$$\frac{q_k^+}{q_0^+}+\frac{q_k^-}{q_0^-}=\frac{q_k}{q_0}-\frac{1}{q_0}\sum_{j+|\beta|=k,j<k}\frac{1}{\beta!}\partial_\xi^\beta q_j^- D_x^\beta q_j^+. \tag{2.39}$$

There is a standard way to associate an exact symbol $q^\pm(x,\xi)$ with the series $\sum_{j\in\mathbb{N}_0} q_j^\pm(x,\xi)$, namely, a convergent sum $q^\pm(x,\xi)=\sum_{j\in\mathbb{N}_0}\eta(\xi/t_j)q_j^\pm(x,\xi)$, where $t_j\to\infty$ sufficiently rapidly (for $\eta(\xi)$, see (2.11)). Any other choice of a symbol with the given asymptotic expansion differs from this by a symbol in $S^{-\infty}_{1,0}(\mathbb{R}^n,\mathbb{R}^{n-1},\mathcal{H}^\pm_{-1})$.

Then one finds by use of the Leibniz formula and regrouping of homogeneous terms of the same order, that $Q-s_0Q^-Q^+$ is a generalized ψdo with symbol in $S^{-\infty}(\mathcal{H}_{-1})$. The last statement follows from Theorem 2.4 ff. $\square$

When q is *even* in ξ, that is,

$$q_j(x,-\xi)=(-1)^j q_j(x,\xi) \text{ for } |\xi|\ge 1, \text{ all } x, \tag{2.40}$$

the property (2.35) holds in any coordinate system.

We furthermore observe the following property.

Theorem 2.7. *For the symbol q^+ constructed in Theorem 2.6, there is a parametrix symbol $\tilde q^+$ with similar symbol properties, such that*

$$q^+\#\tilde q^+\sim 1\sim \tilde q^+\# q^+, \tag{2.41}$$

in the space consisting of symbols in $S^0_{1,0}(\mathbb{R}^n,\mathbb{R}^{n-1},\mathcal{H}_{-1})$ plus functions of x (constant in ξ).

There is a similar result for q^-.

Proof. We apply the standard parametrix construction: With $\tilde q_0^+=1/q_0^+$, we have that

$$q^+\#\tilde q_0^+=1+\sum_{\beta\ne 0}\tfrac{1}{\beta!}\partial_\xi^\beta q_0^+ D_x^\beta \tilde q_0^+ +(q^+-q_0^+)\#\tilde q_0^+\sim 1+r, \tag{2.42}$$

where $r\in S^{-1}_{1,0}(\mathbb{R}^n,\mathbb{R}^{n-1},\mathcal{H}^+)$ is defined from a regrouping of the terms according to homogeneity. Then, defining

$$\tilde r\sim\sum_{k=1}^\infty(-1)^k r^{\# k},$$

where $r^{\# k}\sim r\# r\#\dots\# r$ with k factors, we find that $\tilde q_0^+\#(1+\tilde r)$ is a right parametrix symbol for q^+. Similarly, there is a left parametrix symbol, and they are seen to be equivalent. Thus we can take $\tilde q^+=\tilde q_0^+\#(1+\tilde r)$, and it has the asserted properties. $\square$

Remark 2.8. The constructions in Theorems 2.6 and 2.7 have been developed from [H65], Theorem 2.6.3 ff. in combination with our use of function spaces based on $\mathcal{H}_\pm$ as in [G96], [G09]. The purpose in [H65] was to construct an operator that solves, in the parametrix sense, certain boundary problems for operators such as e.g. $P = \Xi_-^a Q \Xi_+^a$ with nonzero boundary data and 0 data in the interior, generalizing (2.45) below. For Q itself, with $\widetilde{Q}^+ = \operatorname{Op}(\tilde{q}^+)$, it can be shown that the operator $K_{\widetilde{Q}^+} : \varphi(x') \to \widetilde{Q}^+(\varphi(x') \otimes \delta(x_n))$, which is a Poisson operator in the Boutet de Monvel calculus, satisfies:

$$r^+ Q\widetilde{K}_{\widetilde{Q}^+} : \mathcal{E}'(\mathbb{R}^{n-1}) \to C^\infty(\overline{\mathbb{R}}^n_+), \quad \gamma_{-1,0} K_{\widetilde{Q}^+} - I : \mathcal{E}'(\mathbb{R}^{n-1}) \to C^\infty(\mathbb{R}^{n-1}) \tag{2.43}$$

(i.e. $r^+ Q\widetilde{K}_{\widetilde{Q}^+}$ and $\gamma_{-1,0} K_{\widetilde{Q}^+} - I$ are smoothing operators); hence $K_{\widetilde{Q}^+}$ defines a parametrix solution operator to the problem

$$r^+ Qw = 0, \quad \gamma_{-1,0} w = \varphi. \tag{2.44}$$

Here $\gamma_{-1,0}$ is a generalization of $\gamma_{\mu,0}$ to low values of μ, defined but not studied in detail in [G15]. Then for P one finds, setting $w = \Xi_+^a u$, that $\Xi_+^{-a} K_{\widetilde{Q}^+}$ defines a parametrix solution operator to the problem

$$r^+ Pu = 0, \quad \gamma_{a-1,0} u = \varphi; \tag{2.45}$$

here $\Xi_+^{-a} K_{\widetilde{Q}^+}$ can be regarded as a (generalized) Poisson operator of noninteger order. The problem was also discussed in [G15], Th. 6.5; the present construction gives a more direct information. We shall possibly take up the details in another publication.

3. Integration by parts for operators on the half-space

The reader is encouraged to consult the Appendix for notation.

Let P be a classical ψdo on $\mathbb{R}^n$ of order $2a$ ($0 < a < 1$), having the a-transmission property at the boundary of $\mathbb{R}^n_+$. We wish to reduce the expression

$$\int_{\mathbb{R}^n_+} Pu\, \partial_n \bar{u}'\, dx + \int_{\mathbb{R}^n_+} \partial_n u\, \overline{P^* u'}\, dx \tag{3.1}$$

for functions $u, u' \in H^{a(s)}(\overline{\mathbb{R}}^n_+)$ for suitable s, to an integral over the boundary of suitable boundary values, supplied in the x_n-dependent case with an extra integral over $\mathbb{R}^n_+$. The fact that we integrate over $\mathbb{R}^n_+$ implies a restriction r^+ on the integrands, that we therefore need not mention explicitly in the formula.

The central argument will first be presented in a simple constant-coefficient case.

Theorem 3.1. 1° *Let $u, u' \in \mathcal{E}_a(\overline{\mathbb{R}}^n_+)$ with compact support in $\overline{\mathbb{R}}^n_+$. Let $w = r^+ \Xi_+^a u$, $w' = r^+ \Xi_+^a u'$. Then*

$$\int_{\mathbb{R}^n_+} \Xi_-^a e^+ w\, \partial_n \bar{u}'\, dx = (\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + (w, \partial_n w')_{L_2(\mathbb{R}^n_+)}. \tag{3.2}$$

2° *The formula extends to* $u, u' \in H^{a(s)}(\overline{\mathbb{R}}^n_+)$ *for* $s > a + \frac12$, *with dualities:*

$$\langle r^+\Xi^a_- e^+ w, \partial_n u'\rangle_{\overline{H}^{\frac12 - a+\varepsilon}(\mathbb{R}^n_+), \dot H^{a-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+)} = (\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + \langle w, \partial_n w'\rangle_{\overline{H}^{\frac12-\varepsilon}(\mathbb{R}^n_+), \dot H^{-\frac12+\varepsilon}(\overline{\mathbb{R}}^n_+)}, \tag{3.3}$$

for any $0 < \varepsilon \le s - a - \frac12$ *with* $\varepsilon < 1$.

3° *Here, when* $s \ge a + 1$, *then* (3.3) *can be written in the form* (3.2), *all ingredients being locally integrable functions.*

Proof. 1°. First let $u, u' \in \mathcal{E}_a(\overline{\mathbb{R}}^n_+)$ with compact support. Since $u \in H^{a(s)}(\overline{\mathbb{R}}^n_+)$ for any large s, $w = r^+\Xi^a_+ u \in C^\infty(\overline{\mathbb{R}}^n_+) \cap \overline{H}^s(\mathbb{R}^n_+)$ for any s, with $u = \Xi^{-a}_+ e^+ w$ (cf. [G15], Propositions 1.7 and 4.1). Moreover, $r^+\Xi^a_- e^+ w \in C^\infty(\overline{\mathbb{R}}^n_+) \cap \overline{H}^s(\mathbb{R}^n_+)$ for any s. There is similar information for u', w'.

Since $u \in \mathcal{E}_a(\overline{\mathbb{R}}^n_+)$ with compact support, $\partial_n u \in \mathcal{E}_{a-1}(\overline{\mathbb{R}}^n_+)$ with compact support. Here x_n^{a-1} is integrable over compact sets. Altogether, $r^+\Xi^a_- e^+ w\, \partial_n \bar u$ is on $\overline{\mathbb{R}}^n_+$ the product of x_n^{a-1} with a compactly supported smooth function, so the integral is well-defined.

We can also observe that by the identification of $e^+\overline{H}^t(\mathbb{R}^n_+)$ and $\dot H^t(\overline{\mathbb{R}}^n_+)$ for $|t| < \frac12$, $e^+ w' \in \dot H^{\frac12-\varepsilon}(\mathbb{R}^n_+)$ for any $\varepsilon \in]0,1[$, so

$$\partial_n u' = \partial_n \Xi^{-a}_+ e^+ w' \in \partial_n \dot H^{a+\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+) \subset \dot H^{a-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+). \tag{3.4}$$

Then since $r^+\Xi^a_- e^+\Xi^a_+ u \in \overline{H}^{\frac12-a+\varepsilon}(\mathbb{R}^n_+)$, the integral may be written as the duality

$$I = \langle r^+\Xi^a_- e^+ w, \partial_n u'\rangle_{\overline{H}^{\frac12-a+\varepsilon}(\mathbb{R}^n_+), \dot H^{a-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+)}.$$

Now note that by (A.7), $r^+\Xi^a_- e^+ \colon \overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+) \to \overline{H}^{\frac12-a+\varepsilon}(\mathbb{R}^n_+)$ has the adjoint $\Xi^a_+ \colon \dot H^{a-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+) \to \dot H^{-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+)$. We can then continue the calculation of I as follows:

$$I = \langle w, \Xi^a_+ \partial_n u'\rangle_{\overline{H}^{\frac12+\varepsilon}, \dot H^{-\frac12-\varepsilon}} = \langle w, \partial_n \Xi^a_+ u'\rangle_{\overline{H}^{\frac12+\varepsilon}, \dot H^{-\frac12-\varepsilon}} = \langle w, \partial_n e^+ w'\rangle_{\overline{H}^{\frac12+\varepsilon}, \dot H^{-\frac12-\varepsilon}}.$$

Here w' itself is a nice function on $\overline{\mathbb{R}}^n_+$, but the extension $e^+ w'$ to $\mathbb{R}^n$ has the jump $\gamma_0 w'$ at $x_n = 0$, and there holds the formula

$$\partial_n e^+ w' = \gamma_0 w' \otimes \delta(x_n) + e^+ \partial_n w'. \tag{3.5}$$

where $\otimes$ indicates a product of distributions with respect to different variables (x' resp. x_n). It is a distribution version of Green's formula (cf. e.g. [G96] (2.2.38)–(2.2.39)). Recall moreover from distribution theory (cf. e.g. [G09] p. 307) that the "two-sided" trace operator $\widetilde\gamma_0 \colon v(x) \mapsto \widetilde\gamma_0 v = v(x', 0)$ has the mapping $\widetilde\gamma_0^* \colon \varphi(x') \mapsto \varphi(x') \otimes \delta(x_n)$ as adjoint, with continuity properties

$$\widetilde\gamma_0 \colon H^{\frac12+\varepsilon}(\mathbb{R}^n) \to H^\varepsilon(\mathbb{R}^{n-1}), \quad \widetilde\gamma_0^* \colon H^{-\varepsilon}(\mathbb{R}^{n-1}) \to H^{-\frac12-\varepsilon}(\mathbb{R}^n), \text{ for } \varepsilon > 0. \tag{3.6}$$

Here $\widetilde{\gamma}_0^*\varphi$ is supported in $\{x_n=0\}$, hence lies in $\dot H^{-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+)$. We can then write

$$\partial_n e^+w' = \widetilde{\gamma}_0^*(\gamma_0 w') + e^+\partial_n w'. \tag{3.7}$$

Since $w\in \overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)$, it has an extension $W\in H^{\frac12+\varepsilon}(\mathbb{R}^n)$ with $w=r^+W$, and $\gamma_0 w = \widetilde{\gamma}_0 W$. Then

$$\langle w, \widetilde{\gamma}_0^*(\gamma_0 w')\rangle_{\overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+),\dot H^{-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+)} = \langle W, \widetilde{\gamma}_0^*(\gamma_0 w')\rangle_{H^{\frac12+\varepsilon}(\mathbb{R}^n),H^{-\frac12-\varepsilon}(\mathbb{R}^n)};$$

this is verified e.g. by approximating $\widetilde{\gamma}_0^*(\gamma_0 w')$ in $\dot H^{-\frac12-\varepsilon}$-norm by a sequence of functions in $C_0^\infty(\mathbb{R}^n_+)$. Here we can use (3.6) to write

$$\begin{aligned}\langle W, \widetilde{\gamma}_0^*(\gamma_0 w')\rangle_{H^{\frac12+\varepsilon}(\mathbb{R}^n),H^{-\frac12-\varepsilon}(\mathbb{R}^n)} &= \langle \widetilde{\gamma}_0 W, \gamma_0 w'\rangle_{H^\varepsilon(\mathbb{R}^{n-1}),H^{-\varepsilon}(\mathbb{R}^{n-1})}\\ &= \langle \gamma_0 w, \gamma_0 w'\rangle_{H^\varepsilon(\mathbb{R}^{n-1}),H^{-\varepsilon}(\mathbb{R}^{n-1})} = (\gamma_0 w,\gamma_0 w')_{L_2(\mathbb{R}^{n-1})}.\end{aligned}$$

In the last step we used that since both $\gamma_0 w$ and $\gamma_0 w'$ are in $H^\varepsilon(\mathbb{R}^{n-1})\subset L_2(\mathbb{R}^{n-1})$, the duality over the boundary is in fact an $L_2(\mathbb{R}^{n-1})$-scalar product.

Then finally

$$\begin{aligned} I = \langle w,\partial_n e^+w'\rangle_{\overline{H}^{\frac12+\varepsilon},\dot H^{-\frac12-\varepsilon}} &= \langle w, \widetilde{\gamma}_0^*(\gamma_0 w') + e^+\partial_n w'\rangle_{\overline{H}^{\frac12+\varepsilon},\dot H^{-\frac12-\varepsilon}}\\ &= (\gamma_0 w,\gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + \langle w, e^+\partial_n w'\rangle_{\overline{H}^{\frac12+\varepsilon},\dot H^{-\frac12-\varepsilon}}\\ &= (\gamma_0 w,\gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + (w, e^+\partial_n w')_{L_2(\mathbb{R}^n_+)},\end{aligned}$$

where we used that $w'\in\bigcap_s \overline{H}^s(\mathbb{R}^n_+)$. This shows (3.2).

2°. If $u,u'\in H^{a(s)}(\overline{\mathbb{R}}^n_+)$ with $s>a+\frac12$, they are in $\in H^{a(a+\frac12+\varepsilon)}(\overline{\mathbb{R}}^n_+)$ for an $\varepsilon\in\,]0,1[\,$, $\varepsilon\le s-a-\frac12$, and then $w,w'\in \overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)$ by definition. Moreover, by (A.10),

$$\begin{aligned} &u\in e^+x_n^a\overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)+\dot H^{a+\frac12+\varepsilon}(\overline{\mathbb{R}}^n_+),\text{ hence}\\ &\partial_n u\in e^+x_n^{a-1}\overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)+e^+x_n^a\overline{H}^{-\frac12+\varepsilon}(\mathbb{R}^n_+)+\dot H^{a-\frac12+\varepsilon}(\overline{\mathbb{R}}^n_+).\end{aligned}\tag{3.8}$$

(Since $\gamma_0 u=0$, there is no distribution term supported at $\{x_n=0\}$.) On the other hand, since $e^+w=\Xi_+^a u\in e^+\overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)\subset \dot H^{\frac12-\varepsilon'}(\overline{\mathbb{R}}^n_+)$, $u=\Xi_+^{-a}e^+w$ satisfies

$$\begin{aligned} &u\in \dot H^{a+\frac12-\varepsilon'}(\overline{\mathbb{R}}^n_+),\text{ any }\varepsilon'>0,\text{ hence}\\ &\partial_n u\in \dot H^{a-\frac12-\varepsilon'}(\overline{\mathbb{R}}^n_+).\end{aligned}\tag{3.9}$$

There is similar information for u'.

Here we can approximate u,u' in the norm of $H^{a(a+\frac12+\varepsilon)}(\overline{\mathbb{R}}^n_+)$ by compactly supported elements u_k,u_k' of $\mathcal{E}_a(\overline{\mathbb{R}}^n_+)$ (cf. [G15] Prop. 4.1). Then $w_k=r^+\Xi_+^a u_k$ and $w_k'=r^+\Xi_+^a u_k'$ converge in $\overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)$, and in particular, $\partial_n u_k'$ converges in $\dot H^{a-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+)$ and $\partial_n w_k'$ converges in $\overline{H}^{-\frac12+\varepsilon}(\mathbb{R}^n_+)=\dot H^{-\frac12+\varepsilon}(\overline{\mathbb{R}}^n_+)$. This implies (3.3) by passage to the limit, proving 2°.

3°. If $s\ge a+1$, then $w,w'\in \overline{H}^1(\mathbb{R}^n_+)$, so $\partial_n w'\in L_2(\mathbb{R}^n_+)$, and $r^+\Xi^a_-e^+w\in \overline{H}^{1-a}(\mathbb{R}^n_+)\subset L_2(\mathbb{R}^n_+)$. Moreover, by (A.10),

$$\begin{gathered} u\in e^+x_n^a\overline{H}^1(\mathbb{R}^n_+)+\dot H^{a+1}(\overline{\mathbb{R}}^n_+),\text{ hence since }\gamma_0u=0,\\ \partial_nu\in e^+x_n^{a-1}\overline{H}^1(\mathbb{R}^n_+)+e^+x_n^aL_2(\mathbb{R}^n_+)+\dot H^a(\overline{\mathbb{R}}^n_+); \end{gathered}\tag{3.10}$$

so ∂_nu,∂_nu' are functions. □

An immediate consequence is the following integration-by-parts result for fractional Helmholtz operators:

Theorem 3.2. *Let u and u' be as in Theorem 3.1 1° or 3°. Then one has for $m>0$:*

$$\begin{aligned}\int_{\mathbb{R}^n_+}(-\Delta+m^2)^au\,\partial_n\bar u'\,dx+\int_{\mathbb{R}^n_+}\partial_nu\,(-\Delta+m^2)^a\bar u'\,dx&\\ =\Gamma(a+1)^2\int_{\mathbb{R}^{n-1}}\gamma_0(x_n^{-a}u)\,\gamma_0(x_n^{-a}\bar u')\,dx'.&\end{aligned}\tag{3.11}$$

If u and u' are as in Theorem 3.1 2°, the formula holds with dualities, for small $\varepsilon>0$,

$$\begin{aligned}\langle r^+(-\Delta+m^2)^au,\partial_nu'\rangle_{\overline{H}^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}+\langle\partial_nu,r^+(-\Delta+m^2)^au'\rangle_{\dot H^{a-\frac12-\varepsilon},\overline{H}^{\frac12-a+\varepsilon}}&\\ =\Gamma(a+1)^2\int_{\mathbb{R}^{n-1}}\gamma_0(x_n^{-a}u)\,\gamma_0(x_n^{-a}\bar u')\,dx'.&\end{aligned}\tag{3.12}$$

Proof. We have that

$$(-\Delta+m^2)^a=\operatorname{Op}((|\xi|^2+m^2)^a)=\Xi^a_{m,-}\Xi^a_{m,+},\quad \Xi^a_{m,\pm}=\operatorname{Op}((((|\xi'|^2+m^2)^{\frac12}\pm i\xi_n)^a);$$

where $\Xi^a_{m,\pm}$ have exactly the same mapping properties as $\Xi^a_\pm$, which is the case $m=1$. In particular, Theorem 3.1 holds with $\Xi^a_\pm$ replaced by $\Xi^a_{m,\pm}$. It is seen as in [G15], Th. 4.2 and 4.4 that

$$r^+(-\Delta+m^2)u=r^+\Xi^a_{m,-}e^+r^+\Xi^a_{m,+}u,$$

when u satisfies one of the mentioned hypotheses. Set $w=r^+\Xi^a_{m,+}u$, $w'=r^+\Xi^a_{m,+}u'$.

We can then apply Theorem 3.1 to the integrals in the left-hand side of (3.11), resp. the dualities in the left-hand side of (3.12), when u,u' satisfy the respective hypotheses there. This gives e.g. under the weakest hypotheses (in 2°):

$$\begin{aligned}&\langle r^+(-\Delta+m^2)^au,\partial_nu'\rangle_{\overline{H}^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}+\langle\partial_nu,r^+(-\Delta+m^2)^au'\rangle_{\dot H^{a-\frac12-\varepsilon},\overline{H}^{\frac12-a+\varepsilon}}\\ &=\langle r^+\Xi^a_-e^+w,\partial_nu'\rangle_{\overline{H}^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}+\langle\partial_nu,r^+\Xi^a_-e^+w'\rangle_{\dot H^{a-\frac12-\varepsilon},\overline{H}^{\frac12-a+\varepsilon}}\\ &=2(\gamma_0w,\gamma_0w')_{L_2(\mathbb{R}^{n-1})}+\langle w,\partial_nw'\rangle_{\overline{H}^{\frac12-\varepsilon},\dot H^{-\frac12+\varepsilon}}+\langle\partial_nw,w'\rangle_{\dot H^{-\frac12+\varepsilon},\overline{H}^{\frac12-\varepsilon}}.\end{aligned}\tag{3.13}$$

Let w_k and w_k' be sequences in $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)=r^+C^\infty_0(\mathbb{R}^n)$ converging to w resp. w' in $\overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)$ for $k\to\infty$; then $\gamma_0w_k\to\gamma_0w$ in $H^\varepsilon(\mathbb{R}^{n-1})$ and $\partial_nw_k\to\partial_nw$ in $\overline{H}^{-\frac12+\varepsilon}(\mathbb{R}^n_+)$, with similar statements for w'. Now

$$\begin{aligned}\int_{\mathbb{R}^n_+}(w_k\partial_n\bar w_k'+\partial_nw_k\,\bar w_k')\,dx&=\int_{\mathbb{R}^n_+}\partial_n(w_k\bar w_k')\,dx\\ &=-\int_{\mathbb{R}^{n-1}}\gamma_0(w_k\bar w')\,dx'\to=-\int_{\mathbb{R}^{n-1}}\gamma_0w\gamma_0\bar w'\,dx'.\end{aligned}$$

Thus the last two terms in (3.13) contribute with $-(\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})}$, and we find that

$$\begin{aligned}&\langle r^+\Xi^a_- e^+ w, \partial_n u'\rangle_{\overline{H}^{\frac12-a+\varepsilon}, \dot H^{a-\frac12-\varepsilon}} + \langle \partial_n u, r^+\Xi^a_- e^+ w'\rangle_{\dot H^{a-\frac12-\varepsilon}, \overline{H}^{\frac12-a+\varepsilon}}\\ &= (\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})}.\end{aligned} \tag{3.14}$$

Finally, we recall from [G15] Th. 5.1 that

$$\gamma_0 w = \gamma_0(\Xi^a_{m,+} u) = \gamma_{a,0} u = \Gamma(a+1)\gamma_0(x_n^{-a} u). \tag{3.15}$$

Hence

$$(\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} = \Gamma(a+1)^2 \int_{\mathbb{R}^{n-1}} \gamma_0(x_n^{-a} u)\, \gamma_0(x_n^{-a}\bar u')\, dx',$$

and (3.12) follows. Under the hypotheses for 1° and 3° it can be written in the form (3.11). ☐

The theorem also holds for $(-\Delta)^a$ itself (the case $m = 0$), see Corollary 3.4 below.

We now turn to a general elliptic operator P of order $2a$, with symbol having the a-transmission property at the hyperplanes $\{x_n = c\}$, $c \in \mathbb{R}$; this hold in particular when the symbol is even (cf. (2.40)).

Theorem 3.3. *Let P be a classical ψdo on $\mathbb{R}^n$ that is elliptic of order $2a$ for some $0 < a < 1$, with symbol p having the a-transmission property at the hyperplanes $\{x_n = c\}$, $c \in \mathbb{R}$. Let $s_0(x) = p_0(x, 0, 1)$, where p_0 is the principal symbol, and let $P^{(n)}$ denote the commutator $[P, \partial_n]$;*

$$P^{(n)} = P\partial_n - \partial_n P;\ \text{it has symbol } p^{(n)} = -\partial_{x_n} p, \tag{3.16}$$

likewise of order $2a$ and having the a-transmission property at the hyperplanes $\{x_n = c\}$.

For $u, u' \in H^{a(s)}(\overline{\mathbb{R}}^n_+)$, $s \ge a+1$, there holds

$$\begin{aligned}&\int_{\mathbb{R}^n_+} Pu\, \partial_n \bar u'\, dx + \int_{\mathbb{R}^n_+} \partial_n u\, \overline{P^* u'}\, dx\\ &\qquad = \Gamma(a+1)^2 \int_{\mathbb{R}^{n-1}} s_0 \gamma_0(x_n^{-a} u)\, \gamma_0(x_n^{-a}\bar u')\, dx' + \int_{\mathbb{R}^n_+} P^{(n)} u\, \bar u'\, dx.\end{aligned} \tag{3.17}$$

For $s \ge a + \frac12 + \varepsilon$ (for some small ε), the formula holds with the integrals interpreted as dualities:

$$\begin{aligned}&\langle r^+ Pu, \partial_n u'\rangle_{\overline{H}^{\frac12-a+\varepsilon}, \dot H^{a-\frac12-\varepsilon}} + \langle \partial_n u, P^* u'\rangle_{\dot H^{a-\frac12+\varepsilon}, \overline{H}^{\frac12-a-\varepsilon}}\\ &\quad = \Gamma(a+1)^2 \int_{\mathbb{R}^{n-1}} s_0\gamma_0(x_n^{-a} u)\, \gamma_0(x_n^{-a}\bar u')\, dx' + \langle r^+ P^{(n)} u, u'\rangle_{\overline{H}^{\frac12-a+\varepsilon}, \dot H^{a-\frac12-\varepsilon}};\end{aligned} \tag{3.18}$$

the last term is a scalar product $(P^{(n)}u, u')_{L_2(\mathbb{R}^n_+)}$ when $a \le \frac12$.

In particular, when the symbol in independent of x_n, the term with $P^{(n)}$ drops out.

Proof. First let us account for the definition of the terms in (3.17)–(3.18). We already have the information (3.8)–(3.9) on $u, u', \partial_n u$ and $\partial_n u'$. If $u, u' \in H^{a(a+1)}(\overline{\mathbb{R}}^n_+)$, we have the information (3.10).

By [G15] Th. 4.2, r^+P maps $H^{a(s)}(\overline{\mathbb{R}}^n_+)$ continuously into $\overline{H}^{s-2a}(\mathbb{R}^n_+)$. When $s \geq a+1$, this is contained in $\overline{H}^{1-a}(\mathbb{R}^n_+) \subset L_2(\mathbb{R}^n_+)$, so r^+Pu is an L_2-function. When $s \geq a + \frac12 + \varepsilon$, $r^+Pu \in \overline{H}^{\frac12 - a+\varepsilon}(\mathbb{R}^n_+)$; in $L_2(\mathbb{R}^n_+)$ when $a \leq \frac12$. The operator $P^{(n)}$, being of the same type as P, also has these mapping properties.

We see that for $s \geq a+1$, the first and last integrands in (3.17) are functions. For $s > a + \frac12$, the duality

$$\langle r^+Pu, \partial_n u'\rangle_{\overline{H}^{\frac12 -a+\varepsilon}, \dot{H}^{a-\frac12-\varepsilon}}.$$

makes sense for small ε; here r^+Pu is a function when $a \leq \frac12$, and $\partial_n u'$ is a function when $a > \frac12$. In the duality

$$\langle r^+P^{(n)}u, u'\rangle_{\overline{H}^{\frac12 -a+\varepsilon}, \dot{H}^{a-\frac12-\varepsilon}},$$

it is only $P^{(n)}u$ that may not be a function; it will be one when $a \leq \frac12$. (Observe also that since $a - \frac12 \in]-\frac12, \frac12[$, $\overline{H}^{\frac12-a+\varepsilon} \simeq \dot{H}^{\frac12-a+\varepsilon}$ and $\dot{H}^{a-\frac12-\varepsilon} \simeq \overline{H}^{a-\frac12-\varepsilon}$ for small $\varepsilon > 0$.)

The integral with P^* is understood in a similar way (after conjugation).

In the right-hand sides of (3.17)–(3.18), the boundary values $\gamma_0(x_n^{-a}u), \gamma_0(x_n^{-a}u')$ are defined as functions in $H^{a+\frac12+\varepsilon-a-\frac12}(\mathbb{R}^{n-1}) = H^{\varepsilon}(\mathbb{R}^{n-1}) \subset L_2(\mathbb{R}^{n-1})$, by [G15], Th. 5.1.

Now we turn to the proof of the formulas. The detailed arguments will be given under the weakest regularity hypothesis, namely $u, u' \in H^{a(a+\frac12+\varepsilon)}(\overline{\mathbb{R}}^n_+)$.

In the reduction of the operators we shall use $\Lambda^a_\pm$ (cf. [G15]) rather than $\Xi^a_\pm$, in order to have true ψdo's. Then we write

$$P = \Lambda^a_- Q \Lambda^a_+, \quad P^* = \Lambda^a_- Q^* \Lambda^a_+, \tag{3.19}$$

where Q is of order 0. It has the 0-transmission property at $\{x_n = 0\}$, since P is of type a, Λ^{-a}_- is of type 0 and Λ^{-a}_+ is of type $-a$. Now construct Q^+_0 and Q^-_0 from the symbols q^+_0 and q^-_0 (cf. Theorem 2.6, recall that $s_0 = q_0(x, 0, 1)$), and denote

$$Q - s_0 Q^-_0 Q^+_0 = R_1, \quad R = \Lambda^a_- R_1 \Lambda^a_+. \tag{3.20}$$

Here R_1 has order -1, as a generalized ψdo, with symbol in $S^{-1}_{1,0}(\mathbb{R}^n, \mathbb{R}^{n-1}, \mathcal{H}_{-1})$. Indeed, R_1 has the symbol $q - q_0 + q_0 - q^-_0 \# q^+_0$, where $q - q_0$ is a ψdo symbol of order -1 of type 0, and

$$q_0 - q^-_0 \# q^+_0 \sim \sum_{|\alpha|\geq 1} \partial^\alpha_\xi q^-_0 D^\alpha_x q^+_0 / \alpha!,$$

where differentiation with respect to ξ removes the term 1 in q^-_0 and lowers the order, so that the resulting symbol is in $S^{-1}_{1,0}(\mathbb{R}^n, \mathbb{R}^{n-1}, \mathcal{H}_{-1})$.

For the main part of the operator $P_1 = P - R$ we use the factorization

$$P_1 = \Lambda^a_- s_0 Q^-_0 Q^+_0 \Lambda^a_+ = P^- P^+, \quad P^- = \Lambda^a_- s_0 Q^-_0,\ P^+ = Q^+_0 \Lambda^a_+; \tag{3.21}$$

here P^- is a minus-operator, preserving support in $\overline{\mathbb{R}}^n_-$, and P^+ is a plus-operator, preserving support in $\overline{\mathbb{R}}^n_+$. Then we have the decompositions

$$P = P^-P^+ + R, \quad P^* = P^{+*}P^{-*} + R^*.$$

Let us first treat $P_1 = P^-P^+$. We define

$$\begin{gathered} w = r^+P^+u, \quad w' = r^+P^{-*}u', \text{ then} \\ r^+P_1u = r^+P^-e^+r^+P^+u = r^+P^-e^+w, \quad r^+P^*u = r^+P^{+*}e^+r^+P^{-*}u' = r^+P^{+*}e^+w', \end{gathered}$$

as in [G15] Th. 4.2. Here $r^+P_1u, r^+P_1^*u' \in \overline{H}^{\frac12-a+\varepsilon}(\mathbb{R}^n_+)$, and, as noted further above, $u, u' \in \dot H^{a+\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+)$ with $\partial_nu, \partial_nu' \in \dot H^{a-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+)$.

Define $v = r^+\Xi^a_+u$, $v_1 = r^+\Lambda^a_+u$, and recall that by the definition of $H^{a(a+\frac12+\varepsilon)}(\overline{\mathbb{R}}^n_+)$ in [G15],

$$v, v_1 \in \overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+), \text{ with } u = \Xi^{-a}_+e^+v = \Lambda^{-a}_+e^+v_1. \tag{3.22}$$

For w we have that $w = r^+Q^+_0\Lambda^a_+u = r^+Q^+_0e^+v_1$. Here $e^+v_1 \in e^+\overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+) \subset \dot H^{\frac12-\varepsilon'}(\overline{\mathbb{R}}^n_+)$ (any $\varepsilon' > 0$), which allows the conclusion that $w \in \overline{H}^{\frac12-\varepsilon'}(\mathbb{R}^n_+)$, but we need to show that $w \in \overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)$ (and similarly for w'). To do this, we shall use a (rough) parametrix $\widetilde{Q}^-_0 = \mathrm{Op}(1/q^-_0)$ of Q^-_0, cf. Theorem 2.7. It is a minus-operator that satisfies

$$\widetilde{Q}^-_0\, Q^-_0 = I + R_2, \tag{3.23}$$

where R_2 is a minus-operator with symbol in $S^{-1}_{1,0}(\mathbb{R}^n, \mathbb{R}^{n-1}, \mathcal{H}_{-1})$. Denote $r^+P_1u = f$, and recall that $f = r^+\Lambda^a_-s_0Q^-_0e^+w$. Let

$$w_1 = r^+(\widetilde{Q}^-_0s_0^{-1}\Lambda^{-a}_-)e^+f;$$

then since $r^+(\widetilde{Q}^-_0s_0^{-1}\Lambda^{-a}_-)e^+ \colon \overline{H}^s(\mathbb{R}^n_+) \to \overline{H}^{s+a}(\mathbb{R}^n_+)$ for all s, $w_1 \in \overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)$. Now

$$w_1 - w = r^+(\widetilde{Q}^-_0s_0^{-1}\Lambda^{-a}_-)e^+r^+(\Lambda^a_-s_0Q^-_0)e^+w - w = r^+(\widetilde{Q}^-_0Q^-_0)e^+w - w = r^+R_2e^+w;$$

where we used that e^+r^+ in the middle can be left out since the operators are minus-operators, that $\Lambda^{-a}_-\Lambda^a_- = I$, and that (3.23) holds. Here $r^+R_2e^+$ maps w into $\overline{H}^{\frac32-\varepsilon'}(\mathbb{R}^n_+)$, by Theorem 2.4. It follows that $w \in \overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+)$. A similar proof shows this for w'.

Now we can write

$$I_1 \equiv \langle r^+P_1u, \partial_nu'\rangle_{\overline{H}^{\frac12-a+\varepsilon}, \dot H^{a-\frac12-\varepsilon}} = \langle r^+P^-e^+w, \partial_nu'\rangle_{\overline{H}^{\frac12-a+\varepsilon}, \dot H^{a-\frac12-\varepsilon}}.$$

Since $r^+P^-e^+ \colon \overline{H}^{\frac12+\varepsilon}(\mathbb{R}^n_+) \to \overline{H}^{\frac12-a+\varepsilon}(\mathbb{R}^n_+)$ and $P^{-*} \colon \dot H^{a-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+) \to \dot H^{-\frac12-\varepsilon}(\overline{\mathbb{R}}^n_+)$ are adjoints,

$$I_1 = \langle w, P^{-*}\partial_nu'\rangle_{\overline{H}^{\frac12+\varepsilon}, \dot H^{-\frac12-\varepsilon}}.$$

We use here that u' is zero at $x_n = 0$, so that $\partial_n u' = e^+ r^+ \partial_n u'$ (one may identify $\partial_n u'$ with $r^+\partial_n u'$).

The distribution $P^{-*}\partial_n u' \in \dot H^{-\frac12-\varepsilon}(\overline{\mathbb R}^n_+)$ is rewritten as follows:

$$P^{-*}\partial_n u' = \partial_n P^{-*}u' + [P^{-*},\partial_n]u' = \partial_n e^+ w' + P^{-*(n)}u',$$

with the notation $[P^{-*},\partial_n] = P^{-*(n)}$ as in (3.16). Here, as in Theorem 3.1,

$$\partial_n e^+ w' = \gamma_0(w')\otimes\delta(x_n) + e^+\partial_n w',$$

where we moreover note that since $w' \in \overline{H}^{\frac12+\varepsilon}(\overline{\mathbb R}^n_+)$, $e^+\partial_n w'$ is not just in $\dot H^{-\frac12-\varepsilon}(\overline{\mathbb R}^n_+)$, but is in $\dot H^{-\frac12+\varepsilon}(\overline{\mathbb R}^n_+) \simeq \overline{H}^{-\frac12+\varepsilon}(\mathbb R^n_+)$, and $\gamma_0 w' \in H^\varepsilon(\mathbb R^{n-1})$. Insertion of the expressions in I_1 and integration by parts as in Theorem 3.1 gives:

$$\begin{aligned} I_1 &= \langle w, \gamma_0(w')\otimes\delta(x_n) + e^+\partial_n w' + P^{-*(n)}u'\rangle_{\overline{H}^{\frac12-\varepsilon},\dot H^{-\frac12+\varepsilon}} \\ &= (\gamma_0 w,\gamma_0 w')_{L_2(\mathbb R^{n-1})} + \langle w,\partial_n w'\rangle_{\overline{H}^{\frac12-\varepsilon},\dot H^{-\frac12+\varepsilon}} + \langle w, P^{-*(n)}u'\rangle_{\overline{H}^{\frac12-\varepsilon},\dot H^{-\frac12+\varepsilon}}. \end{aligned} \tag{3.24}$$

It is shown in the same way (in fact it can be concluded from the above by interchanging P_1 and P_1^*, u and u', and conjugating), that

$$I_2 \equiv \langle \partial_n u, r^+P_1^* u'\rangle_{\dot H^{a-\frac12+\varepsilon},\overline{H}^{\frac12-a-\varepsilon}},$$

satisfies

$$I_2 = (\gamma_0 w,\gamma_0 w')_{L_2(\mathbb R^{n-1})} + \langle \partial_n w, w'\rangle_{\dot H^{-\frac12+\varepsilon},\overline{H}^{\frac12-\varepsilon}} + \langle P^{+(n)}u, w'\rangle_{\dot H^{-\frac12+\varepsilon},\overline{H}^{\frac12-\varepsilon}}, \tag{3.25}$$

where $P^{+(n)}$ stands for $[P^+,\partial_n]$ as in (3.16).

Taking the two contributions together, we find that

$$\begin{aligned} I_1+I_2 &= 2(\gamma_0 w,\gamma_0 w')_{L_2(\mathbb R^{n-1})} + \langle w,\partial_n w'\rangle_{\overline{H}^{\frac12-\varepsilon},\dot H^{-\frac12+\varepsilon}} + \langle \partial_n w, w'\rangle_{\dot H^{-\frac12+\varepsilon},\overline{H}^{\frac12-\varepsilon}} \\ &\quad + \langle w, P^{-*(n)}u'\rangle_{\overline{H}^{\frac12-\varepsilon},\dot H^{-\frac12+\varepsilon}} + \langle P^{+(n)}u, w'\rangle_{\dot H^{-\frac12+\varepsilon},\overline{H}^{\frac12-\varepsilon}} \\ &= (\gamma_0 w,\gamma_0 w')_{L_2(\mathbb R^{n-1})} + I_3, \text{ where} \\ I_3 &= \langle P^+u, P^{-*(n)}u'\rangle_{\overline{H}^{\frac12-\varepsilon},\dot H^{-\frac12+\varepsilon}} + \langle P^{+(n)}u, P^{-*}u'\rangle_{\dot H^{-\frac12+\varepsilon},\overline{H}^{\frac12-\varepsilon}}; \end{aligned} \tag{3.26}$$

here we used the calculation after (3.13) to reduce the first line to a single boundary integral, and collected the last two terms in I_3. This will now be further reduced.

Observe that $P^{+(n)}$ has symbol equal to $-\partial_{x_n}$ of the symbol of $P^+ = \Lambda^a_+ Q^+_0$, so it is a plus-operator, continuous from $\dot H^s(\overline{\mathbb R}^n_+)$ to $\dot H^{s-a}(\overline{\mathbb R}^n_+)$ for all s, with an adjoint $r^+(P^{+(n)})^*e^+$ going from $\overline{H}^{a-s}(\mathbb R^n_+)$ to $\overline{H}^{-s}(\mathbb R^n_+)$ for all s. $P^{-*(n)}$ has similar properties.

In particular, $P^{+(n)}u = P^{+(n)}\Xi_+^{-a}e^+v$ is in $\dot H^{\frac12-\varepsilon'}(\overline{\mathbb R}^n_+)$, cf. (3.22), and so is $P^{-*(n)}u'$, so the dualities in I_3 identify with $L_2(\mathbb R^n_+)$-scalar products:

$$I_3 = (P^+u, P^{-*(n)}u')_{L_2(\mathbb R^n_+)} + (P^{+(n)}u, P^{-*}u')_{L_2(\mathbb R^n_+)}.$$

The adjoint of $P^{-*(n)}$ is $r^+P^{-(n)}e^+$, since $[P^{-*},\partial_n]^* = [P^-,\partial_n]$. Then in view of the mapping properties,

(3.27)

$$\begin{aligned} I_3 &= \langle r^+P^{-(n)}e^+r^+P^+u, u'\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}} + \langle r^+P^-e^+r^+P^{+(n)}u, u'\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}\\ &= \langle r^+(P^{-(n)}e^+r^+P^+ + r^+P^-e^+r^+P^{+(n)})u, u'\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}. \end{aligned}$$

We now use moreover, that

$$r^+P^-e^+r^+P^{+(n)}u = r^+P^-P^{+(n)}u,\quad r^+P^{-(n)}e^+r^+P^+u = r^+P^{-(n)}P^+u$$

(because of the support-preserving properties, as in [G15] Th. 4.2), so that

$$I_3 = \langle r^+(P^-P^{+(n)} + P^{-(n)}P^+)u, u'\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}.$$

Here we can perform a little calculation on the ψdo's on $\mathbb R^n$:

$$\begin{aligned} P^-P^{+(n)} + P^{-(n)}P^+ &= P^-P^+\partial_n - P^-\partial_nP^+ + P^-\partial_nP^+ - \partial_nP^-P^+\\ &= P^-P^+\partial_n - \partial_nP^+P^- = P_1^{(n)}, \end{aligned}\tag{3.28}$$

showing that in fact

$$I_3 = \langle r^+P_1^{(n)}u, u'\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}.$$

Inserting this in (3.26), we reach the conclusion that

$$I_1 + I_2 = (\gamma_0(P^+u), \gamma_0(P^{-*}u'))_{L_2(\mathbb R^{n-1})} + \langle r^+P_1^{(n)}u, u'\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}.\tag{3.29}$$

The boundary term can be further clarified as follows: Let $v = r^+\Xi_+^a u$ as in (3.22). We know from [G15] (cf. e.g. Cor. 5.3) that $\gamma_0 v = \Gamma(a+1)\gamma_0((x_n^{-a})u)$. In view of Theorem 2.5, we have that

$$q_0^+(x,\xi',\xi_n) = 1 + f(x,\xi',\xi_n)$$

where f is in $\mathcal H^+$ as a function of ξ_n; $f\in S^0_{1,0}(\mathbb R^n,\mathbb R^{n-1},\mathcal H^+)$. Hence

$$Q_0^+ = I + F,\quad F = \mathrm{Op}(f).$$

Moreover, $\Lambda_+^a = (1+\Psi)\Xi_+^a$, where Ψ has symbol $\psi(\xi)$ in $\mathcal H^+$ with respect to ξ_n, cf. [G15] (1.16) and Lemma 6.6; $\psi\in S^0_{1,0}(\mathbb R^n,\mathbb R^{n-1},\mathcal H^+)$. It follows that

$$Q_0^+\Lambda_+^a = (I+F)(I+\Psi)\Xi_+^a = (I+F_1)\Xi_+^a,$$

where F_1 has symbol $f_1\in S^0_{1,0}(\mathbb{R}^n,\mathbb{R}^{n-1},\mathcal{H}^+)$. (One could also deal with the factors $I+F$ and $I+\Psi$ in two successive steps, to avoid using Leibniz products.) By the rules of the Boutet de Monvel calculus,

$$\gamma_0(F_1v)=(2\pi)^{-n}\int_{\mathbb{R}^{n-1}}e^{ix'\cdot\xi'}\int_{\mathbb{R}}f_1(x,\xi',\xi_n)\mathcal{F}(e^+v(x',x_n))\,d\xi_nd\xi'=0. \tag{3.30}$$

(Briefly recalled, the reason is that both f_1 and $\mathcal{F}(e^+v(x',x_n))$ are in $\mathcal{H}^+$ as functions of ξ_n — the latter because e^+v is supported in $\overline{\mathbb{R}}^n_+$; then their product is $O(\langle\xi_n\rangle^{-2})$ and holomorphic in $\mathbb{C}_-$, so the integral over $\mathbb{R}$ can be transformed to a closed contour in $\mathbb{C}_-$ and therefore vanishes.) It follows that

$$\gamma_0(P^+u)=\gamma_0(Q_0^+\Lambda^a_+u)=\gamma_0(v+F_1v)=\gamma_0v=\gamma_0(\Xi^a_+u)=\Gamma(a+1)\gamma_0(x_n^{-a}u). \tag{3.31}$$

As a slight variant, we also have, with $v'=\Xi^a_+u'$:

$$\begin{aligned}\gamma_0(P^{-*}u')&=\gamma_0(Q_0^{-*}\bar s_0\Lambda^a_+u')=\gamma_0(Q_0^{-*}\bar s_0(I+\Psi)v')=\gamma_0(\bar s_0v'+F_2v')\\&=\gamma_0(\bar s_0v')=\bar s_0\Gamma(a+1)\gamma_0(x_n^{-a}u'),\end{aligned} \tag{3.32}$$

where $Q_0^{-*}\bar s_0(I+\Psi)=\bar s_0I+F_2$, and also F_2 has symbol in $\mathcal{H}^+$ w.r.t. ξ_n, hence does not contribute. (Recall that s_0 is a function of x, namely $s_0(x)=q_0(x,0,1)=p_0(x,0,1)$; in the final formula it is just its value on $\{x_n=0\}$ that enters.)

We conclude that

$$(\gamma_0(P^+u),\gamma_0(P^{-*}u'))_{L_2(\mathbb{R}^{n-1})}=\Gamma(a+1)^2\int_{\mathbb{R}^{n-1}}\gamma_0(x_n^{-a}u)\,s_0\gamma_0(x_n^{-a}\bar u')\,dx', \tag{3.33}$$

whereby

$$I_1+I_2=\Gamma(a+1)^2\int_{\mathbb{R}^{n-1}}s_0\,\gamma_0(x_n^{-a}u)\,\gamma_0(x_n^{-a}\bar u')\,dx'+\langle r^+P_1^{(n)}u,u'\rangle_{\overline{H}^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}. \tag{3.34}$$

Finally, we must also treat the contribution from $R=\Lambda^a_-R_1\Lambda^a_+$. As already noted, the symbol $r_1(x,\xi)$ of R_1 is in $\mathcal{H}_{-1}$ as a function of ξ_n, so we can apply the projections h^+ and h^-, decomposing

$$r_1(x,\xi)=r_1^+(x,\xi)+r_1^-(x,\xi),\quad r_1^\pm\in S^{-1}_{1,0}(\mathbb{R}^n,\mathbb{R}^{n-1},\mathcal{H}^\pm_{-1}). \tag{3.35}$$

Denote the hereby defined operators $R_1^\pm$; $R_1=R_1^++R_1^-$. Then when we set $S^-=\Lambda^a_-R_1^-$, $S^+=R_1^+\Lambda^a_+$, R is decomposed as

$$R=\Lambda^a_-R_1^-\Lambda^a_++\Lambda^a_-R_1^+\Lambda^a_+=S^-\Lambda^a_++\Lambda^a_-S^+; \tag{3.36}$$

a sum of two operators that are products of a minus-operator and a plus-operator. To each of these products, we can apply the same method as we did to P^-P^+. This reduces the corresponding integrals to scalar products over the boundary plus commutator contributions:

$$\begin{aligned}I_4&\equiv\langle r^+Ru,\partial_nu'\rangle_{\overline{H}^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}+\langle\partial_nu,r^+R^*u'\rangle_{\dot H^{a-\frac12+\varepsilon},\overline{H}^{\frac12-a-\varepsilon}}\\&=(\gamma_0(\Lambda^a_+u),\gamma_0(S^{-*}u'))_{L_2(\mathbb{R}^{n-1})}+\langle\Lambda^a_+u,S^{-*(n)}u'\rangle_{\overline{H}^{\frac12-\varepsilon},\dot H^{-\frac12+\varepsilon}}\\&\quad+(\gamma_0(S^+u),\gamma_0(\Lambda^a_+u'))_{L_2(\mathbb{R}^{n-1})}+\langle S^{+(n)}u,\Lambda^a_+u'\rangle_{\dot H^{-\frac12+\varepsilon},\overline{H}^{\frac12-\varepsilon}}.\end{aligned} \tag{3.37}$$

(The dualities in the second and third line reduce to L_2-scalar products since S^- and S^+ are of negative order.) Since R_1^{-*} and R_1^+ have symbols in $\mathcal{H}^+$ as functions of ξ_n, the boundary values of $S^{-*}u'$ and S^+u are zero, so only the commutator terms survive. These are reduced in a similar way as in the treatment of P_1, to give

$$I_4 = (r^+ R^{(n)}u, u')_{L_2(\mathbb{R}^n_+)}.$$

Collecting all the terms, we find (3.18). As accounted for in the beginning of the proof, it can be written in the form (3.17) when $u, u' \in H^{a(a+1)}(\overline{\mathbb{R}}^n_+)$. $\square$

Observe in particular:

Corollary 3.4. *For $u, u' \in H^{a(s)}(\overline{\mathbb{R}}^n_+)$ with $s > a + \frac{1}{2}$,*

$$(3.38)\quad \langle r^+(-\Delta)^a u, \partial_n u'\rangle_{\overline{H}^{\frac12 - a+\varepsilon}, \dot{H}^{a-\frac12-\varepsilon}} + \langle \partial_n u, r^+(-\Delta)^a u'\rangle_{\dot{H}^{a-\frac12-\varepsilon}, \overline{H}^{\frac12-a+\varepsilon}} = \Gamma(a+1)^2 \int_{\mathbb{R}^{n-1}} \gamma_0(x_n^{-a}u)\, \gamma_0(x_n^{-a}\bar{u}')\, dx'.$$

for small $\varepsilon > 0$. When $s \geq a+1$, this can be written as

$$(3.39)\quad \int_{\mathbb{R}^n_+} (-\Delta)^a u\, \partial_n \bar{u}'\, dx + \int_{\mathbb{R}^n_+} \partial_n u\, (-\Delta)^a \bar{u}'\, dx \;=\; \Gamma(a+1)^2 \int_{\mathbb{R}^{n-1}} \gamma_0(x_n^{-a}u)\, \gamma_0(x_n^{-a}\bar{u}')\, dx'.$$

Proof. Write

$$(3.40)\qquad (-\Delta)^a = P + \mathcal{S},\ \text{ where } P = \mathrm{Op}(\eta(\xi)|\xi|^{2a}),\ \mathcal{S} = \mathrm{Op}((1-\eta(\xi))|\xi|^{2a});$$

$\eta(\xi)$ denoting an excision function as in (2.11). Then P satisfies the hypotheses of Theorem 3.3, so (3.38) holds for this operator.

Now consider $\mathcal{S}$. Its symbol $s(\xi) = (1-\eta(\xi))|\xi|^{2a}$ is bounded and supported in $\overline{B}_1 = \{|\xi| \leq 1\}$. The same holds for all the symbols $s_\alpha = \xi^\alpha(1-\eta(\xi))|\xi|^{2a}$, $\alpha \in \mathbb{N}_0^n$, so they all define bounded operators in $H^t(\mathbb{R}^n)$, for all $t \in \mathbb{R}$. Since $\mathrm{Op}(s_\alpha) = D^\alpha \mathcal{S} = \mathcal{S}D^\alpha$, we see that $\mathcal{S}$ and its compositions with D^α are smoothing operators, going from $H^\infty(\mathbb{R}^n) = \bigcup_t H^t(\mathbb{R}^n)$ to $H^{-\infty}(\mathbb{R}^n) = \bigcap_t H^t(\mathbb{R}^n)$.

Recall from (3.9) that $u \in \dot{H}^{a+\frac12-\varepsilon'}(\overline{\mathbb{R}}^n_+)$, $\partial_n u \in \dot{H}^{a-\frac12-\varepsilon'}(\overline{\mathbb{R}}^n_+)$ for any $\varepsilon' > 0$; here we can choose ε' so that $\sigma = a - \frac12 - \varepsilon' \in \,]-\frac12, \frac12[$. Then $\mathcal{S}u \in H^{-\infty}(\mathbb{R}^n)$; and

$$(3.41)\qquad \langle r^+\mathcal{S}u, \partial_n u'\rangle_{\overline{H}^{-\sigma}, \dot{H}^\sigma} = \langle \mathcal{S}u, \partial_n u'\rangle_{H^{-\sigma}(\mathbb{R}^n), H^\sigma(\mathbb{R}^n)},$$

since $\mathcal{S}u = e^+r^+\mathcal{S}u + e^-r^-\mathcal{S}u$, where the terms are in $\dot{H}^{|\sigma|} = \overline{H}^{|\sigma|}$ over $\overline{\mathbb{R}}^n_+$ resp. $\overline{\mathbb{R}}^n_-$, and $e^-r^-\mathcal{S}u$ vanishes on $\partial_n u'$. In the last expression in (3.41), ∂_n can be moved to the left-hand side with a minus, and $\mathcal{S}$ can be moved to the right-hand side replaced by $\mathcal{S}^*$ (of the same type), with suitable adaptation of the duality indications. Then we find that

$$\langle r^+\mathcal{S}u, \partial_n u'\rangle_{\overline{H}^{-\sigma}, \dot{H}^\sigma} + \langle \partial_n u, r^+\mathcal{S}^* u'\rangle_{\dot{H}^\sigma, \overline{H}^{-\sigma}} = 0,$$

and when this is added to the integration by parts formula for P, we find (3.38).

When $s \geq a+1$, then u and $\partial_n u$ are functions, and so are r^+Pu and $\mathcal{S}u$, with similar statements for u'. Then the formula can be written as in (3.39). $\square$

4. Integration by parts over bounded smooth domains

In this part, we consider a classical ψdo P of order $2a$ on $\mathbb{R}^n$ and its restriction to a bounded smooth subset Ω. Assuming that the symbol is even (cf. (2.40)), we have that it satisfies the a-transmission condition in any direction at all points, hence at the boundary of any choice of Ω. The indications $r^\pm$ and $e^\pm$ now pertain to the embedding $\Omega\subset\mathbb{R}^n$.

We begin with a simple integration-by-parts formula, that can be shown by reduction to operators of order 0.

Theorem 4.1. *Let P be a classical ψdo on $\mathbb{R}^n$ of order $2a$ for some $a>0$, with even symbol. Then for $u,u'\in H^{a(s)}(\overline\Omega)$, $s\ge a$,*

$$\langle r^+Pu,u'\rangle_{\overline H^{-a}(\Omega),\dot H^a(\overline\Omega)}-\langle u,r^+P^*u'\rangle_{\dot H^a(\overline\Omega),\overline H^{-a}(\Omega)}=0; \tag{4.1}$$

when $s\ge 2a$, this can also be written

$$\int_\Omega Pu\,\bar u'\,dx-\int_\Omega u\,\overline{P^*u'}\,dx=0. \tag{4.2}$$

Proof. We shall apply the families of order-reducing operators $\Lambda_+^{(t)}$ and $\Lambda_{-,+}^{(t)}$, $t\in\mathbb{R}$, introduced in [G15] and recalled in the Appendix, chosen such that $\Lambda_{-,+}^{(t)}\colon\overline H_p^s(\Omega)\to\overline H_p^{s-a}(\Omega)$ and $\Lambda_+^{(t)}\colon\dot H_{p'}^{a-s}(\overline\Omega)\to\dot H_{p'}^{-s}(\overline\Omega)$ are adjoints for all $s\in\mathbb{R}$. Recall that $H_p^{a(s)}(\overline\Omega)=\Lambda_+^{(-a)}e^+\overline H_p^{s-a}(\Omega)$. We restrict the attention to the case $p=2$.

Since P is even, it has the a-transmission property at any boundary; then the operator

$$Q=\Lambda_-^{(-a)}P\Lambda_+^{(-a)}, \tag{4.3}$$

is a ψdo of order 0 having the 0-transmission property at the boundary of Ω. Recall that r^+P maps $H^{a(s)}(\overline\Omega)$ continuously into $\overline H^{s-2a}(\Omega)$ for all $s>a-\frac12$, cf. [G15] Th. 4.2.

Let

$$w=r^+\Lambda_+^{(a)}u,\quad w'=r^+\Lambda_+^{(a)}u';$$

they are in $\overline H^{s-a}(\Omega)$, which identifies with a subset of $L_2(\Omega)$ since $s\ge a$. Then

$$u=\Lambda_+^{(-a)}e^+w,\; u'=\Lambda_+^{(-a)}e^+w'\in\dot H^a(\overline\Omega)$$

(using that $\Lambda_+^{(-a)}$ lifts $e^+L_2(\Omega)$ to $\dot H^a(\overline\Omega)$), and r^+Pu, $r^+P^*u'\in\overline H^{s-2a}(\Omega)\subset\overline H^{-a}(\Omega)$. (Since u is an L_2-function supported in $\overline\Omega$, we identify it with r^+u.) Moreover (cf. [G15]),

$$\begin{aligned}r^+Pu&=r^+\Lambda_-^{(a)}e^+r^+Q\Lambda_+^{(a)}u=r^+\Lambda_-^{(a)}e^+r^+Qe^+w,\\ r^+P^*u'&=r^+\Lambda_-^{(a)}e^+r^+Q^*\Lambda_+^{(a)}u'=r^+\Lambda_-^{(a)}e^+r^+Q^*e^+w'.\end{aligned} \tag{4.4}$$

Now since $r^+\Lambda_-^{(a)}e^+$ and $\Lambda_+^{(a)}$ are adjoints,

$$\langle r^+Pu,u'\rangle_{\overline H^{-a},\dot H^a}=\langle r^+\Lambda_-^{(a)}e^+r^+Qe^+w,\Lambda_+^{(-a)}w'\rangle_{\overline H^{-a},\dot H^a}=(r^+Qe^+w,w')_{L_2(\Omega)}.$$

There is a similar formula for P^*, so we find

(4.5)
$$\langle r^+Pu,u'\rangle_{\overline{H}^{-a},\dot H^a}-\langle u,r^+P^*u'\rangle_{\dot H^a,\overline{H}^{-a}}=(r^+Qe^+w,w')_{L_2(\Omega)}-(w,r^+Q^*e^+w')_{L_2(\Omega)}.$$

Since Q is of order 0, the adjoint of r^+Qe^+ in $L_2(\Omega)$ is $r^+Q^*e^+$, and

$$(r^+Qe^+w,w')_{L_2(\Omega)}-(w,r^+Q^*e^+w')_{L_2(\Omega)}=0. \tag{4.6}$$

This shows (4.1).

When $s\ge 2a$, r^+Pu and $r^+P^*u'\in L_2(\Omega)$, so the formula can be written as in (4.2). □

The formula can be extended to suitable $L_p,L_{p'}$-dualities.

Our main aim is to show extensions of the integration-by-parts formula in Theorem 3.3 to the curved situation.

First there is a result in the spirit of Theorem 3.1.

Theorem 4.2. *Let P^- be an operator of order a (i.e., continuous from $H^s(\mathbb{R}^n)$ to $H^{s-a}(\mathbb{R}^n)$ for all $s\in\mathbb{R}$) such that $r^+P^-e^+$ maps $\overline{H}^s(\Omega)$ continuously to $\overline{H}^{s-a}(\Omega)$ with adjoint $P^{-*}\colon \dot H^{a-s}(\overline\Omega)\to\dot H^{-a}(\overline\Omega)$ for all $s\in\mathbb{R}$. Assume that the commutator*

$$P^{-(j)}=P^-\partial_j-\partial_jP^-$$

has similar mapping properties. Let $w,w'\in\overline{H}^s(\overline\Omega)$ with $s\ge\frac12+\varepsilon$ for some small $\varepsilon>0$, and assume that $w'=r^+P^{-*}u'$ for some $u'\in H^{a(s+a)}(\overline\Omega)$ with $P^{-*}u'=e^+w'$. Then

$$\langle r^+P^-e^+w,\partial_ju'\rangle_{\overline{H}^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}=\int_{\partial\Omega}\nu_j\,\gamma_0w\,\gamma_0\bar w'\,d\sigma+\langle w,\partial_jw'\rangle_{\overline{H}^{\frac12-\varepsilon},\dot H^{-\frac12+\varepsilon}}+(w,P^{-*(j)}u')_{L_2(\Omega)}, \tag{4.7}$$

where $\nu_j(x)$ is the j'th component of the interior normal vector $\nu(x)$ at $x\in\partial\Omega$.

Proof. Recall the standard Gauss-Green formula

$$-\int_\Omega\partial_j\varphi\,dx=\int_{\partial\Omega}\nu_j\gamma_0\varphi\,d\sigma, \tag{4.8}$$

where $\gamma_0\varphi$ is the restriction of φ to $\partial\Omega$ and $d\sigma$ is the induced measure on $\partial\Omega$; it holds for sufficiently regular functions φ. We can write it as a distribution formula on $\mathbb{R}^n$ (with sesquiliear duality):

$$\langle\partial_j1_\Omega,\varphi\rangle_{\mathbb{R}^n}=-\langle 1_\Omega,\partial_j\varphi\rangle_{\mathbb{R}^n}=\langle 1,\nu_j\widetilde\gamma_0\varphi\rangle_{\partial\Omega}\text{ for }\varphi\in C_0^\infty(\mathbb{R}^n), \tag{4.9}$$

where the last brackets is a duality over $\partial\Omega$ consistent with the scalar product in $L_2(\partial\Omega,d\sigma)$. For accuracy, we denote by $\widetilde\gamma_0$ the restriction operator going from functions on $\mathbb{R}^n$ to functions on $\partial\Omega$ (sometimes called the two-sided trace operator); it is this one that has nice adjoint properties. In fact,

$$\widetilde\gamma_0\colon H^s(\mathbb{R}^n)\to H^{s-\frac12}(\partial\Omega)\text{ has an adjoint }\widetilde\gamma_0^*\colon H^{\frac12-s}(\partial\Omega)\to H^{-s}(\mathbb{R}^n)\text{ for }s>\tfrac12, \tag{4.10}$$

and (4.9) shows that $\partial_j 1_\Omega = \widetilde{\gamma}_0^* \nu_j$.

There is also a version with two functions W and φ: When $W \in C_0^\infty(\mathbb{R}^n)$, $\partial_j(1_\Omega W) = (\partial_j 1_\Omega)W + 1_\Omega \partial_j W$, so

$$\begin{aligned}\langle \partial_j(1_\Omega W) - 1_\Omega \partial_j W, \varphi\rangle_{\mathbb{R}^n} &= \langle (\partial_j 1_\Omega)W, \varphi\rangle_{\mathbb{R}^n} = \langle \partial_j 1_\Omega, \overline{W}\varphi\rangle_{\mathbb{R}^n} = \langle 1, \nu_j \widetilde{\gamma}_0(\overline{W}\varphi)\rangle_{\partial\Omega} \\ &= \langle 1, \nu_j \widetilde{\gamma}_0(\overline{W})\,\widetilde{\gamma}_0(\varphi)\rangle_{\partial\Omega} = \langle \nu_j \widetilde{\gamma}_0(W), \widetilde{\gamma}_0\varphi\rangle_{\partial\Omega} = \langle \widetilde{\gamma}_0^*(\nu_j \widetilde{\gamma}_0(W)), \varphi\rangle_{\mathbb{R}^n},\end{aligned}$$

showing that

$$\partial_j(1_\Omega W) = 1_\Omega \partial_j W + \widetilde{\gamma}_0^*(\nu_j \widetilde{\gamma}_0(W)).$$

Setting $r^+W = w$, we find the formula

$$\partial_j e^+ w = e^+ \partial_j w + \widetilde{\gamma}_0^*(\nu_j \gamma_0 w). \tag{4.11}$$

It extends by continuity to more general functions, namely $w \in \overline{H}^{\frac12+\varepsilon}(\Omega)$ with $\gamma_0 w \in H^\varepsilon(\partial\Omega)$.

For the left-hand side in (4.7) we then find:

$$\begin{aligned}&\langle r^+P^-e^+w, \partial_j u'\rangle_{\overline{H}^{\frac12-a+\varepsilon}, \dot{H}^{a-\frac12-\varepsilon}} = \langle w, P^{-*}\partial_j u'\rangle_{\overline{H}^{\frac12+\varepsilon}, \dot{H}^{-\frac12-\varepsilon}} \\ &= \langle w, \partial_j P^{-*}u' + P^{-*(j)}u'\rangle_{\overline{H}^{\frac12+\varepsilon}, \dot{H}^{-\frac12-\varepsilon}} = \langle w, \partial_j e^+w' + P^{-*(j)}u'\rangle_{\overline{H}^{\frac12+\varepsilon}, \dot{H}^{-\frac12-\varepsilon}} \\ &= \langle w, e^+\partial_j w' + \widetilde{\gamma}_0^*(\nu_j\gamma_0 w') + P^{-*(j)}u'\rangle_{\overline{H}^{\frac12+\varepsilon}, \dot{H}^{-\frac12-\varepsilon}} \\ &= \int_{\partial\Omega} \nu_j \gamma_0 w \gamma_0 \bar{w}'\, d\sigma + \langle w, e^+\partial_j w'\rangle_{\overline{H}^{\frac12-\varepsilon}, \dot{H}^{-\frac12+\varepsilon}} + (w, P^{-*(j)}u')_{L_2(\Omega)}.\end{aligned}$$

Here we used the information on adjoints and inserted (4.11) applied to w'; the dualitiy indications could be changed since $e^+\partial_j w'$ and $P^{-*(j)}u'$ lie in better spaces $\dot{H}^{-\frac12+\varepsilon}$, resp. $\dot{H}^{\frac12+\varepsilon}$. $\square$

To treat the full problem, we shall use local coordinates.

Let Ω be a smooth bounded subset of $\mathbb{R}^n$. Then $\overline{\Omega}$ has a finite cover by bounded open sets $U_0, \dots, U_{I_0}$ with diffeomorphisms $\kappa_i : U_i \to V_i$, V_i bounded open in $\mathbb{R}^n$, such that $U_i \cap \Omega$ is mapped to $V_i \cap \mathbb{R}^n_+$ and $U_i \cap \partial\Omega$ is mapped to $V_i \cap \partial\overline{\mathbb{R}}^n_+$; as usual we write $\partial\overline{\mathbb{R}}^n_+ = \mathbb{R}^{n-1}$. When P is a ψdo on $\mathbb{R}^n$, its application to functions supported in U_i carries over to functions on V_i as a ψdo $\widetilde{P}^{(i)}$ defined by

$$\widetilde{P}^{(i)} v = P(v \circ \kappa_i) \circ \kappa_i^{-1}, \quad v \in C_0^\infty(V_i). \tag{4.12}$$

Remark 4.3. A useful choice near $\partial\Omega$ is where we provide the $(n-1)$-dimensional manifold $\partial\Omega$ with coordinate charts $\kappa_i' : U_i' \to V_i' \subset \mathbb{R}^{n-1}$, $i = 1, \dots, I_0$, and consider a tubular neighborhood $\Sigma_r = \{x' + t\nu(x') \mid x' \in \partial\Omega, |t| < r\}$, where $\nu(x') = (\nu_1(x'), \dots, \nu_n(x'))$ is the interior normal to $\partial\Omega$ at $x' \in \partial\Omega$, and r is taken so small that the mapping $x' + t\nu(x') \mapsto (x', t)$ is a diffeomorphism from Σ_r to $\partial\Omega \times\,]-r, r[$. Then for each coordinate patch κ_i', we can use the mapping $\kappa_i : x' + tn(x') \mapsto (\kappa_i'(x'), t)$ as the diffeomorphism in dimension n; κ_i goes from U_i to V_i, where

$$U_i = \{x' + tn(x') \mid x' \in U_i', |t| < r\}, \quad V_i = V_i' \times\,]-r, r[\,. \tag{4.13}$$

The advantage is that the normal $\nu(x')$ at $x' \in \partial\Omega$ is carried over to the normal $(0,1)$ at $(\kappa_i'(x'),0)$. Moreover, for points $x \in \Sigma_{r,+} = \Sigma_r \cap \Omega$, t is a good approximation to the distance function $d(x) = \operatorname{dist}(x,\partial\Omega)$; their difference goes to 0 for $t \to 0$.

We can supply these charts with a chart consisting of the identity mapping on an open set U_0 containing $\Omega \setminus \overline{\Sigma}_{r,+}$, with $\overline{U}_0 \subset \Omega$, to get a full cover of $\overline{\Omega}$.

Together with the cover by local coordinate charts there exists an associated partition of unity $\varphi_0, \dots, \varphi_{I_0}$ such that each φ_i is in $C_0^\infty(U_i)$ taking values in $[0,1]$, and $\sum_{0\le i\le i_0} \varphi_i(x) = 1$ for $x \in \overline{\Omega}$. It will be convenient in the following to have the more refined concept of a partition of unity *subordinate* to a system of local coordinates, where any two functions are supported in one of the U_i's. This fact was originally used in Seeley [S69], proofs are given (in more complicated cases) in [G96], Appendix, and [G09], Ch. 8. For the convenience of the reader we provide a proof here.

Lemma 4.4. *There exists a system of coordinate charts $\kappa_i \colon U_i \to V_i$, $i = 0, \dots, I_1$, and a subordinate partition of unity ϱ_j, $j = 1, \dots, J_0$ (with values in $[0,1]$ and sum 1 on $\overline{\Omega}$), such that for each pair $k, l \le J_0$ there is an $i = i(k,l) \le I_1$ such that* $\operatorname{supp}\varrho_k \cup \operatorname{supp}\varrho_l \subset U_i$.

Proof. We start out with an arbitrary cover by coordinate charts $\kappa_i \colon U_i \to V_i$, $i = 0, \dots, I_0$. By the compactness of $\overline{\Omega}$, there is a $\delta > 0$ such that any subset of $\overline{\Omega}$ with diameter $\le \delta$ is contained in one of the U_i's. Cover $\overline{\Omega}$ with a finite system of open balls B_j with radius $\le \delta/4$, $j = 1, \dots, J_0$. When B_{j_1} and B_{j_2} are two such balls, we have two possibilities:

1) If $B_{j_1} \cap B_{j_2} \ne \emptyset$, it has diameter $\le \delta$, hence lies in a set U_i, take the first such i. We shall adjoin the set $U' = B_{j_1} \cup B_{j_2}$ to our system, using the mapping κ_i to define a coordinate mapping κ' from U' to $V' = \kappa_i(B_{j_1} \cup B_{j_2})$.

2) If $B_{j_1} \cap B_{j_2} = \emptyset$, the balls lie in two possibly different sets U_{i_1} and U_{i_2} (take the first i_1 and first i_2 that occur); then we shall adjoin the coordinate neighborhood $U' = B_{j_1} \cup B_{j_2}$ to the given system using as coordinate transformation the mapping κ_{i_1} on B_{j_2} and κ_{i_2} on B_{j_2}. Here we may have to make a translation τ of the image $\kappa_{i_2}(B_{j_2})$ to make it disjoint from $\kappa_{i_i}(B_{j_1})$. In this way we get a coordinate chart κ' from U' to $V' = \kappa_{i_1}(B_{j_1}) \cup \tau\kappa_{i_2}(B_{j_2})$.

We do this for all pairs j_1, j_2 and enumerate the resulting coordinate charts $\kappa' \colon U' \to V'$ by numbers $i = I_0 + 1, \dots, I_1$; then we get an extended cover of $\overline{\Omega}$ by coordinate charts $\kappa_i \colon U_i \to V_i$, $i = 0, \dots, I_1$.

Finally, let ϱ_j, $j = 1, \dots, J_0$, be a partition of unity associated with the cover B_j, $j = 1, \dots, J_0$ (i.e. with $\varrho_j \in C_0^\infty(B_j)$ for each j), then any two functions ϱ_k, ϱ_l have their support in one of the open sets in the extended cover. □

We shall now show:

Theorem 4.5. *Let P be a classical elliptic ψdo on $\mathbb{R}^n$ of order $2a$ with even symbol, $0 < a < 1$. Then for $u, u' \in H^{a(s)}(\overline{\Omega})$ with $s \ge a+1$ there holds, for $j = 1, \dots, n$:*

$$\begin{aligned}\int_\Omega Pu\, \partial_j \bar u'\, dx + \int_\Omega \partial_j u\, \overline{P^* u'}\, dx \\ = \Gamma(a+1)^2 \int_{\partial\Omega} s_0 \nu_j \gamma_0(d^{-a}u)\, \gamma_0(d^{-a}\bar u')\, d\sigma + \int_\Omega P^{(j)} u\, \bar u'\, dx,\end{aligned} \tag{4.14}$$

where $s_0(x)$ is the value of the principal symbol of P at $(x, \nu(x))$ for $x \in \partial\Omega$, and $P^{(j)} = P\partial_j - \partial_j P$.

The term with $P^{(j)}$ vanishes if P is independent of x_j (in particular, when P is translation-invariant).

The formula extends to the case $s > a+\frac12$, with the integrals over Ω replaced by dualities:

$$\begin{aligned}\langle r^+Pu, \partial_j u'\rangle_{\overline{H}^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}} &+ \langle \partial_j u, P^*u'\rangle_{\dot H^{a-\frac12+\varepsilon},\overline{H}^{\frac12-a-\varepsilon}}\\ &= \Gamma(a+1)^2\int_{\partial\Omega}\nu_j s_0\gamma_0(x_n^{-a}u)\,\gamma_0(x_n^{-a}\bar u')\,d\sigma + \langle r^+P^{(j)}u, u'\rangle_{\overline{H}^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}};\end{aligned} \tag{4.15}$$

the last term is a scalar product $(P^{(j)}u, u')_{L_2(\Omega)}$ when $a\le\frac12$.

Proof. For a transparent notation, we formulate the proof in the case $s\ge a+1$; the extensions to dualities for lower s follow easily (as in Theorem 3.3).

Starting with a choice of coordinate charts as in Remark 4.3, we use Lemma 4.4 to extend it to a covering of $\overline\Omega$ with a system of coordinate patches $\kappa_i\colon U_i\to V_i\subset\mathbb R^n$, $i=0,\dots,I_1$, such that there is a subordinate partition of unity ϱ_j, $j=1,\dots,J_0$, where for any pair of indices $k,l\le J_0$ there is a U_i, $i=i(k,l)$, such that ϱ_k and ϱ_l have support in U_i. We can moreover choose real functions $\psi_k,\psi_l\in C_0^\infty(U_i)$ such that $\psi_k\varrho_k=\varrho_k$, $\psi_l\varrho_l=\varrho_l$ (i.e., they are 1 on the respective supports). Then

$$\begin{aligned}\int_\Omega(Pu\,\partial_j\bar u' + \partial_j u\,\overline{P^*u'})\,dx &= \sum_{k,l\le J_0}\int_\Omega(P\varrho_k u\,\partial_j\bar\varrho_l u' + \partial_j\varrho_k u\,\overline{P^*\varrho_l u'})\,dx\\ &= \sum_{k,l\le J_0}\int_\Omega(P\psi_k\varrho_k u\,\partial_j\psi_l\varrho_l\bar u' + \partial_j\psi_k\varrho_k u\,\overline{P^*\psi_l\varrho_l u'})\,dx\\ &= \sum_{k,l\le J_0}\int_\Omega(P_{kl}u_k\,\partial_j\bar u_l' + \partial_j u_k\,\overline{P_{kl}^*u_l'})\,dx,\end{aligned}\tag{4.16}$$

where

$$P_{kl}=\psi_l P\psi_k,\quad P_{kl}^*=\psi_k P^*\psi_l,\quad u_k=\varrho_k u,\quad u_l'=\varrho_l u'.\tag{4.17}$$

For each pair (k,l) we treat the term by use of the coordinate map for U_i, $i=i(k,l)$. Denote by $\widetilde P_{kl}$ the operator on $V_i\subset\mathbb R^n$ that P_{kl} carries over to; it has compact kernel support in $V_i\times V_i$. In detail, $\widetilde P_{kl}=\widetilde\psi_l^{(i)}\widetilde P^{(i)}\widetilde\psi_k^{(i)}$, cf. (4.12). The parity property of the symbol, hence the a-transmission property, is preserved under the coordinate transformation. By Theorem 2.7 applied to $\widetilde P^{(i)}$, $\widetilde P_{kl}$ has a decomposition into a product of $\pm$-factors and a lower-order term:

$$\widetilde P_{kl}=\widetilde P_{kl}^-\widetilde P_{kl}^+ + \widetilde S_{kl},\ \text{ in detail } \widetilde P_{kl}^-=\widetilde\psi_l^{(i)}\widetilde P^{(i)-},\ \widetilde P_{kl}^+=\widetilde P^{(i)+}\widetilde\psi_k^{(i)},\tag{4.18}$$

where $\widetilde P_{kl}^\pm$ preserve support in $\overline{\mathbb R}^n_\pm$, respectively, and $\widetilde S_{kl}$ is of order $2a-1$ with a structure like S in Theorem 3.3, with compact kernel support in $V_i\times V_i$. We can moreover assume that $\widetilde P_{kl}^\pm$ have compact kernel supports in $V_i\times V_i$ since multiplication by a smooth cutoff function that is 1 on the supports of $\widetilde\psi_k^{(i)}$, $\widetilde\psi_l^{(i)}$, changes the operator by a smoothing term.

Now all this is carried back to U_i by the coordinate transformation; $\widetilde P_{kl}^\pm$ are carried over to operators $P_{kl}^\pm$, and $\widetilde{\mathcal S}_{kl}$ is carried over to $\mathcal S_{kl}$. The property that $\widetilde P_{kl}^\pm$ preserve supports in $\overline{\mathbb{R}}^n_\pm$, respectively, carries over to the property that $P^\pm_{kl}$ preserve support in $\overline{\Omega}$ resp. $\complement\Omega$. Then we have the adjoint mapping properties (where r^+ and e^+ are defined relative to $\overline{\Omega}\subset\mathbb{R}^n$):

$$
\begin{aligned}
&r^+P^-_{kl}e^+\colon \overline{H}^s_p(\Omega)\to\overline{H}^{s-a}_p(\Omega) \text{ and } {P^-_{kl}}^*\colon \dot H^{a-s}_{p'}(\overline{\Omega})\to\dot H^{-s}_{p'}(\overline{\Omega}) \text{ are adjoints,}\\
&r^+{P^+_{kl}}^*e^+\colon \overline{H}^s_p(\Omega)\to\overline{H}^{s-a}_p(\Omega) \text{ and } P^+_{kl}\colon \dot H^{a-s}_{p'}(\overline{\Omega})\to\dot H^{-s}_{p'}(\overline{\Omega}) \text{ are adjoints.}
\end{aligned}
\tag{4.19}
$$

With this preparation, we can calculate as follows: Denote $r^+P^+_{kl}u_k=w$, $r^+{P^-_{kl}}^*u'_l=w'$. Then

$$
I=\int_{\Omega\cap U_i}(P^-_{kl}P^+_{kl}u_k\,\partial_ju'_l+\partial_ju_k\,\overline{{P^+_{kl}}^*{P^-_{kl}}^*u'_l})\,dx=\int_{\Omega\cap U_i}(P^-_{kl}e^+w\,\partial_ju'_l+\partial_ju_k\,\overline{{P^+_{kl}}^*e^+w'})\,dx.
$$

We apply Theorem 4.2 to the first term, and a conjugated variant to the second term, obtaining

$$
\begin{aligned}
I&=2\int_{\partial\Omega\cap U_i}\nu_j\,\gamma_0w\,\gamma_0\bar w'\,d\sigma+\int_{\Omega\cap U_i}(w\,\partial_j\bar w'+\partial_jw\,\bar w'+w\,\overline{[{P^-_{kl}}^*,\partial_j]u'_l}+[P^+_{kl},\partial_j]u_k\,\bar w')\,dx\\
&=\int_{\partial\Omega\cap U_i}\nu_j\,\gamma_0w\,\gamma_0\bar w'\,d\sigma+\int_{\Omega\cap U_i}[P^-_{kl}e^+r^+P^+_{kl},\partial_j]u_k\,\bar u'_l\,dx.
\end{aligned}
$$

For the second line it was used that $\int_{\Omega\cap U_i}(w\,\partial_j\bar w'+\partial_jw\,\bar w')\,dx'$ gives another copy of $\int_{\partial\Omega\cap U_i}\nu_j\,\gamma_0w\,\gamma_0\bar w'\,d\sigma$ with a minus sign, and the two terms with commutators were reduced to a single term as in the proof of Theorem 3.3.

For the term with S_{kl} we proceed as in Theorem 3.3, concluding that it gives no boundary contribution, only a commutator term that can be added to the one with $P^-_{kl}e^+r^+P^+_{kl}$.

This leads to the formula

$$
\int_\Omega(P_{kl}u_k\,\partial_j\bar u'_l+\partial_ju_k\,\overline{P^*_{kl}u'_l})\,dx=\int_{\partial\Omega\cap U_i}\nu_j\,\gamma_0w\,\gamma_0\bar w'\,d\sigma+\int_{\Omega\cap U_i}[P_{kl},\partial_j]u_k\,\bar u'_l\,dx.
\tag{4.20}
$$

The boundary contributions from $P^\pm_{kl}$ are found from the values of the functions in the localized situation. Here $\gamma_0(P^+_{kl}u_k)$ comes from

$$
\widetilde P^{(i)+}\widetilde\psi^{(i)}_k\,\widetilde{\varrho_ku}^{(i)}|_{x_n=0}=\widetilde P^{(i)+}\widetilde{\varrho_ku}^{(i)}|_{x_n=0}=\lim_{x_n\to0+}\Gamma(a+1)x_n^{-a}\,\widetilde{\varrho_ku}^{(i)},
$$

by calculations as in (3.31); recall that $\psi_k\varrho_k=\varrho_k$. This carries over to $\partial\Omega$ as $\Gamma(a+1)\lim_{d\to0}(d^{-a}\varrho_ku)$, since $\widetilde d^{(i)}/x_n\to1$ for $x_n\to0$. Similarly, cf. (3.32), $\gamma_0({P^-_{kl}}^*u'_l)=\Gamma(a+1)s_0\gamma_0(d^{-a}\varrho_lu')$. We conclude that

$$
\int_{\partial\Omega\cap U_i}\nu_j\gamma_0(P^+_{kl}u_k)\gamma_0(\overline{{P^-_{kl}}^*u'_l})\,d\sigma=\Gamma(a+1)^2\int_{\partial\Omega\cap U_i}\nu_js_0\gamma_0(d^{-a}\varrho_ku)\gamma_0(d^{-a}\varrho_l\bar u')\,d\sigma.
$$

We have then obtained:

$$
\begin{aligned}
&\int_{\Omega\cap U_i}(P_{kl}u_k\,\partial_j\bar u'_l+\partial_ju_k\,\overline{P^*_{kl}u'_l})\,dx\\
&=\Gamma(a+1)^2\int_{\partial\Omega\cap U_i}\nu_js_0\gamma_0(d^{-a}\varrho_ku)\gamma_0(d^{-a}\varrho_l\bar u')\,d\sigma+\int_{\Omega\cap U_i}[P_{kl},\partial_j]u_k\,\bar u'_l\,dx,
\end{aligned}
$$

for each pair (k,l), and when we sum over k and l, using that $\sum_k \varrho_k = \sum_l \varrho_l = 1$ on $\overline{\Omega}$, we find (4.14).

The extension to dualities in (4.15), when $s > a + \frac{1}{2}$, follows when one formulates the detailed study of P_{kl} in terms of dualities as in Theorem 3.3. $\square$

The validity extends to suitable Hölder spaces. To get a very efficient statement, we can apply the general result of [G14] Th. 4.2, Ex. 4.3, for Hölder-Zygmund spaces, showing that r^+P defines a Fredholm operator for $s > a - 1$:

$$r^+P : C_*^{a(s)}(\overline{\Omega}) \to \overline{C}_*^{s-2a}(\Omega). \tag{4.21}$$

There is also a regularity result stating that when $u \in \dot{C}_*^t(\overline{\Omega})$ for some $t > a - 1$ (in particular when $u \in e^+L_\infty(\Omega)$), then $r^+Pu \in \overline{C}_*^{s-2a}(\Omega)$ implies $u \in C_*^{a(s)}(\overline{\Omega})$. We recall that $\overline{C}_*^s(\Omega)$ equals the Hölder space $C^s(\overline{\Omega})$ when $s > 0$, $s \notin \mathbb{N}$; cf. also (A.3). Here $C_*^{a(s)} = \Lambda_+^{(-a)} e^+ \overline{C}_*^{s-a}(\Omega)$, where $\Lambda_+^{(t)}$ is an order-reducing operator on $\mathbb{R}^n$ preserving support in $\overline{\Omega}$, as recalled in (A.7) and used in the proof of Theorem 4.1. These operators apply also to C_*^s-spaces by [G14].

To assure that r^+Pu is bounded and $\partial_j u$ is integrable on Ω, we take $s = 1 + a + \varepsilon$ with $\varepsilon > 0$. Then $r^+Pu \in \overline{C}^{1-a+\varepsilon}(\Omega)$, and (when $1 + a + \varepsilon \notin \mathbb{N}$)

$$u \in C_*^{a(1+a+\varepsilon)}(\overline{\Omega}) \subset e^+ d^a \overline{C}^{1+\varepsilon}(\Omega), \tag{4.22}$$

with $\partial_j u \in e^+ d^{a-1} \overline{C}^{1+\varepsilon}(\Omega) + e^+ d^a \overline{C}^{\varepsilon}(\Omega) \subset L_1(\Omega)$. Since the various spaces are invariant under C^∞-coordinate changes, the proof of Theorem 4.5 carries through for such functions.

We have hereby obtained:

Corollary 4.6. *Formula* (4.14) *holds also when* $u, u' \in C_*^{a(1+a+\varepsilon)}(\overline{\Omega})$, *some* $\varepsilon > 0$.

This is assured when $u, u' \in e^+L_\infty(\Omega)$ *and* $r^+Pu, r^+P^*u' \in \overline{C}^{1-a+\varepsilon'}(\overline{\Omega})$ *(*$\varepsilon' = \varepsilon$ *when* $1 + a + \varepsilon \notin \mathbb{N}$, $\varepsilon' > \varepsilon$ *when* $1 + a + \varepsilon \in \mathbb{N}$*).*

The assumption on r^+Pu in the corollary is a little more general than the assumption in [RS14b, RSV15] which take $r^+Pu \in C^{0,1}(\overline{\Omega})$. On the other hand, these authors work under a weaker smoothness hypothesis on Ω (namely that it is $C^{1,1}$).

The assumptions in Theorm 4.5 are a considerable generalization.

The advantage of referring to $H^{a(s)}(\overline{\Omega})$ and $C_*^{a(s)}(\overline{\Omega})$ is that these scales of spaces do not depend on a choice of P, but are the appropriate solution spaces for the Dirichlet problem for *all* classical elliptic ψdo's P of order $2a$ with even symbol.

The results apply for example to $(-\Delta)^a$ and to a'th powers A^a of second-order strongly elliptic differential operators A with C^∞-coefficients. Seeley [S67] showed that A^a is a classical ψdo of order $2a$, with a symbol constructed via the resolvent; it is even. (Since we are considering the operators in a neighborhood of the compact set $\overline{\Omega}$, we need not worry about global estimates.) For $(-\Delta)^a$ one can instead remark that the symbol may be written $|\xi|^{2a} = |\xi|^{2a}\eta(\xi) + |\xi|^{2a}(1 - \eta(\xi))$ with an excision function η (cf. (2.11)), and proceed in a similar way as in Corollary 3.4.

As a consequence of the above results, we can show a Pohozaev-type formula for selfadjoint x-independent operators of order $2a$ with even symbol, allowing lower-order terms:

Theorem 4.7. *Let P be a classical elliptic selfadjoint ψdo on $\mathbb{R}^n$ of order $2a$ $(0<a<1)$ with even symbol independent of x. Then for real u satifying hypotheses as in Theorem 4.5 or Corollary 4.5a there holds:*

(4.23)
$$2\int_\Omega Pu\,(x\cdot\nabla u)\,dx=\Gamma(a+1)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\,\gamma_0(d^{-a}u)^2\,d\sigma-n\int_\Omega Pu\,u\,dx+\int_\Omega P_1u\,u\,dx;$$
$$P_1=\operatorname{Op}(\xi\cdot\nabla_\xi p(\xi)).$$

In particular, when p is homogeneous of degree $2a$ (i.e., equals its principal part), then $\xi\cdot\nabla_\xi p=2ap$, so the formula takes the form

$$2\int_\Omega Pu\,(x\cdot\nabla u)\,dx=\Gamma(a+1)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\,\gamma_0(d^{-a}u)^2\,d\sigma+(2a-n)\int_\Omega Pu\,u\,dx. \tag{4.24}$$

When $u\in H^{a(s)}(\overline\Omega)$ with $a+\frac12<s<a+1$, some integrals in the formulas are replaced by dualities:

$$\begin{aligned}&\langle r^+Pu,x\cdot\nabla u\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}+\langle x\cdot\nabla u,Pu\rangle_{\dot H^{a-\frac12+\varepsilon},\overline H^{\frac12-a-\varepsilon}}\\&\quad=\Gamma(a+1)^2\int_{\partial\Omega}(x\cdot\nu)s_0\gamma_0(d^{-a}u)^2\,d\sigma-n\langle r^+Pu,u\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}+\int_\Omega P_1u\,u\,dx.\end{aligned} \tag{4.25}$$

Proof. The calculation goes as follows:

$$\begin{aligned}2\int_\Omega(x\cdot\nabla u)\,Pu\,dx&=\int_\Omega(x\cdot\nabla u)\,Pu\,dx+\int_\Omega Pu\,(x\cdot\nabla u)\,dx\\&=\sum_{j=1}^n\int_\Omega(\partial_j(x_ju)\,Pu-uPu+x_jPu\,\partial_ju)\,dx\\&=\sum_{j=1}^n\int_\Omega(\partial_j(x_ju)\,Pu+P(x_ju)\,\partial_ju+[x_j,P]u\,\partial_ju-u\,Pu)\,dx\\&=\Gamma(a+1)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\,\gamma_0(d^{-a}u)^2\,d\sigma+\int_\Omega(\sum_j[x_j,P]u\,\partial_ju-nu\,Pu)\,dx.\end{aligned} \tag{4.26}$$

Here $[x_j,P]=\operatorname{Op}(i\partial_{\xi_j}p(\xi))$, which is a classical ψdo of order $2a-1$, again with even symbol (having the a-transmission property at $\partial\Omega$), so $[x_j,P]u\in H^{\frac32-a+\varepsilon}(\Omega)$ resp. $C^{2-a+\varepsilon}(\overline\Omega)$, and $\partial_j[x_j,P]u\in H^{\frac12-a+\varepsilon}(\Omega)$ resp. $C^{1-a+\varepsilon}(\overline\Omega)$, under the hypotheses in Theorem 4.5 resp. Corollary 4.6 (by [G15], Th. 4.2, resp. [G14], Th. 3.2(1)). Then

$$\int_\Omega[x_j,P]u\,\partial_ju\,dx+\int_\Omega\partial_j[x_j,P]u\,u\,dx=\int_{\partial\Omega}\nu_j\gamma_0([x_j,P]u)\gamma_0u\,dx=0, \tag{4.27}$$

since $\gamma_0u=0$, so $\int_\Omega[x_j,P]u\,\partial_ju\,dx=-\int_\Omega\partial_j[x_j,P]u\,u\,dx$. Moreover, $\sum_j\partial_j[x_j,P]=\sum_j\operatorname{Op}(i\xi_ji\partial_{\xi_j}p(\xi))=-\operatorname{Op}(\xi\cdot\nabla_\xi p)$; it is selfadjoint. Hence

$$\int_\Omega\sum_j[x_j,P]u\,\partial_ju\,dx=\int_\Omega\operatorname{Op}(\xi\cdot\nabla_\xi p)u\,u\,dx,$$

and when we insert this in (4.26), we find the desired formula. (4.24) follows, since Euler's formula for homogeneous functions in this case shows that $\xi \cdot \nabla_\xi p(\xi) = 2ap(\xi)$.

This shows (4.23) and (4.24) when $u \in H^{a(s)}(\overline{\Omega})$ with $s \geq a+1$; the extension to lower s holds in view of Theorem 4.5. Here we note that P_1 is of order $2a-1$, hence maps $H^{a(a+\frac12+\varepsilon)}(\overline{\Omega})$ into $\overline{H}^{\frac32-a+\varepsilon}(\Omega) \subset L_2(\mathbb{R}^n_+)$, so the expression with P_1 is an ordinary L_2-scalar product. $\square$

Since P is assumed selfadjoint elliptic in Theorem 4.7, the principal symbol p_0 is real $\neq 0$ away from zero, so it is no restriction to assume that it is positive (when $n \geq 2$). Note that there are no sign restrictions on the lower-order terms, so the full operator P need not be positive.

Formula (4.24) was shown in [RS14b] for $P = (-\Delta)^a$, and for more general positive homogeneous symbols in [RSV15] (under somewhat different hypotheses). It leads to a Pohozaev-type formula (generalizing a formula of Pohozaev [P65] for Δ) that is used to obtain uniqueness and (non)existence results. We similarly find from (4.23):

Corollary 4.8. *Let P be as in Theorem 4.7, and let u be a bounded real solution of the problem*

$$r^+Pu = f(u) \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}, \tag{4.28}$$

where f is a real $C^{0,1}$-function. Let $F(t) = \int_0^t f(s)\,ds$. Then

$$-2n\int_\Omega F(u)\,dx + n\int_\Omega u\,f(u)\,dx = \int_\Omega P_1 u\,u\,dx + \Gamma(1+a)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\gamma_0(d^{-a}u)^2\,d\sigma, \tag{4.29}$$

where $P_1 = \operatorname{Op}(\xi\cdot\nabla_\xi p(\xi))$.

Proof. Since u is bounded, so is $f(u)$; then $u \in \dot{C}^a(\overline{\Omega})$. Then since $F(0) = 0$, $F(u) \in \dot{C}^a(\overline{\Omega})$. We have that

$$(x\cdot\nabla)F(u) = \sum_{j=1}^n x_j\partial_j F(u) = \sum_{j=1}^n x_j F'(u)\partial_j u = f(u)(x\cdot\nabla u),$$
$$(x\cdot\nabla)F(u) = \sum_{j=1}^n \partial_j(x_jF(u)) - nF(u).$$

Then since the integral over Ω of $\partial_j(x_jF(u))$ is zero,

$$\int_\Omega (x\cdot\nabla u)f(u)\,dx = \int_\Omega (x\cdot\nabla)F(u)\,dx = -n\int_\Omega F(u)\,dx.$$

Insertion of this and the formula $f(u) = r^+Pu$ in (4.23) leads to (4.29). $\square$

Example 4.9. The fractional Helmholtz (or Schrödinger) operator $P = (-\Delta + m^2)^a$, $0 < a < 1$ and $m > 0$, has the symbol $p(\xi) = (|\xi|^2 + m^2)^a$ of order $2a$. It is not homogeneous, but has the (classical) expansion in homogeneous terms

$$p(\xi) \sim |\xi|^{2a} + am^2|\xi|^{2a-2} + \tfrac12 a(a-1)m^4|\xi|^{2a-4} + \dots,$$

and it is even. Here

$$\xi \cdot \nabla p(\xi) = 2a|\xi|^2(|\xi|^2 + m^2)^{a-1} > 0 \text{ for } \xi \neq 0,$$

and $P_1 = \mathrm{Op}(\xi \cdot \nabla p(\xi)) = 2a(-\Delta)(-\Delta + m^2)^{a-1}$ is a positive operator on $\mathbb{R}^n$.

Let us see what this gives for an eigenvalue problem

$$r^+ Pu = \lambda u \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega},$$

for some $\lambda \in \mathbb{R}$ and bounded real u. With $f(u) = \lambda u$, $F(u) = \frac{1}{2}\lambda u^2$, so the first two integrals in (4.29) cancel out, giving

$$0 = \int_\Omega P_1 u\, u\, dx + \Gamma(1+a)^2 \int_{\partial\Omega} (x \cdot \nu)\, s_0 \gamma_0 (d^{-a} u)^2\, d\sigma.$$

Since P_1 is positive, this allows the conclusion

$$\gamma_0(d^{-a}u) = 0 \implies \int_\Omega P_1 u\, u\, dx = 0 \implies u \equiv 0, \tag{4.30}$$

a kind of unique continuation principle.

Example 4.10. For the operator in Example 4.9,

$$P_1 = 2a(-\Delta)(-\Delta + m^2)^{a-1} = 2a(-\Delta + m^2)^a - 2am^2(-\Delta + m^2)^{a-1} = 2aP - P_2,$$
$$\text{with } P_2 = 2am^2(-\Delta + m^2)^{a-1},$$

here P_2 is again a positive operator. Thus for bounded real solutions of (4.28), equation (4.29) can be written in the form

(4.31)

$$-2n \int_\Omega F(u)\, dx + (n-2a) \int_\Omega u\, f(u)\, dx + \int_\Omega P_2 u\, u\, dx = \Gamma(1+a)^2 \int_{\partial\Omega} (x \cdot \nu)\, s_0 \gamma_0 (d^{-a}u)^2\, d\sigma.$$

Consider the case $f(u) = u|u|^{r-1} = \operatorname{sign} u\, |u|^r$ with an $r > 1$. Here since $F(u) = \frac{1}{r+1}|u|^{r+1}$, (4.31) takes the form

$$\tfrac{-2n+(n-2a)(r+1)}{r+1} \int_\Omega |u|^{r+1}\, dx + \int_\Omega P_2 u\, u\, dx = \Gamma(1+a)^2 \int_{\partial\Omega} (x \cdot \nu)\, s_0 \gamma_0 (d^{-a}u)^2\, d\sigma. \tag{4.32}$$

Consider a starshaped domain Ω ($n \geq 2$), we can assume that 0 is a center. Then $x \cdot \nu \leq 0$ on $\partial\Omega$ (recall that our ν is the interior normal). Note that

$$[-2n + (n-2a)(r+1) =]\ (n-2a)r - (n+2a) \gtreqless 0 \iff r \gtreqless \tfrac{n+2a}{n-2a}.$$

In the critical and supercritical cases $r \geq \frac{n+2a}{n-2a}$ we thus have that if u is a bounded solution (hence is in $\dot{C}^a(\overline{\Omega})$), then the left-hand side of (4.32) is > 0 unless $u \equiv 0$, and the right-hand side is ≤ 0.

This shows *nonexistence of nontrivial solutions, when* $r \geq \frac{n+2a}{n-2a}$.

There is a treatment of existence questions in [RS15a], which goes beyond the case of homogeneous integral operator kernels, by allowing nonnegative kernels with certain growth estimates on rays. It is possible that the result of this example can be proved in that framework (one would have to verify the hypotheses of Th. 1.1(b) there).

One can also ask for generalizations of the formula with $x \cdot \nabla$ to variable-coefficient operators and nonselfadjoint cases. Here we can show:

Theorem 4.11. *Let P be a classical elliptic ψdo on $\mathbb{R}^n$ of order $2a$ $(0<a<1)$ with even symbol. Then for u, u' as in Theorem 4.5 or Corollary 4.6 there holds:*

(4.33)
$$\int_\Omega (Pu\,(x\cdot\nabla\bar u') + (x\cdot\nabla u)\,\overline{P^*u'})\,dx = \Gamma(a+1)^2\int_{\partial\Omega}(x\cdot\nu)s_0\gamma_0(d^{-a}u)\,\gamma_0(d^{-a}\bar u')\,d\sigma$$
$$- n\int_\Omega Pu\,\bar u'\,dx + \int_\Omega [P, x\cdot\nabla]u\,\bar u'\,dx;$$
$$[P, x\cdot\nabla] = \operatorname{Op}(\xi\cdot\nabla_\xi p(x,\xi)) - \operatorname{Op}(x\cdot\nabla_x p(x,\xi)).$$

When $u\in H^{a(s)}(\overline\Omega)$ with $a+\frac12<s<a+1$, some integrals are replaced by dualities:

$$\text{(4.34)}\quad \langle r^+Pu, x\cdot\nabla u'\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}} + \langle x\cdot\nabla u, P^*u'\rangle_{\dot H^{a-\frac12+\varepsilon},\overline H^{\frac12-a-\varepsilon}}$$
$$= \Gamma(a+1)^2((x\cdot\nu)s_0\gamma_0(d^{-a}u), \gamma_0(d^{-a}u'))_{L_2(\mathbb{R}^n)}$$
$$- n\langle r^+Pu, u'\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}} + \langle r^+[P, x\cdot\nabla]u, u'\rangle_{\overline H^{\frac12-a+\varepsilon},\dot H^{a-\frac12-\varepsilon}}.$$

Proof. We proceed in much the same way as in the proof of Theorem 4.7:

$$\int_\Omega Pu\,(x\cdot\nabla\bar u')\,dx + \int_\Omega (x\cdot\nabla u)\,\overline{P^*u'}\,dx$$
$$= \sum_{j=1}^n \int_\Omega (x_jPu\,\partial_j\bar u' + \partial_j(x_ju)\,\overline{P^*u'} - u\,\overline{P^*u'})\,dx$$
$$\text{(4.35)}\qquad = \sum_{j=1}^n \int_\Omega (P(x_ju)\,\partial_j\bar u' + [x_j,P]u\,\partial_j\bar u' + \partial_j(x_ju)\,\overline{P^*u'} - Pu\,\bar u')\,dx$$
$$= \Gamma(a+1)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\,\gamma_0(d^{-a}u)\gamma_0(d^{-a}\bar u')\,d\sigma + \int_\Omega\sum_j [P,\partial_j]x_ju\,\bar u'$$
$$+ \int_\Omega (\sum_j [x_j,P]u\,\partial_j\bar u' - nPu\,\bar u')\,dx.$$

In the third line we have applied Theorem 4.1 to $u\,\overline{P^*u'}$, and in the fourth line we have applied Theorem 4.5 to the terms $P(x_ju)\,\partial_j\bar u'$ and $\partial_j(x_ju)\,\overline{P^*u'}$. As in (4.27), $\int_\Omega [x_j,P]u\,\partial_j\bar u'\,dx$ is turned into $-\int_\Omega \partial_j[x_j,P]u\,\bar u'\,dx$. Moreover,

$$[P,\partial_j]x_ju - \partial_j[x_j,P]u = P\partial_jx_ju - \partial_jx_jPu = [P, x_j\partial_j]u,$$

so the two commutator integrals with ∂_j and x_j together give $\int_\Omega [P, x\cdot\nabla]u\,\bar u'\,dx$. For the symbols, since $[P,\partial_j]$ has symbol $-\partial_{x_j}p$ and $[x_j,P]$ has symbol $i\partial_{\xi_j}p$ (by the formula for the Leibniz product, cf. (2.34)),

$$\text{symbol}([P,\partial_j]x_j - \partial_j[x_j,P]) = -\partial_{x_j}p\,\#\,x_j - i\xi_j\,\#\,i\partial_{\xi_j}p$$
$$= -x_j\partial_{x_j}p - (-i)\partial_{\xi_j}\partial_{x_j}p + \xi_j\partial_{\xi_j}p + (-i)\partial_{x_j}\partial_{\xi_j}p = -x_j\partial_{x_j}p + \xi_j\partial_{\xi_j}p,$$

so $[P, x\cdot\nabla]$ has symbol $\xi\cdot\nabla_\xi p(x,\xi) - x\cdot\nabla_x p(x,\xi)$.

This shows (4.33), with (4.34) valid for low s as in Theorem 4.5. $\square$

Note that the "price" one pays for having the symbol depend on x is a term $-\int_\Omega \operatorname{Op}(x\cdot\nabla_x p)u\,\bar u'\,dx$. The result applies for example to fractional powers of the magnetic Schrödinger operator.

Appendix: Spaces and pseudodifferential operators

We here collect the notation and concepts from the theory of pseudodifferential operators that will be used, including some results from [G15,G14]. Since the set-up is explained in a much more elaborate form there, in particular in [G15], we shall just give a brief summary here.

A pseudodifferential operator (ψdo) P on $\mathbb{R}^n$ is defined from a symbol $p(x,\xi)$ on $\mathbb{R}^n\times\mathbb{R}^n$ by

$$Pu = p(x,D)u = \operatorname{Op}(p(x,\xi))u = (2\pi)^{-n}\int e^{ix\cdot\xi}p(x,\xi)\hat u\,d\xi = \mathcal{F}^{-1}_{\xi\to x}(p(x,\xi)\hat u(\xi)); \tag{A.1}$$

here $\mathcal{F}$ is the Fourier transform $(\mathcal{F}u)(\xi)=\hat u(\xi)=\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,dx$. We refer to textbooks such as Hörmander [H85], Taylor [T91], Grubb [G09] for the rules of calculus. [G09] moreover gives an account of the Boutet de Monvel calculus of pseudodifferential boundary problems, cf. also e.g. [G96]. A standard choice is to take p in the symbol space $S^r_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, consisting of C^∞-functions $p(x,\xi)$ such that $\partial_x^\beta\partial_\xi^\alpha p(x,\xi)$ is $O(\langle\xi\rangle^{r-|\alpha|})$ for all α,β, for some $r\in\mathbb{R}$; then p and P have order r. Also more general symbol spaces will be used in this paper. When P is a ψdo on $\mathbb{R}^n$, $P_+=r^+Pe^+$ denotes its truncation to $\mathbb{R}^n_+$, or to Ω, depending on the context.

Let $1<p<\infty$ (with $1/p'=1-1/p$), then the L_p-Sobolev spaces (Bessel-potential spaces) are defined for $s\in\mathbb{R}$ by

$$\begin{aligned} H_p^s(\mathbb{R}^n) &= \{u\in\mathcal{S}'(\mathbb{R}^n)\mid \mathcal{F}^{-1}(\langle\xi\rangle^s\hat u)\in L_p(\mathbb{R}^n)\},\\ \dot H_p^s(\overline\Omega) &= \{u\in H_p^s(\mathbb{R}^n)\mid \operatorname{supp}u\subset\overline\Omega\},\\ \overline H_p^s(\Omega) &= \{u\in\mathcal{D}'(\Omega)\mid u=r^+U \text{ for some } U\in H_p^s(\mathbb{R}^n)\}; \end{aligned} \tag{A.2}$$

here $\operatorname{supp}u$ denotes the support of u. The definition is also used with $\Omega=\mathbb{R}^n_+$. In most current texts, $\overline H_p^s(\Omega)$ is denoted $H_p^s(\Omega)$ without the overline (that was introduced along with the notation $\dot H$ in [H65,H85]), but we keep it here since it is practical in indications of dualities, and makes the notation more clear in formulas where both types occur. When $p=2$, the mention of p is usually left out.

We recall that $\overline H_p^s(\Omega)$ and $\dot H_{p'}^{-s}(\overline\Omega)$ are dual spaces with respect to a sesquilinear duality extending the $L_2(\Omega)$-scalar product, written e.g.

$$\langle f,g\rangle_{\overline H_p^s(\Omega),\dot H_{p'}^{-s}(\overline\Omega)},\text{ or just } \langle f,g\rangle_{\overline H_p^s,\dot H_{p'}^{-s}}.$$

There is a wealth of other interesting scales of spaces, the Triebel-Lizorkin and Besov spaces $F^s_{p,q}$ and $B^s_{p,q}$, where the problems can be studied; see details in [G14]. In the present work, we shall just use the Hölder-Zygmund spaces $B^s_{\infty\infty}$, also denoted C^s_*. These are interesting because $C^s_*(\mathbb{R}^n)$ equals the Hölder space $C^s(\mathbb{R}^n)$ when $s\in\mathbb{R}_+\setminus\mathbb{N}$. There are similar statements for derived spaces over $\mathbb{R}^n_+$ and Ω, and again the conventions $\overline C$ and $\dot C$ are used for spaces of restricted resp. supported functions. For integer values one has, with $C^k_b(\mathbb{R}^n)$ denoting the space of functions with bounded continuous derivatives up to order k,

$$\begin{aligned} &C^k_b(\mathbb{R}^n)\subset C^{k-1,1}(\mathbb{R}^n)\subset C^k_*(\mathbb{R}^n)\subset C^{k-0}(\mathbb{R}^n) \text{ when } k\in\mathbb{N},\\ &C^0_b(\mathbb{R}^n)\subset L_\infty(\mathbb{R}^n)\subset C^0_*(\mathbb{R}^n), \end{aligned} \tag{A.3}$$

and similar statements for derived spaces.

We use the notation $\bigcup_{\varepsilon>0} H_p^{s+\varepsilon}(\mathbb{R}^n) = H_p^{s+0}(\mathbb{R}^n)$, $\bigcap_{\varepsilon>0} H_p^{s-\varepsilon}(\mathbb{R}^n) = H_p^{s-0}(\mathbb{R}^n)$, applied in a similar way for the other scales of spaces.

A ψdo P is called classical (or polyhomogeneous) when the symbol p has an asymptotic expansion $p(x,\xi) \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi)$ with p_j homogeneous in ξ of degree $m-j$ for all j, and $p(x,\xi) - \sum_{j<J} p_j(x,\xi) \in S_{1,0}^{m-J}(\mathbb{R}^n\times\mathbb{R}^n)$ for all J. Then P has order m. One can even allow m to be complex (with complex homogeneities, $p_j(x,t\xi) = t^{m-j}p(x,\xi)$ for $|\xi|\ge 1$, $t\ge 1$); then p and its remainders are in $S_{1,0}^{\operatorname{Re} m-J}(\mathbb{R}^n\times\mathbb{R}^n)$; the operator and symbol are still said to be of order m.

Here there is an additional definition, introduced by Hörmander in [H65,H85]: P satisfies *the μ-transmission condition* at $\partial\Omega$ (in short: is of type μ) for some $\mu\in\mathbb{C}$ when, in local coordinates,

$$\partial_x^\beta\partial_\xi^\alpha p_j(x,-\nu) = e^{\pi i(m-2\mu-j-|\alpha|)}\partial_x^\beta\partial_\xi^\alpha p_j(x,\nu), \tag{A.4}$$

for all $x\in\partial\Omega$, all j,α,β, where ν denotes the interior normal to $\partial\Omega$ at x. The implications of the μ-transmission condition were a main subject of [G15].

A special role in the theory is played by the *order-reducing operators*. There is a simple definition of operators $\Xi_\pm^\mu$ on $\mathbb{R}^n$

$$\Xi_\pm^\mu = \operatorname{Op}(([\xi']\pm i\xi_n)^\mu) \tag{A.5}$$

(or with $[\xi']$ replaced by $\langle\xi'\rangle$); they preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively. Here the function $([\xi']\pm i\xi_n)^\mu$ does not satisfy all the estimates required for the class $S_{1,0}^{\operatorname{Re}\mu}(\mathbb{R}^n\times\mathbb{R}^n)$, but the operators are useful for many purposes. There is a more refined choice $\Lambda_\pm^\mu$ [G90,G15], with symbols $\lambda_\pm^\mu(\xi)$ that do satisfy all the estimates for $S_{1,0}^{\operatorname{Re}\mu}(\mathbb{R}^n\times\mathbb{R}^n)$. The symbols have holomorphic extensions in ξ_n to the complex halfspaces $\mathbb{C}_\mp = \{z\in\mathbb{C} \mid \operatorname{Im} z \lessgtr 0\}$, and hence the operators preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively; operators with that property are called plus- resp. minus-operators. There is also a pseudodifferential definition $\Lambda_\pm^{(\mu)}$ adapted to the situation of a smooth domain Ω.

It is elementary to see by the definition of the spaces $H_p^s(\mathbb{R}^n)$ in terms of Fourier transformation, that the operators define homeomorphisms for all s:

$$\Xi_\pm^\mu\colon H_p^s(\mathbb{R}^n) \xrightarrow{\sim} H_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n), \quad \Lambda_\pm^\mu\colon H_p^s(\mathbb{R}^n) \xrightarrow{\sim} H_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n) \tag{A.6}$$

(and so does of course $\Xi^\mu = \operatorname{Op}(\langle\xi\rangle^\mu)$). The special interest is that the plus/minus operators also define homeomorphisms related to $\overline{\mathbb{R}}^n_+$ and $\overline{\Omega}$:

$$\begin{aligned} \Xi_+^\mu\colon \dot H_p^s(\overline{\mathbb{R}}^n_+) &\xrightarrow{\sim} \dot H_p^{s-\operatorname{Re}\mu}(\overline{\mathbb{R}}^n_+), \quad \Xi_{-,+}^\mu\colon \overline{H}_p^s(\mathbb{R}^n_+) \xrightarrow{\sim} \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+);\\ \Lambda_+^{(\mu)}\colon \dot H_p^s(\overline{\Omega}) &\xrightarrow{\sim} \dot H_p^{s-\operatorname{Re}\mu}(\overline{\Omega}), \quad \Lambda_{-,+}^{(\mu)}\colon \overline{H}_p^s(\Omega) \xrightarrow{\sim} \overline{H}_p^{s-\operatorname{Re}\mu}(\Omega); \end{aligned} \tag{A.7}$$

for all $s\in\mathbb{R}$; here $\Xi_{-,+}^\mu$ resp. $\Lambda_{-,+}^{(\mu)}$ is short for $r^+\Xi_-^\mu e^+$ resp. $r^+\Lambda_-^{(\mu)}e^+$, suitably extended to large negative s (cf. Rem. 1.1 and Th. 1.3 in [G15]). The first line in (A.7) also holds with Ξ replaced by Λ.

One has moreover, that the operators Ξ_+^μ and $r^+\Xi_-^{\overline{\mu}}e^+$ identify with each other's adjoints over $\overline{\mathbb{R}}_+^n$, because of the support preserving properties; more precisely,

(A.8)
$$\Xi_+^\mu\colon \dot H_{p'}^{\operatorname{Re}\mu - s}(\overline{\mathbb{R}}_+^n) \to \dot H_{p'}^{-s}(\overline{\mathbb{R}}_+^n) \text{ and } r^+\Xi_-^{\overline{\mu}}e^+\colon \overline{H}_p^s(\mathbb{R}_+^n) \to \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}_+^n) \text{ are adjoints,}$$

for $1 < p < \infty$ and $1/p + 1/p' = 1$, all $s \in \mathbb{R}$. The same holds for the operators $\Lambda_+^\mu, \Lambda_{-,+}^{\overline{\mu}}$, and there is a similar statement for $\Lambda_+^{(\mu)}$ and $\Lambda_{-,+}^{(\overline{\mu})}$ relative to the set Ω.

The following special spaces introduced by Hörmander [H65] (for $p = 2$), cf. [G15], are particularly adapted to μ-transmission operators P:

$$\begin{aligned} \mathcal{E}_\mu(\overline{\Omega}) &= e^+\{u(x) = d(x)^\mu v(x) \mid v \in C^\infty(\overline{\Omega})\}, \\ H_p^{\mu(s)}(\overline{\mathbb{R}}_+^n) &= \Xi_+^{-\mu} e^+ \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}_+^n), \quad s > \operatorname{Re}\mu - 1/p', \\ H_p^{\mu(s)}(\overline{\Omega}) &= \Lambda_+^{(-\mu)} e^+ \overline{H}_p^{s-\operatorname{Re}\mu}(\Omega), \quad s > \operatorname{Re}\mu - 1/p'. \end{aligned} \tag{A.9}$$

Namely, r^+P (of order m) maps them into $C^\infty(\overline{\Omega})$, $\overline{H}_p^{s-\operatorname{Re}m}(\mathbb{R}_+^n)$, resp. $\overline{H}_p^{s-\operatorname{Re}m}(\Omega)$ (cf. [G15] Sections 1.3, 2, 4), and they represent, when P is elliptic, the solution space for the Dirichlet problem (1.3) in the current scale. In the first line of (A.9), $\operatorname{Re}\mu > -1$ (for other μ, cf. [G15]) and $d(x)$ is a C^∞-function vanishing to order 1 at $\partial\Omega$ and positive on Ω, e.g. $d(x) = \operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$. One has that $H_p^{\mu(s)}(\overline{\Omega}) \supset \dot H_p^s(\overline{\Omega})$, and the distributions are locally in H_p^s on Ω, but at the boundary they in general have a singular behavior (cf. [G15] Th. 5.4):

$$H_p^{\mu(s)}(\overline{\Omega}) \begin{cases} = \dot H_p^s(\overline{\Omega}) \text{ if } s \in \,]\operatorname{Re}\mu - 1/p', \operatorname{Re}\mu + 1/p[\,, \\ \subset e^+ d^\mu \overline{H}_p^{s-\operatorname{Re}\mu}(\Omega) + \dot H_p^s(\overline{\Omega}) \text{ if } s > \operatorname{Re}\mu + 1/p. \end{cases} \tag{A.10}$$

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