MODULI OF CONTINUITY FOR VISCOSITY SOLUTIONS

XIAOLONG LI

ABSTRACT. In this paper, we investigate the moduli of continuity for viscosity solutions of a wide class of nonsingular quasilinear evolution equations and also for the level set mean curvature flow, which is an example of singular degenerate equations. We prove that the modulus of continuity is a viscosity subsolution of some one dimensional equation. This work extends B. Andrews' recent result on moduli of continuity for smooth spatially periodic solutions.

1. Introduction

Given a function $u : \mathbb{R}^n \to \mathbb{R}$, any function $f : [0, \infty) \to \mathbb{R}_+$ satisfying

$$|u(y) - u(x)| \leq 2f\left(\frac{|y-x|}{2}\right)$$

for all x and y is called a modulus of continuity of u. The (optimal) modulus of continuity ω of u is defined by

$$\omega(s) = \sup\left\{\frac{u(y)-u(x)}{2}\Big|\frac{|y-x|}{2} = s\right\}.$$

The estimate of modulus of continuity has been studied by B. Andrews and J. Clutterbuck in several papers [2] [3]. B. Andrews and J. Clutterbuck [4], B. Andrews and L. Ni [5] and L. Ni [8] have also studied the modulus of continuity for heat equations on manifolds.

More precisely, B. Andrews and J. Clutterbuck considered the following quasi-linear evolution equation

$$u_t = a^{ij}(Du, t)D_iD_ju + b(Du, t) \tag{1.1}$$

where $A(p, t) = \left(a^{ij}(p, t)\right)$ is positive semi-definite. Under the assumption that there exists a continuous function $\alpha : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ with

$$0 < \alpha(R, t) \leq R^2 \inf_{|p|=R,(v\cdot p)\neq 0} \frac{v^T A(p,t)v}{(v\cdot p)^2}, \tag{1.2}$$

They showed [3, Theorem 3.1] that the modulus of continuity of a regular periodic solution to (1.1) is a viscosity subsolution of the one dimensional equation

$$\phi_t = \alpha(|\phi'|, t)\phi''. \tag{1.3}$$

Note that their result is applicable to any anisotropic mean curvature flow and can be used to obtain gradient estimate and thus existence and uniqueness of (1.1).

The first result of this paper is that the same holds for viscosity solutions of (1.1) when (1.2) holds and $a^{ij}, b : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ are continuous functions.

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Theorem 1.1. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a continuous periodic viscosity solution of* (1.1)*. Then the modulus of continuity $\omega(s,t) = \sup\left\{ \frac{u(y,t)-u(x,t)}{2} \middle| \frac{|y-x|}{2} = s \right\}$ of u is a viscosity subsolution of the one dimensional equation* (1.3)*.*

We also study the modulus of continuity for singular evolution equations. As summarized in a recent survey [1] by B. Andrews, for the isotropic flows of the form

$$u_t = \left[a(|Du|) \frac{D_i u D_j u}{|Du|^2} + b(|Du|) \left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2} \right) \right] D_i D_j u, \tag{1.4}$$

the modulus of continuity of a spatially periodic smooth solution of (1.4) is a viscosity subsolution of the corresponding one-dimensional heat equation $\omega_t = a(\omega')\omega''$. Note that equation (1.4) covers the classical heat equation, the graphical mean curvature flow and the p-Laplace heat equation with suitable choices of a and b. When (1.4) is nonsingular, it is covered by (1.1). When it is singular, it has to be treated differently since there are various definitions for viscosity solutions of singular equations. We will focus on the particular case $a = 0$ and $b = 1$, which corresponds to the level set mean curvature flow:

$$u_t = \left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2} \right) D_i D_j u. \tag{1.5}$$

Equation (1.5) was studied by L. Evans and J. Spruck in [7]. They gave a definition of viscosity solution and proved that for an initial data g that is continuous and constant on $\mathbb{R}^n \cap \{|x| \geq S\}$, there exists a unique viscosity solution u that is continuous and constant on $\mathbb{R}^n \cap \{|x| \geq R\}$, with R depending only on S. We will recall their definition in Section 2.

We prove the following theorem:

Theorem 1.2. *Let $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a viscosity solution of* (1.5) *with continuous initial data g that is a constant on $\mathbb{R}^n \cap \{|x| \geq S\}$. Then the modulus of continuity $\omega(s,t) = \sup\left\{ \frac{u(y,t)-u(x,t)}{2} \middle| \frac{|y-x|}{2} = s \right\}$ of u is a viscosity subsolution of $\omega_t = \max\{0, \frac{1}{4}(\omega'' + |\omega''|)\}$ on $(0,\infty) \times (0,\infty)$.*

As an immediate consequence, we have that any concave modulus of continuity for the initial data is preserved by the level set mean curvature flow.

Corollary 1.3. *Let $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a viscosity solution of* (1.5) *with continuous initial data g that is a constant on $\mathbb{R}^n \cap \{|x| \geq S\}$. Assume ϕ is nonnegative, concave and satisfies $|g(y) - g(x)| \leq 2\phi\left(\frac{|y-x|}{2}\right)$ for all x, y, then*

$$|u(y,t) - u(x,t)| \leq 2\phi\left(\frac{|y-x|}{2}\right)$$

for all x, y and $t \geq 0$.

Proof of Corollary 1.3. Since the function $\phi_\epsilon = \phi + \epsilon e^t$ satisfies

$$\partial_t \phi_\epsilon > 0 = \max\{0, \frac{1}{4}(\phi'' + |\phi''|)\},$$

so it cannot touch ω from above by Theorem 1.2. □

2. Definitions of Viscosity Solutions

We give definition of a viscosity solution for the general equation

$$u_t + F(x, t, u, \nabla u, \nabla^2 u) = 0 \tag{2.1}$$

assuming $F : \mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} \to \mathbb{R}$ is continuous and degenerate elliptic. Let O be an open subset of $\Omega \times (0, T)$. We write $z = (x, t)$ and $z_0 = (x_0, t_0)$.

The following notations are useful:

$$\mathrm{USC}(O) = \{u : O \to \mathbb{R} |\ u \text{ is upper semicontinuous }\},$$

$$\mathrm{LSC}(O) = \{u : O \to \mathbb{R} |\ u \text{ is lower semicontinuous }\},$$

Definition 2.1. (i) A function $u \in \mathrm{USC}(O)$ is a viscosity subsolution of (2.1) in O if for any $\phi \in C^\infty(O)$ such that $u - \phi$ has a local maximum at $z_0 \in O$, then

$$\phi_t(z_0) + F(z_0, u(z_0), \nabla\phi(z_0), \nabla^2\phi(z_0)) \le 0.$$

(ii) A function $u \in \mathrm{LSC}(O)$ is a viscosity supersolution of (2.1) in O if for any $\phi \in C^\infty(O)$ such that $u - \phi$ has a local minimum at $z_0 \in O$, then

$$\phi_t(z_0) + F(z_0, u(z_0), \nabla\phi(z_0), \nabla^2\phi(z_0)) \ge 0.$$

(iii) A viscosity solution of (2.1) in O is defined to be a continuous function that is both a viscosity subsolution and a viscosity supersolution of (2.1) in O.

We have an equivalent definition in terms of parabolic semijets. Assume $u \in \mathrm{USC}(O)$ and $z_0 \in O$. The parabolic superjet of u at z_0, denoted by $\mathcal{P}^{2,+}u(z_0)$, is defined by

$$\begin{aligned}\mathcal{P}^{2,+}u(z_0) =&\{(\tau, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} | u(z) \le u(z_0) + \tau(t - t_0) + p \cdot (x - x_0)\\ &+ \frac{1}{2}(x - x_0)^T X(x - x_0) + o(|x - x_0|^2 + |t - t_0|) \text{ as } z \to z_0\}.\end{aligned}$$

The parabolic subjet of $u \in \mathrm{LSC}(O)$ at z_0, denoted by $\mathcal{P}^{2,-}u(z_0)$, is defined by

$$\mathcal{P}^{2,-}u(z_0) = -\mathcal{P}^{2,+}(-u)(z_0).$$

Definition 2.2. (i) A function $u \in \mathrm{USC}(O)$ is a viscosity subsolution of (2.1) in O if for all $(x, t) \in O$ and $(\tau, p, X) \in \mathcal{P}^{2,+}u(x, t)$,

$$\tau + F(z, u(z), p, X) \le 0.$$

(ii) A function $u \in \mathrm{LSC}(O)$ is a viscosity supersolution of (2.1) in O if for all $(x, t) \in O$ and $(\tau, p, X) \in \mathcal{P}^{2,-}u(x, t)$,

$$\tau + F(z, u(z), p, X) \ge 0.$$

Remark 2.3. In the above definitions, since F is continuous, we can replace $\mathcal{P}^{2,+}u(z_0)$ and $\mathcal{P}^{2,-}u(z_0)$ by $\overline{\mathcal{P}}^{2,+}u(z_0)$ and $\overline{\mathcal{P}}^{2,-}u(z_0)$ respectively, where the closures are defined by

$$\begin{aligned}\overline{\mathcal{P}}^{2,+}u(z_0) = \{&(\tau, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} |\text{ there is a sequence } (z_j, \tau_j, p_j, X_j)\\ &\text{such that } (\tau_j, p_j, X_j) \in \mathcal{P}^{2,+}u(z_j)\\ &\text{and } (z_j, u(z_j), \tau_j, p_j, X_j) \to (z_0, u(z_0), \tau, p, X) \text{ as } j \to \infty\}.\\ \overline{\mathcal{P}}^{2,-}u(z_0) = -&\overline{\mathcal{P}}^{2,+}(-u)(z_0).\end{aligned}$$

For singular equations, there are different ways to define viscosity solutions. For the level set mean curvature flow (1.5), we use the definition given by L. Evans and J. Spruck in [7], where viscosity solutions are called weak solutions.

Definition 2.4. A continuous and bounded function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a viscosity subsolution of (1.5) if for any $\phi \in C^\infty(\mathbb{R}^{n+1})$ such that $u - \phi$ has a local maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, then we have

$$\begin{cases} \phi_t \le \left(\delta_{ij} - \frac{D_i\phi D_j\phi}{|D\phi|^2}\right) D_i D_j \phi \text{ at } (x_0, t_0) \\ \text{if } D\phi(x_0, t_0) \neq 0, \end{cases}$$

and

$$\begin{cases} \phi_t \le (\delta_{ij} - \eta_i\eta_j)\, D_i D_j \phi \text{ at } (x_0, t_0) \\ \text{ for some } \eta \in \mathbb{R}^n \text{ with } |\eta| \le 1, \text{ if } D\phi(x_0, t_0) = 0 \end{cases}$$

Definition 2.5. A continuous and bounded function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a viscosity supersolution of (1.5) if for any $\phi \in C^\infty(\mathbb{R}^{n+1})$ such that $u - \phi$ has a local minimum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, then we have

$$\begin{cases} \phi_t \ge \left(\delta_{ij} - \frac{D_i\phi D_j\phi}{|D\phi|^2}\right) D_i D_j \phi \text{ at } (x_0, t_0) \\ \text{if } D\phi(x_0, t_0) \neq 0, \end{cases}$$

and

$$\begin{cases} \phi_t \ge (\delta_{ij} - \eta_i\eta_j)\, D_i D_j \phi \text{ at } (x_0, t_0) \\ \text{ for some } \eta \in \mathbb{R}^n \text{ with } |\eta| \le 1, \text{ if } D\phi(x_0, t_0) = 0 \end{cases}$$

Definition 2.6. A continuous and bounded function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a viscosity solution of (1.5) provided u is both a viscosity subsolution and a viscosity supersolution.

We also have alternative definitions in terms of parabolic semijets.

Definition 2.7. A continuous and bounded function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a viscosity subsolution of (1.5) if for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and $(\tau, p, X) \in \mathcal{P}^{2,+}u(x, t)$,

$$\tau \le \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) X_{ij} \text{ if } p \neq 0$$

and

$$\tau \le (\delta_{ij} - \eta_i\eta_j) X_{ij} \text{ for some } |\eta| \le 1, \text{ if } p = 0.$$

Definition 2.8. A continuous and bounded function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a viscosity supersolution of (1.5) if for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and $(\tau, p, X) \in \mathcal{P}^{2,-}u(x, t)$,

$$\tau \ge \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) X_{ij} \text{ if } p \neq 0$$

and

$$\tau \ge (\delta_{ij} - \eta_i\eta_j) X_{ij} \text{ for some } |\eta| \le 1, \text{ if } p = 0.$$

Remark 2.9. One can replace $\mathcal{P}^{2,+}u(z_0)$ and $\mathcal{P}^{2,-}u(z_0)$ by $\overline{\mathcal{P}}^{2,+}u(z_0)$ and $\overline{\mathcal{P}}^{2,-}u(z_0)$ respectively in the above definitions for the reason of continuity.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We must show that if ϕ is a smooth function such that $\omega-\phi$ has a local maximum at (s_0,t_0) for $s_0>0$ and $t_0>0$, then at (s_0,t_0)

$$\phi_t \le \alpha(|\phi'|,t)\phi''.$$

Since u is continuous and periodic, there exist points x_0 and y_0 with $|y_0-x_0|=2s_0$ attaining the supremum,

$$\omega(s_0,t_0)=\frac{u(y_0,t_0)-u(x_0,t_0)}{2}.$$

Define

$$Z(x,y,t):=u(y,t)-u(x,t)-2\phi\left(\frac{|y-x|}{2},t\right).$$

In view of the definition of ω, we obtain that

$$Z(x,y,t)\le Z(x_0,y_0,t_0)$$

for all $|y-x|$ close to $2s_0$ and t close to t_0. Thus Z has a local maximum at (x_0,y_0,t_0). Since Z is continuous, by the parabolic version maximum principle for semicontinuous functions [6, Theorem 8.3], for any $\lambda>0$, there exist $X,Y\in S^{n\times n}$ such that

$$(b_1,2D_y\phi(s_0,t_0),X)\in\overline{\mathcal{P}}^{2,+}u(y_0,t_0),$$

$$(-b_2,-2D_x\phi(s_0,t_0),Y)\in\overline{\mathcal{P}}^{2,-}u(x_0,t_0),$$

$$b_1+b_2=2\phi_t(s_0,t_0),$$

$$-\left(\lambda^{-1}+\|M\|\right)I\le\begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix}\le M+\lambda M^2, \tag{3.1}$$

where

$$M=2D^2\phi=2\begin{pmatrix}D_y^2\phi & D_{y,x}^2\phi\\ D_{x,y}^2\phi & D_x^2\phi\end{pmatrix}=\begin{pmatrix}B & -B\\ -B & B\end{pmatrix},$$

with $B=2D_y^2\phi(s_0,t_0)$.

To simplify, we choose an orthonormal basis of $\mathbb{R}^n$ with $e_n=\frac{y-x}{|y-x|}$, then

$$2D_y\phi(s_0,t_0)=-2D_x\phi(s_0,t_0)=\phi'(s_0,t_0)e_n.$$

$$B=\begin{pmatrix}\frac{\phi'}{2s_0} & & & \\ & \ddots & & \\ & & \frac{\phi'}{2s_0} & \\ & & & \frac{1}{2}\phi''\end{pmatrix}.$$

Since u is both a subsolution and a supersolution of (1.1), we have

$$b_1\le tr(A(\phi' e_n)X)+b(\phi' e_n)$$

$$-b_2\ge tr(A(\phi' e_n)Y)+b(\phi' e_n)$$

By choosing a symmetric matrix C such that $\begin{pmatrix} A & C \\ C & A \end{pmatrix} \geq 0$, we obtain using (3.1)

$$\begin{aligned} 2\phi_t(s_0,t_0) = b_1 + b_2 &\leq tr\,(A(\phi' e_n)(X-Y)) = tr \begin{pmatrix} A & C \\ C & A \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \\ &\leq tr \begin{pmatrix} A & C \\ C & A \end{pmatrix} \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \lambda tr \begin{pmatrix} A & C \\ C^T & A \end{pmatrix} \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}^2 \\ &= 2tr\,((A-C)B) + 4\lambda tr\,\left((A-C)B^2\right) \end{aligned}$$

Taking $C = A - 2\alpha(|\phi'|)e_n \otimes e_n$, it's easy to verify $\begin{pmatrix} A & C \\ C & A \end{pmatrix} \geq 0$ due to (1.2). Thus at (s_0, t_0)

$$\phi_t \leq \alpha(|\phi'|)\phi'' + \lambda\alpha(|\phi'|)(\phi'')^2 \tag{3.2}$$

Since $\lambda > 0$ is arbitrary, we get

$$\phi_t \leq \alpha(|\phi'|)\phi''$$

at (s_0, t_0). □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Suppose that ϕ is a smooth function such that $\omega - \phi$ has a strict local maximum at (s_0, t_0) with $s_0 > 0$ and $t_0 > 0$. As in the proof of Theorem 1.1, we arrive at that the function

$$Z(x,y,t) := u(y,t) - u(x,t) - 2\phi\left(\frac{|y-x|}{2}, t\right).$$

has a local maximum at (x_0, y_0, t_0). Again we use an orthonormal frame with $e_n = \frac{y-x}{|y-x|}$. The maximum principle for semicontinuous functions [6, Theorem 8.3] implies that for any $\lambda > 0$, there exist $X, Y \in S^{n\times n}$ such that

$$(b_1, \phi'(s_0,t_0)e_n, X) \in \overline{\mathcal{P}}^{2,+}u(y_0,t_0)$$

$$(-b_2, \phi'(s_0,t_0)e_n, Y) \in \overline{\mathcal{P}}^{2,-}u(x_0,t_0)$$

$$b_1 + b_2 = 2\phi_t(s_0,t_0)$$

$$-\left(\lambda^{-1} + \|M\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M + \lambda M^2, \tag{4.1}$$

where

$$M = \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

with

$$B = 2D_y^2\phi(s_0,t_0) = \begin{pmatrix} \frac{\phi'}{2s_0} & & & \\ & \ddots & & \\ & & \frac{\phi'}{2s_0} & \\ & & & \frac{1}{2}\phi'' \end{pmatrix}.$$

For any vector $p \in \mathbb{R}^n$, we have

$$p^T X p - p^T Y p = (p,p)^T \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} (p,p) \leq (p,p)^T (M + \lambda M^2)(p,p) = 0$$

Therefore $X \leq Y$. For simiplicity, we denote $A(p) = I - \frac{p \otimes p}{|p|^2}$. If $\phi'(s_0, t_0) \neq 0$, then by the definition of viscosity solution of (1.5),

$$b_1 \leq tr(A(\phi' e_n)X),$$
$$-b_2 \geq tr(A(\phi' e_n)Y).$$

Using the fact $X \leq Y$, we get

$$\phi_t = \frac{1}{2}(b_1 + b_2) \leq \frac{1}{2} tr\left(A(\phi' e_n)(X - Y)\right) \leq 0.$$

If $\phi'(s_0, t_0) = 0$, then it follows from the definition of a viscosity solutions that for some ξ, η with $|\xi|, |\eta| \leq 1$,

$$b_1 \leq tr(A(\xi)X),$$
$$-b_2 \geq tr(A(\eta)Y).$$

In view of (4.1), we have $X \leq B + 2\lambda B^2$ and $-Y \leq B + 2\lambda B^2$. Thus

$$tr((A(\xi)X)) \leq tr\left((A(\xi)(B + 2\lambda B^2)\right) = \frac{1}{2}(1 - \xi_n^2)\left((\phi'' + \lambda(\phi'')^2\right),$$

$$tr((A(\eta)Y)) \geq tr\left((A(\eta)(-B - 2\lambda B^2)\right) = -\frac{1}{2}(1 - \eta_n^2)\left((\phi'' + \lambda(\phi'')^2\right).$$

Therefore,

$$2\phi_t = b_1 + b_2 \leq tr(A(\xi)X) - tr(A(\eta)Y) \leq \left(1 - \frac{\xi_n^2 + \eta_n^2}{2}\right)\left(\phi'' + \lambda(\phi'')^2\right).$$

Letting $\lambda \to 0$ yields

$$\phi_t \leq \frac{1}{4}\left(\phi'' + |\phi''|\right).$$

□

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Department of Mathematics, University of California, San Diego, La Jolla, CA 92093

E-mail address: xil117@ucsd.edu