WELL-POSEDNESS OF THE PRANDTL EQUATION IN SOBOLEV SPACE WITHOUT MONOTONICITY

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ABSTRACT. We study the well-posedness theory for the Prandtl boundary layer equation on the half plane with initial data in Sobolev spaces. We consider a class of initial data which admit the non-degenerate critical points, so it is not monotonic. For this kind of initial data, we prove the local-in-time existence, uniqueness and stability of solutions for the nonlinear Prandtl equation in weighted Sobolev space. We use the energy method to prove the existence of solution by a parabolic regularizing approximation. The nonlinear cancellation properties of the Prandtl equations and non-degeneracy of the critical points are the main ingredients to establish a new energy estimate. Our result improves the classical local well-posedness results for the initial data that are monotone, analytic or Gevery class, and it will also help us to understand the ill-posedness and instability results for the Prandtl equation.

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1. Introduction

In this work, we study the initial-boundary value problem for the Prandtl boundary layer equation in two dimension, which reads

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = \partial_y^2 u, \quad t > 0,\, (x, y) \in \mathbb{R}^2_+, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = U(t, x), \\ u|_{t=0} = u_0(x, y)\,, \end{cases}$$

where $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2;\, y > 0\}$, $u(t, x, y)$ represents the tangential velocity, $v(t, x, y)$ normal velocity. $p(t, x)$ and $U(t, x)$ are the values on the boundary of the Euler's pressure and Euler's tangential velocity and determined by the Bernoulli's law:

$$\partial_t U(t, x) + U(t, x)\partial_x U(t, x) + \partial_x p = 0.$$

The Prandtl boundary layer equation was firstly derived formally by Ludwing Prandtl in 1904 ([20]). From the mathematical point of view, the well-posedness and justification of the Prandtl boundary layer theory don't have satisfactory theory yet, and remain open for general cases. During the past century, lots of mathematicians have investigated this problems. The Russian school has contributed a lot to the boundary layer theory and their works were collected in [19]. Up to now, the existence theory for the Prandtl boundary layer equation has been achieved only when the initial data belong to some special functional spaces: 1) the analytic space [16, 22, 23, 26]; 2) Sobolev spaces or Hölder spaces under monotonicity assumption [1, 14, 18, 19, 24]; 3) recently ([7]) in Gevrey class of index $\frac{7}{4}$ with non-degenerate critical points. At the same time, there are many unstability or blow up results [4, 5, 6, 8, 10, 11, 21]. In particular, Gérard-Varet and Dormy [6] constructed a highly persuasive profile to reveal that the linearized Prandtl equation around the shear flow with a non-degenerate critical point is ill-posed in the sense of Hadamard in Sobolev spaces, see also [11] for nonlinear extension.

In this work, we consider the uniform outflow $U(t, x) = 1$ which implies $p_x = 0$. In other words the following problem for the Prandtl equation is considered :

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = \partial_y^2 u, \quad t > 0,\, (x, y) \in \mathbb{R}^2_+, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = 1, \\ u|_{t=0} = u_0(x, y)\,. \end{cases} \tag{1.1}$$

The aim of this work is to prove the local well-posedness in Sobolev spaces of the system (1.1) around a **non-monotonic shear profile**. More precisely, we assume that

$$u_0(x, y) = u_0^s(y) + \tilde{u}_0(x, y),$$

where u_0^s satisfies the following conditions: for some even integer $m \geq 6$, and $k > 1$, $u_0^s \in C^{m+4}([0, +\infty[)$, $\lim\limits_{y\to+\infty} u_0^s(y) = 1$ with the compatibility condition

$(\partial_y^{2p} u_0^s)(0) = 0,\ 0 \le 2p \le m+4$, and that there exists $0 < a < +\infty, c_1, c_2 > 0$ such that

$$\begin{cases} \partial_y u_0^s(a) = 0,\ \partial_y^2 u_0^s(a) \neq 0,\ \text{and } \partial_y u_0^s(y) \neq 0,\ \text{for any } y \in \bar{\mathbb{R}}_+ \setminus \{a\}; \\ c_1 \langle y \rangle^{-k} \le |\partial_y u_0^s(y)| \le c_2 \langle y \rangle^{-k},\ \forall\, y \ge a+1, \\ |\partial_y^p u_0^s(y)| \le c_2 \langle y \rangle^{-k-p+1},\ \forall\ y \ge 0,\ 0 \le p \le m+4, \end{cases} \tag{1.2}$$

where $\langle y \rangle = (1+|y|^2)^{1/2}$. Therefore, u_0^s has a **non-degenerate critical point** at $a \in \mathbb{R}_+$ and, in particular, it is not monotonic on $\mathbb{R}_+$.

We introduce the weighted Sobolev spaces with the following norm, for $\lambda \in \mathbb{R}, m \in \mathbb{N}$,

$$\|f\|^2_{H^m_\lambda(\mathbb{R}^2_+)} = \sum_{|\alpha_1+\alpha_2| \le m} \int_{\mathbb{R}^2_+} \langle y \rangle^{2\lambda+2\alpha_2} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} f|^2 dxdy\,. \tag{1.3}$$

We use also the notation

$$\|f\|^2_{L^2_\lambda(\mathbb{R}^2_+)} = \int_{\mathbb{R}^2_+} \langle y \rangle^{2\lambda} |f|^2 dxdy\,,$$

and H^m stands for the usual Sobolev space.

Let $u^s(t,y)$ be the solution of the heat equation (2.1) with initial data $u_0^s(y)$, then $(u^s, 0)$ is a shear flow of the system (1.1). We will construct the solution of the Prandtl equation (1.1) as a perturbation of this shear flow. For the initial-boundary values problem (1.1), the existence of the smooth solution requires the high order compatibility conditions for the initial data u_0, see Proposition 2.3 below for the precise description.

The main result of this paper is stated as following :

Theorem 1.1. *Let $m \ge 6$ be an even integer, $k > 1, 0 \le \ell < \frac{1}{2}, k+\ell > \frac{3}{2}$. Assume that u_0^s satisfies (1.2), the initial data $(u_0 - u_0^s) \in H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)$ and $(u_0 - u_0^s)$ satisfies the compatibility condition up to order $m+2$. Then there exists $T > 0$ such that if*

$$\|(u_0 - u_0^s)\|_{H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)} \le \delta_0, \tag{1.4}$$

for some $\delta_0 > 0$ small enough, the initial-boundary value problem (1.1) admits a unique solution (u, v) with

$$(u - u^s) \in L^\infty([0,T]; H^m_{k+\ell-1}(\mathbb{R}^2_+)),\ \partial_y(u-u^s) \in L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+)), \tag{1.5}$$

and

$$v \in L^\infty([0,T]; L^\infty(\mathbb{R}_{y,+}; H^{m-1}(\mathbb{R}_x)),\ \ \partial_y v \in L^\infty([0,T]; H^{m-1}_{k+\ell-1}(\mathbb{R}^2_+)).$$

Moreover, we have the stability with respect to the initial data in the following sense: given any two initial data

$$u_0^1 = u_0^s + \tilde{u}_0^1, \quad u_0^2 = u_0^s + \tilde{u}_0^2,$$

if u_0^s satisfies (1.2) and $\tilde{u}_0^1, \tilde{u}_0^2$ satisfies (1.4), then the solutions u^1 and u^2 of (1.1) with initial data u_0^1 and u_0^2 respectively satisfy,

$$\|u^1 - u^2\|_{L^\infty([0,T]; H^{m-2}_{k+\ell-1}(\mathbb{R}^2_+))} \le C \|(u_0^1 - u_0^2)\|_{H^m_{2k+\ell-1}(\mathbb{R}^2_+)}.$$

Remark 1.2. *.*

(1) *It is not difficult to see from the proof of Theorem 1.1 that the above main results hold also for the problem defined on the torus $\mathbb{T}$ for the x-variable.*

(2) *For the notational simplicity, we assume here only one non-degenerate critical point for the initial shear profile u_0^s. In fact, the above main results also hold if u_0^s admits finite non-degenerate critical points $0 < a_1 < a_2 < \cdots < a_N < +\infty$.*
(3) *Comparing the order of regularity and the decay rate of initial data (1.4) with those of the solution (1.5), we observe that there are a loss of the regularity and also a loss of the decay rate. While the loss of the regularity seems natural for a degenerate parabolic type equation, the loss of decay rate of order k (which is the decay rate of shear flow) is a special phenomenon for the Prandtl equation.*
(4) *Under the assumption of Theorem 1.1, there are many initial data u_0 with a curve of non-degenerate critical points : for some smooth curve $a(x) > 0$,*
$$(\partial_y u_0)(x, a(x)) = 0, \text{ with } (\partial_y^2 u_0)(x, a(x)) > 0, \ \forall x \in \mathbb{R}.$$
See some precise examples in Appendix B.
(5) *It is well-known that to obtain the smooth solution of the initial-boundary value problem, the assumption on the high order compatibility condition is neccesary. We will make it precise in Proposition 2.3 and Remark 3.3. See [2] and [3] on the Prandtl equation under the incompatibility condition.*

Now, we give some comments on the differences and compatibility between our results and the ill-posedness results. In [6] and [11], the authors showed that the linear and nonlinear Prandtl equations around the shear profile with non degenerate critical point are ill-posed in the sense of Hadamard in the Sobolev space with exponential weight. But the results of our Theorem 1.1 observe the contrary phenomenon. We think that it is because we work in the Sobolev space with polynomial type weight, and our shear flow has a lower bound of finite order as in (1.2) when $y \to +\infty$. This means that $(u_0^s - 1)$ tends to 0 with a finite order decay rate $k > 1$. Consequently, we can close the energy estimate with a loss of finite order decay. However, it is not possible to apply this approach to the shear profile considered in [6] and [11] which is of exponential decay. So we can't say that there is contradiction between our Theorem 1.1 and the results of [6, 11], even though in all of those works, the shear profile has non-degenerate critical points.

As pointed out by Masmoudi and Wong in [18] and confirmed by Theorem 1.1, we believe that the suitable function space to study the Prandtl equation is Sobolev space with polynomial type weight defined in (1.3). For the existence of the solution to the Cauchy problem, the loss of derivatives and loss of decay relative to the initial data seem natural for the degenerate parabolic type equations. But on the other hand, under the monotonic assumption, the Prandtl equation is hypo-elliptic, and for $t > 0$, any Sobolev solution belongs to the Gevrey class of index $3(1 + k)$ (see [13]), where the decay rate k gives an effective reflection for the Gevery index. See also [25] for the more general nonlinear hypo-elliptic equations which yields also the C^∞ smoothness of the solution of the Prandtl equation.

The lifespan T of solution is not required to be very small in our Theorem 1.1. In fact, it is equal to the lifespan T_1 of shear flow $u^s(t, y)$ such that it preserves the monotonicity and convexcity on $[0, T_1]$ (see Lemma 2.1). So if the initial data of the shear flow u_0^s is uniformly monotonic on $\mathbb{R}_+$ in the following sense:
$$\begin{cases} c_1\langle y\rangle^{-k} \le |\partial_y u_0^s(y)| \le c_2\langle y\rangle^{-k}, \ \forall\, y \ge 0, \\ |\partial_y^p u_0^s(y)| \le c_2\langle y\rangle^{-k-p+1}, \ \forall\ y \ge 0, \ 1 \le p \le m+2, \end{cases} \tag{1.6}$$

for certain $c_1, c_2 > 0$, we have the following existence of almost global-in-time solution.

Theorem 1.3. *Let $m \geq 6$ be an even integer, $k > 1, 0 \leq \ell < \frac{1}{2}, k+\ell > \frac{3}{2}$. Assume that u_0^s satisfies (1.6), the initial data $(u_0 - u_0^s) \in H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)$ and $(u_0 - u_0^s)$ satisfies the compatibility condition up to order $m+2$. Then for any $T > 0$, there exists $\delta_0 > 0$ small enough such that if*

$$\|(u_0 - u_0^s)\|_{H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)} \leq \delta_0,$$

then the initial-boundary value problem (1.1) admits a unique solution (u, v) with

$$(u - u^s) \in L^\infty([0,T]; H^m_{k+\ell-1}(\mathbb{R}^2_+)),\ \ \partial_y(u - u^s) \in L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))$$

and

$$v \in L^\infty([0,T]; L^\infty(\mathbb{R}_{y,+}; H^{m-1}(\mathbb{R}_x))),\ \ \partial_y v \in L^\infty([0,T]; H^{m-1}_{k+\ell-1}(\mathbb{R}^2_+)).$$

It also has the stability with respect to the initial data as in Theorem 1.1.

Remark 1.4. *About the global-in-time solution of the Prandtl equation, Xin and Zhang ([24]) proved the global existence of weak solutions under the monotonic assumption, (see [15] for 3-D case), and it is still an open problem for the existence of the smooth global-in-time solution. On the other hand, in the analytical frame, Zhang-Zhang [26] (see also [12]) recently get an almost global solution. In this sense, Theorem 1.3 improves the previous results under the monotonic assumption, it is the almost global-in-time solutions in Sobolev spaces.*

This article is arranged as follows. In Section 2, we explain the main difficulties for the study of the Prandtl equation and present an outline of our approach. In Section 3, we study the approximate solutions to (1.1) by a parabolic regularization. In Section 4, we prepare some technical tools and two formal transformations for the Prandtl equations, so that the new transformed equations possess some nonlinear cancellation properties. Sections 5-6 are dedicated to the uniform estimates of approximate solutions obtained in Section 3. We prove finally the main theorem in Section 7-8.

Notations: The letter C stands for various suitable constants, independent with functions and the special parameters, which may vary from line to line and step to step. When it depends on some crucial parameters in particular, we put a sub-index such as C_ϵ etc, which may also vary from line to line.

2. Outline of our approach and preliminary

2.1. **Difficulties and our approach.** Now, we explain the main difficulties in proving Theorem 1.1, and present the strategies of our approach.

It is well-known that the major difficulty for the study of the Prandtl equation (1.1) is the term $v\,\partial_y u$, where the vertical velocity behaves like

$$v(t,x,y) = -\int_0^y \partial_x u(t,x,\tilde{y})d\tilde{y},$$

by using the divergence free condition and boundary conditions. So it introduces a loss of x-derivative, then the standard energy estimates do not work. This explains why there are few existence results in the literatures.

Recalling that in [1] (see also [18]), under the monotone assumption $\partial_y u > 0$, we divide the Prandtl equations by $\partial_y u$ and then take derivative with respect to y, to obtain an equation of the new unknown function

$$g = \left(\frac{u}{\partial_y u}\right)_y .$$

In the new equation, the term v disappears by using the divergence free condition. We call this property *the first type nonlinear cancellation*. By using this cancellation, under the monotone assumption, we can establish an energy estimate for the new equation in a suitable weighted Sobolev space to deduce the existence of classical solutions. Now, under the assumption of Theorem 1.1, we use this properties together with a cut-off functions to treat the monotonic part, and establish a partial energy estimate.

On the other hand, near the non-degenerate critical points, saying $\partial_y^2 u > 0$ on a small domain, we consider the vorticity equation ($w = \partial_y u$) of (1.1),

$$\partial_t w + u\partial_x w + v\partial_y w = \partial_y^2 w.$$

Since $\partial_y w = \partial_y^2 u > 0$, we can try the same approach as the monotone case. But it doesn't work well because the divergence free condition can't be used here to make v disappear. In [9, 10], for the hydrostatic Euler equation, E. Grenier proposed another approach(see also [17]), after dividing the vorticity equation by $\sqrt{\partial_y w}$, we get then an equation of the following form

$$\partial_t h + u\partial_x h + v\sqrt{\partial_y w} = \partial_y^2 h + \text{commutators}, \quad \text{on a convex domain},$$

with $h = \frac{w}{\sqrt{\partial_y w}}$. One can then take advantage of the following *second type nonlinear cancellation*

$$\begin{aligned}\int v\sqrt{\partial_y w}\, h dxdy &= \int v\,\partial_y u dxdy = -\int (\partial_y v)\, u dxdy \\ &= \int (\partial_x u)\, u dxdy = \frac{1}{2}\int \partial_x (u^2) dxdy = 0\,.\end{aligned}$$

This approach works well for the hydrostatic Euler equation, because, in that case, the integration is taken over $\mathbb{T}\times]0,1[$ and $\partial_y w > 0$ everywhere. But for the Prandtl equation, it is defined over $\mathbb{T}\times\mathbb{R}_+$ or $\mathbb{R}\times\mathbb{R}_+$, and the condition $\partial_y w > 0$ holds only on a small sub-domain, so we can't use directly this approach. In the Prandtl case, we then need to introduce a cut-off function, and in this case, *a new problem appears!* The above calculation doesn't preserve the cancellation property. There are remaining terms from the commutators with cut-off functions. So it is standard that we can only close the energy estimate in Gevery class(see [7]) with a suitably chosen cut-off functions in Gevery class.

In this work, our approach is to combine above two type nonlinear cancellations with carefully chosen cut-off functions. We decompose $\mathbb{R}\times\mathbb{R}_+$ into two parts, namely monotonic parts and convex part. We prove the energy estimates on each part. Of course, we can't close the energy estimate on each part separately. But it is a miracle that we can close a full energy estimate when combining those two partial energy estimates together. By using this full energy estimate, we prove then the existence of classical solutions.

In order to prove the existence of solutions, we will construct an approximate scheme and study the parabolic regularized Prandtl equation (3.1), which preserves the nonlinear structure of the original Prandlt equation (1.1), as well as the nonlinear cancellation properties. Then by an uniform energy estimate for the approximate solutions, the existence of solutions to the original Prandlt equation (1.1) follows. This energy estimate also implies the uniqueness and the stability. The uniform energy estimate for the approximate solutions is the main duty of this paper.

2.2. **Analysis of shear flow.** We write the solution (u, v) of system (1.1) as

$$u(t,x,y) = u^s(t,y) + \tilde{u}(t,x,y),\ v(t,x,y) = \tilde{v}(t,x,y),$$

where $u^s(t,y)$ is the solution of the following heat equation

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = 0, \\ u^s|_{y=0} = 0, \lim\limits_{y\to+\infty} u^s(t,y) = 1, \\ u^s|_{t=0} = u_0^s(y). \end{cases} \tag{2.1}$$

Then (1.1) can be written as

$$\begin{cases} \partial_t \tilde{u} + (u^s + \tilde{u})\partial_x \tilde{u} + \tilde{v}(u_y^s + \partial_y \tilde{u}) = \partial_y^2 \tilde{u}, \\ \partial_x \tilde{u} + \partial_y \tilde{v} = 0, \\ \tilde{u}|_{y=0} = \tilde{v}|_{y=0} = 0,\ \lim\limits_{y\to+\infty} \tilde{u} = 0, \\ \tilde{u}|_{t=0} = \tilde{u}_0(x,y)\,. \end{cases} \tag{2.2}$$

The equation of vorticity $\tilde{w} = \partial_y \tilde{u}$ reads

$$\begin{cases} \quad \partial_t \tilde{w} + (u^s + \tilde{u})\partial_x \tilde{w} + v(u_{yy}^s + \partial_y \tilde{w}) = \partial_y^2 \tilde{w}, \\ \quad \partial_y \tilde{w}|_{y=0} = 0, \\ \quad \tilde{w}|_{t=0} = \tilde{w}_0. \end{cases}$$

We first study the shear flow, and give a precise version for the condition on u_0^s: there exists $k > 1, a > 0$ and $a_0, c_0 > 0$ with $a_0 \le \frac{a}{20}$ such that

$$\begin{cases} u_0^s \in C^{m+4}([0,+\infty[),\ \ \lim\limits_{y\to+\infty} u_0^s(y) = 1; \\ (\partial_y^{2p} u_0^s)(0) = 0, \quad 0 \le 2p \le m+4; \\ \partial_y u_0^s(a) = 0,\ \partial_y^2 u_0^s(a) \ne 0,\ \text{and}\ \partial_y u_0^s(y) \ne 0,\ \forall y \in \bar{\mathbb{R}}_+ \setminus \{a\}\,; \\ |\partial_y u_0^s(y)| \ge 2c_0,\ \ 0 \le y \le a - a_0\,; \\ |\partial_y^2 u_0^s(y)| \ge 2c_0^2,\ \ a - 6a_0 \le y \le a + 6a_0 \\ c_0 \langle y \rangle^{-k} \le |\partial_y u_0^s(y)| \le c_0^{-1} \langle y \rangle^{-k},\ \ \ y \ge a + a_0\,; \\ |\partial_y^p u_0^s(y)| \le c_0^{-1} \langle y \rangle^{-k-p+1},\ \ y \ge 0,\ 1 \le p \le m+4\,. \end{cases} \tag{2.3}$$

We have first,

Lemma 2.1. *Assume that u_0^s satisfies (2.3). There exists a $T_1 > 0$ such that the solution $u^s(t,y)$ to the initial-boundary value problem (2.1) satisfies all the following properties on $[0,T_1] \times \mathbb{R}^+$,*

(1) $u^s \in L^\infty([0,T_1]; C^{m+4}([0,+\infty[)) \cap C^1([0,T_1]; C^{m+2}([0,+\infty[))$;
$(\partial_y^{2p} u^s)(t,0) = 0,\ \ 0 \le 2p \le m+2,\ \ \lim\limits_{y\to+\infty} u^s(t,y) = 1,\ \ t \in [0,T_1]$;

(2) *there exists* $\alpha \in C^{\frac{m}{2}+1}([0,T_1])$, $\alpha(0)=a$, *such that* $a-\frac{a_0}{2} \le \alpha(t) \le a+\frac{a_0}{2}$ *and* $(\partial_y u^s)(t,\alpha(t))=0$;
(3) $|(\partial_y u^s)(t,y)| \ge c_0,\ 0\le y\le a-2a_0,\ t\in[0,T_1]$;
(4) $|(\partial_y^2 u^s)(t,y)| \ge c_0^2,\ a-5a_0\le y\le a+5a_0,\ t\in[0,T_1]$;
(5) $\frac{c_0}{2}\langle y\rangle^{-k} \le |(\partial_y u^s)(t,y)| \le 2c_0^{-1}\langle y\rangle^{-k},\quad y\ge a+2a_0,\ t\in[0,T_1]$, *and* $|\partial_y^p u^s(t,y)| \le 2c_0^{-1}\langle y\rangle^{-k-p+1},\ \ y\ge 0,\ t\in[0,T_1],\ 1\le p\le m+4$.

Proof. Firstly, the solution of (2.1) can be written as

$$
\begin{aligned}
u^s(t,y) &= \frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde y)^2}{4t}} - e^{-\frac{(y+\tilde y)^2}{4t}}\Big)u_0^s(\tilde y)d\tilde y\\
&= \frac{1}{\sqrt{\pi}}\Big(\int_{-\frac{y}{2\sqrt t}}^{+\infty} e^{-\xi^2}u_0^s(2\sqrt t\xi+y)d\xi - \int_{\frac{y}{2\sqrt t}}^{+\infty} e^{-\xi^2}u_0^s(2\sqrt t\xi-y)d\xi\Big),
\end{aligned}
$$

which gives

$$
\begin{aligned}
\partial_t u^s(t,y) =& \frac{1}{\sqrt{\pi t}}\Big(\int_{-\frac{y}{2\sqrt t}}^{+\infty} \xi\, e^{-\xi^2}(\partial_y u_0^s)(2\sqrt t\xi+y)d\xi\\
&\qquad - \int_{\frac{y}{2\sqrt t}}^{+\infty} \xi\, e^{-\xi^2}(\partial_y u_0^s)(2\sqrt t\xi-y)d\xi\Big).
\end{aligned}
$$

By using $\partial_y^{2j}u_0^s(0)=0$ for $0\le 2j\le m+4$, it follows

$$
\begin{aligned}
\partial_y^p u^s(t,y) =& \frac{1}{\sqrt{\pi}}\Big(\int_{-\frac{y}{2\sqrt t}}^{+\infty} e^{-\xi^2}(\partial_y^p u_0^s)(2\sqrt t\xi+y)d\xi\\
&\quad + (-1)^{p+1}\int_{\frac{y}{2\sqrt t}}^{+\infty} e^{-\xi^2}(\partial_y^p u_0^s)(2\sqrt t\xi-y)d\xi\Big)\\
=& \frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde y)^2}{4t}} + (-1)^{p+1}e^{-\frac{(y+\tilde y)^2}{4t}}\Big)(\partial_y^p u_0^s)(\tilde y)d\tilde y,
\end{aligned}
\tag{2.4}
$$

for all $1\le p\le m+4$.

This gives the conclusion (1). The conclusions (3) and (4) can be obtained by the continuity of $\partial_y u^s(t,y)$ and $\partial_y^2 u^s(t,y)$ on $[0,T]\times[0,a+6a_0]$, where the smallness of $T_1>0$ is required. For the conclusion (2), we use (4) and the implicit function theorem to the equation

$$
(\partial_y u^s)(t,y(t))=0,\ \ (t,y(t))\in[0,T_1]\times[a-a_0/2,\, a+a_0/2],\ \ \partial_y u^s(0,a)=0,
$$

where $T_1>$ is again required to be small. In fact, for (2), the curve $\alpha(t)$ is the solution of the following ordinary differential equation

$$
\begin{cases}
\alpha'(t) = \frac{(\partial_y^3 u^s)(t,\alpha(t))}{(\partial_y^2 u^s)(t,\alpha(t))}, \quad t\in[0,T_1],\\
\alpha(0)=a,
\end{cases}
$$

where $\frac{(\partial_y^3 u^s)(t,y)}{(\partial_y^2 u^s)(t,y)} \in C^{\frac{m}{2}}([0,T_1]\times[a-a_0/2,\, a+a_0/2])$.

For the upper bound of the conclusion (5), (2.4) implies

$$
\begin{aligned}
|\partial_y^p u^s(t,y)| &\le \frac{1}{c_0 2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde y)^2}{4t}} + e^{-\frac{(y+\tilde y)^2}{4t}}\Big)\langle\tilde y\rangle^{-k-p+1}d\tilde y\\
&\le \frac{1}{c_0 2\sqrt{\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{(y+\tilde y)^2}{4t}}\langle\tilde y\rangle^{-k-p+1}d\tilde y\,.
\end{aligned}
$$

Using now Peetre's inequality, for any $\lambda \in \mathbb{R}$

$$\tilde{c}_0 \langle y\rangle^{\lambda} \langle y+\tilde{y}\rangle^{-|\lambda|} \le \langle \tilde{y}\rangle^{\lambda} \le \tilde{c}_0^{-1} \langle y\rangle^{\lambda} \langle y+\tilde{y}\rangle^{|\lambda|}, \tag{2.5}$$

we get then, with $\lambda = -k-p+1$,

$$|\partial_y^p u^s(t,y)| \le \tilde{c}_0^{-1} c_0^{-1} (1+t)^{\frac{k+p-1}{2}} \langle y\rangle^{-k-p+1}.$$

For the lower bound of $|\partial_y u^s(t,y)|$ on $y \ge a+2a_0$, we use the maximal principal, similar to [18]. We suppose that $\partial_y u_0^s(y) > 0$ for $y \ge R_0 = a+2a_0$, denote $H(t,y) = \langle y\rangle^k \partial_y u^s(t,y)$ (if $\partial_y u_0^s(y) < 0$ for $y \ge R_0$ and take $H(t,y) = -\langle y\rangle^k \partial_y u^s(t,y)$). Then $H(t,y)$ satisfies the following equation

$$\begin{cases} \partial_t H(t,y) - \partial_y^2 H(t,y) + h_1(y) H_y(t,y) + h_2(y) H(t,y) = 0, \\ H|_{y=R_0} = \langle R_0\rangle^k \partial_y u^s(t,R_0), \\ H|_{t=0} = \langle y\rangle^k \partial_y u_0^s(y), \end{cases}$$

where

$$h_1(y) = 2\frac{\partial_y \langle y\rangle^k}{\langle y\rangle^k},\ \ h_2(y) = -2\left(\frac{\partial_y \langle y\rangle^k}{\langle y\rangle^k}\right)^2 + \frac{\partial_y^2 \langle y\rangle^k}{\langle y\rangle^k}.$$

Firstly noticing

$$|h_1(y)| \le 2k, \quad |h_2(y)| \le 8k^2,$$

then we have

$$\begin{aligned} \partial_t (H(t,y) e^{-8k^2 t}) - \partial_y^2 (H(t,y) e^{-8k^2 t}) + h_1(y) \partial_y (H(t,y) e^{-8k^2 t}) \\ + \left(h_2(y) + 8k^2\right)(H(t,y) e^{-8k^2 t}) = 0, \end{aligned}$$

where the key point is that $h_2(y) + 8k^2 \ge 0$.

Denoting

$$b(T_1) = \min\left\{ \min_{0\le s\le T_1} H(s,R_0),\ \min_{y\ge R_0} H(0,y)\right\} \ge \frac{3c_0}{4},$$

and, for $\nu > 0$

$$E(t,y) = \left(H(t,y) - b(T_1)\right) e^{-8k^2 t} + (8k^2 b(T_1) + \nu\, 2k)t + \nu \ln(1+y),$$

also $E(t,y)$ satisfies

$$\partial_t E(t,y) - \partial_y^2 E(t,y) + h_1(y) E_y(t,y) + \left(h_2(y) + 8k^2\right) E(t,y) \ge 0.$$

Using the upper bound of (5), $|H(t,y)|_{L^\infty([0,T_1]\times\mathbb{R}^+)} \le 2c_0^{-1}$, setting

$$R_\nu = \exp\left\{ \frac{\left(|H(t,y)|_{L^\infty([0,T]\times\mathbb{R}^+)} + b(T_1)\right)}{\nu} \right\} - 1,$$

we have

$$\begin{aligned} E(t,y)|_{y=R_\nu} = \left(H(t,R_\nu) - b(T_1)\right) e^{-8k^2 t} + (8k^2 b(T_1) + \nu\, 2k)t \\ + \left(|H(t,y)|_{L^\infty([0,T_1]\times\mathbb{R}^+)} + b(T_1)\right) \ge 0, \end{aligned}$$

and the choose of $b(T_1)$ imply also

$$E(t,y)|_{y=R_0} \ge 0,\ t \in [0,T_1], \text{ and } E(t,y)|_{t=0} \ge 0,\ y \ge R_0.$$

Now thanks to the maximal principal of heat equation, we have

$$E(t,y)\geq 0,\ (t,y)\in[0,T_1]\times[R_0,R_\nu].$$

Let $\nu\to 0$, we get

$$H(t,y)\geq\frac{3c_0}{4}-(8k^2)te^{8k^2t}.$$

Then we can choose T_1 such that

$$(8k^2)T_1e^{8k^2T_1}\leq\frac{c_0}{4},$$

then

$$H(t,y)_{\geq}\frac{c_0}{2},\ (t,y)\in[0,\,T_1]\times[R_0,\infty[.$$

□

If the initial date of shear flow u_0^s is uniformly monotone on $\mathbb{R}_+$, we have

Corollary 2.2. *Assume that the initial date u_0^s satisfy (1.6), then for any $T>0$, there exist $\tilde c_1,\tilde c_2,\tilde c_3>0$ such that the solution $u^s(t,y)$ of the initial boundary value problem (2.1) satisfies*

$$\begin{cases}\tilde c_1\langle y\rangle^{-k}\leq|\partial_y u^s(t,y)|\leq\tilde c_2\langle y\rangle^{-k},\ \forall\,(t,y)\in[0,T]\times\bar{\mathbb{R}}_+,\\ |\partial_y^p u^s(t,y)|\leq\tilde c_3\langle y\rangle^{-k-p+1},\ \forall\ (t,y)\in[0,T]\times\bar{\mathbb{R}}_+,\ 1\leq p\leq m+4,\end{cases}\tag{2.6}$$

where $\tilde c_1,\tilde c_2,\tilde c_3$ depend on T.

Proof. We only need to prove the first estimate, using (2.4) for $p=1$,

$$\begin{aligned}\partial_y u^s(t,y)=&\frac{1}{\sqrt{\pi}}\Big(\int_{-\frac{y}{2\sqrt t}}^{+\infty}e^{-\xi^2}(\partial_y u_0^s)(2\sqrt t\xi+y)d\xi\\ &+\int_{\frac{y}{2\sqrt t}}^{+\infty}e^{-\xi^2}(\partial_y u_0^s)(2\sqrt t\xi-y)d\xi\Big)\\ =&\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde y)^2}{4t}}+e^{-\frac{(y+\tilde y)^2}{4t}}\Big)(\partial_y u_0^s)(\tilde y)d\tilde y\,.\end{aligned}$$

Thanks to the monotonic assumption (1.6), we have that

$$\begin{aligned}\partial_y u^s(t,y)&\approx\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde y)^2}{4t}}+e^{-\frac{(y+\tilde y)^2}{4t}}\Big)\langle\tilde y\rangle^{-k}d\tilde y\\ &\approx\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(y+\tilde y)^2}{4t}}\langle\tilde y\rangle^{-k}d\tilde y\,.\end{aligned}$$

Using again Peetre's inequality (2.5) for $\lambda=-k$, we get then

$$(1+t)^{-\frac{k}{2}}\langle y\rangle^{-k}\lesssim\partial_y u^s(t,y)\lesssim(1+t)^{\frac{k}{2}}\langle y\rangle^{-k}.$$

So we get (2.6) with

$$\tilde c_1=c_1(1+T)^{-\frac{k}{2}},\ \tilde c_2=c_2(1+T)^{\frac{k}{2}}.\tag{2.7}$$

□

2.3. **Compatibility conditions and reduction of boundary data.** We give now the precise version of the compatibility condition for the nonlinear system (2.2) and the reduction properties of boundary data.

Proposition 2.3. *Let $m \geq 6$ be an even integer, and assume that $\tilde{u}$ is a smooth solution of the system (2.2), then the initial data $\tilde{u}_0$ have to satisfy the following compatibility conditions up to order $m+2$:*

$$\begin{cases} \tilde{u}_0(x,0) = 0, \quad (\partial_y^2 \tilde{u}_0)(x,0) = 0, \ \forall x \in \mathbb{R}, \\ (\partial_y^4 \tilde{u}_0)(x,0) = \big(\partial_y u_0^s(0) + (\partial_y \tilde{u}_0)(x,0)\big)(\partial_y \partial_x \tilde{u}_0)(x,0), \forall x \in \mathbb{R}, \end{cases} \tag{2.8}$$

and for $4 \leq 2p \leq m$,

$$(\partial_y^{2(p+1)} \tilde{u}_0)(x,0) = \sum_{q=2}^{p} \sum_{(\alpha,\beta)\in\Lambda_q} C_{\alpha,\beta} \prod_{j=1}^{q} \partial_x^{\alpha_j} \partial_y^{\beta_j+1} \big(u_0^s + \tilde{u}_0\big)\big|_{y=0}, \ \ \forall x \in \mathbb{R}, \tag{2.9}$$

where

$$\begin{aligned} \Lambda_q = \Big\{ &(\alpha,\beta) = (\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_q) \in \mathbb{N}^q \times \mathbb{N}^q; \\ &\alpha_j + \beta_j \leq 2p-1, \ \ 1 \leq j \leq q; \ \sum_{j=1}^{q} 3\alpha_j + \beta_j = 2p+1; \\ &\sum_{j=1}^{q} \beta_j \leq 2p-2, \ \ 0 < \sum_{j=1}^{q} \alpha_j \leq p-1 \Big\}. \end{aligned} \tag{2.10}$$

Remark that for $\alpha_j > 0$, we have $\partial_x^{\alpha_j}\partial_y^{\beta_j+1}\big(u^s + \tilde{u}\big) = \partial_x^{\alpha_j}\partial_y^{\beta_j+1}\tilde{u}$. So the condition $0 < \sum_{j=1}^{q} \alpha_j$ implies that, for each terms of (2.9), there is at last one factor like $\partial_x^{\alpha_j}\partial_y^{\beta_j+1}\tilde{u}_0$.

Proof. By the assumption of this Proposition, $\tilde{u}$ is a smooth solution. If we need the existence of the trace of $\partial_y^{m+2}\tilde{u}$ on $y=0$, then we at least need to assume that $\tilde{u} \in L^\infty([0,T]; H^{m+3}_{k+\ell-1}(\mathbb{R}^2_+))$.

Recalling the boundary condition in (2.2):

$$\tilde{u}(t,x,0) = 0, \quad \tilde{v}(t,x,0) = 0, \quad (t,x) \in [0,T] \times \mathbb{R},$$

then the following is obvious:

$$(\partial_t \partial_x^n \tilde{u})(t,x,0) = 0, \quad (\partial_t \partial_x^n \tilde{v})(t,x,0) = 0, \quad (t,x) \in [0,T]\times\mathbb{R}, \ 0 \leq n \leq m.$$

Thus the first result of (2.8) is exactly the compatibility of the solution with the initial data at $t=0$. For the second result of (2.8), using the equation of (2.2), we find that, fro $0 \leq n \leq m$

$$(\partial_y^2 \partial_x^n \tilde{u})(t,x,0) = 0, \quad (\partial_t \partial_y^2 \partial_x^n \tilde{u})(t,x,0) = 0, \quad (t,x) \in [0,T]\times\mathbb{R}.$$

Derivating the equation of (2.2) with y,

$$\partial_t \partial_y \tilde{u} + \partial_y\big((u^s + \tilde{u})\partial_x \tilde{u}\big) + \partial_y\big(\tilde{v}(u^s_y + \partial_y \tilde{u})\big) = \partial_y^3 \tilde{u},$$

observing

$$\left(\partial_y\big((u^s + \tilde{u})\partial_x \tilde{u}\big) + \partial_y\Big(\tilde{v}(u^s_y + \partial_y \tilde{u})\Big)\right)\Big|_{y=0} = 0,$$

then we get

$$(\partial_t\partial_y\tilde{u}))|_{y=0} = (\partial_y^3\tilde{u}_\epsilon)|_{y=0}.$$

Derivating again the equation of (2.2) with y,

$$\partial_t\partial_y^2\tilde{u} + \partial_y^2\Big((u^s+\tilde{u})\partial_x\tilde{u}\Big) + \partial_y^2\Big(\tilde{v}(u^s_y+\partial_y\tilde{u})\Big) = \partial_y^4\tilde{u},$$

using Leibniz formula

$$\begin{aligned}
&\partial_y^2\Big((u^s+\tilde{u})\partial_x\tilde{u}\Big) + \partial_y^2\Big(\tilde{v}(u^s_y+\partial_y\tilde{u})\Big)\\
&= (\partial_y^2(u^s+\tilde{u}))\partial_x\tilde{u} + (\partial_y^2\tilde{v})(u^s_y+\partial_y\tilde{u})\\
&\quad + (u^s+\tilde{u})\partial_y^2\partial_x\tilde{u} + \tilde{v}\partial_y^2(u^s_y+\partial_y\tilde{u})\\
&\qquad + 2(\partial_y(u^s+\tilde{u}))\partial_y\partial_x\tilde{u} + 2(\partial_y\tilde{v})\partial_y(u^s_y+\partial_y\tilde{u}),
\end{aligned}$$

thus,

$$(\partial_y^4\tilde{u})(t,x,0) = \Big(u^s_y(t,0) + (\partial_y\tilde{u})(t,x,0)\Big)(\partial_y\partial_x\tilde{u})(t,x,0),$$

and

$$\begin{aligned}
(\partial_t\partial_y^4\tilde{u})(t,x,0) =& \Big(\partial_y u^s(t,0) + (\partial_y\tilde{u})(t,x,0)\Big)\Big((\partial_y^3\partial_x\tilde{u})(t,x,0)\Big)\\
&+ \Big(\partial_y^3 u^s(t,0) + (\partial_y^3\tilde{u})(t,x,0)\Big)\Big((\partial_y\partial_x\tilde{u})(t,x,0)\Big).
\end{aligned} \tag{2.11}$$

For $p=2$, we have

$$\partial_t\partial_y^4\tilde{u} + \partial_y^4\Big((u^s+\tilde{u})\partial_x\tilde{u}\Big) + \partial_y^4\Big(\tilde{v}(u^s_y+\partial_y\tilde{u})\Big) = \partial_y^6\tilde{u},$$

using Leibniz formula

$$\begin{aligned}
&\partial_y^4\Big((u^s+\tilde{u})\partial_x\tilde{u}\Big) + \partial_y^4\Big(\tilde{v}(u^s_y+\partial_y\tilde{u})\Big)\\
&= (\partial_y^4(u^s+\tilde{u}))\partial_x\tilde{u} + (\partial_y^4\tilde{v})(u^s_y+\partial_y\tilde{u})\\
&\quad + (u^s+\tilde{u})\partial_y^4\partial_x\tilde{u} + \tilde{v}\partial_y^4(u^s_y+\partial_y\tilde{u})\\
&\qquad + \sum_{1\le j\le 3} C^4_j\Big((\partial_y^j(u^s+\tilde{u}))\partial_y^{4-j}\partial_x\tilde{u} + (\partial_y^j\tilde{v})\partial_y^{4-j}(u^s_y+\partial_y\tilde{u})\Big),
\end{aligned}$$

thus, by (2.11)

$$\begin{aligned}
(\partial_y^6\tilde{u})(t,x,0) &= (\partial_t\partial_y^4\tilde{u})(t,x,0) - (\partial_y^3\partial_x u)(u^s_y+\partial_y\tilde{u})(t,x,0)\\
&\quad + \sum_{1\le j\le 3} C^4_j\Big((\partial_y^j(u^s+\tilde{u}))\partial_y^{4-j}\partial_x\tilde{u} + (\partial_y^j\tilde{v})\partial_y^{4-j}(u^s_y+\partial_y\tilde{u})\Big)(t,x,0)\\
&= \quad \Big(\partial_y^3 u^s(t,0) + (\partial_y^3\tilde{u})(t,x,0)\Big)\Big((\partial_y\partial_x\tilde{u})(t,x,0)\Big)\\
&\quad + \sum_{1\le j\le 3} C^4_j\Big((\partial_y^j(u^s+\tilde{u}))\partial_y^{4-j}\partial_x\tilde{u} - (\partial_y^{j-1}\partial_x\tilde{u})\partial_y^{4-j}(u^s_y+\partial_y\tilde{u})\Big)(t,x,0).
\end{aligned} \tag{2.12}$$

Taking the values at $t=0$, we have proven (2.9) for $p=2$. The case of $p\ge 3$ is then by induction. □

Remark 2.4. *By the similar methods, we can prove that if $\tilde{u}$ is a smooth solution of the system* (2.2)*, then we have*

$$\begin{cases} \tilde{u}(t,x,0)=0,\ (\partial_y^2\tilde{u})(t,x,0)=0,\ \forall(t,x)\in[0,T]\times\mathbb{R},\\ (\partial_y^4\tilde{u})(t,x,0)=\big(u_y^s(t,0)+(\partial_y\tilde{u})(t,x,0)\big)(\partial_y\partial_x\tilde{u})(t,x,0),\forall(t,x)\in[0,T]\times\mathbb{R}, \end{cases}$$

and for $4\le 2p\le m$,

$$(\partial_y^{2(p+1)}\tilde{u})(t,x,0)=\sum_{q=2}^{p}\sum_{(\alpha,\beta)\in\Lambda_q}C_{\alpha,\beta}\prod_{j=1}^{q}\partial_x^{\alpha_j}\partial_y^{\beta_j+1}\Big(u^s(t,0)+\tilde{u}(t,x,0)\Big),\tag{2.13}$$

for all $(t,x)\in[0,T]\times\mathbb{R}$, where Λ_q is defined in (2.10)*.*

See Lemma 5.9 of [18] *and Lemma 4 of* [7] *for the similar results.*

Remark that the condition $0<\sum_{j=1}^{q}\alpha_j$ implies that, for each terms of (2.13), there is at last one factor like $\partial_x^{\alpha_j}\partial_y^{\beta_j+1}\tilde{u}(t,x,0)$.

3. The approximate solutions

To prove the existence of solution of the Prandtl equation, we study a parabolic regularized equation for which we can get the existence by using the classical energy method.

3.1. **Nonlinear regularized Prandtl equation.** In this section, we study the following nonlinear regularized Prandtl equation, for $0<\epsilon\le 1$,

$$\begin{cases} \partial_t\tilde{u}_\epsilon+(u^s+\tilde{u}_\epsilon)\partial_x\tilde{u}_\epsilon+v_\epsilon(u_y^s+\partial_y\tilde{u}_\epsilon)=\partial_y^2\tilde{u}_\epsilon+\epsilon\partial_x^2\tilde{u}_\epsilon,\\ \partial_x\tilde{u}_\epsilon+\partial_y v_\epsilon=0,\\ \tilde{u}_\epsilon|_{y=0}=v_\epsilon|_{y=0}=0,\ \lim\limits_{y\to+\infty}\tilde{u}_\epsilon=0,\\ \tilde{u}_\epsilon|_{t=0}=\tilde{u}_{0,\epsilon}=\tilde{u}_0+\epsilon\mu_\epsilon\,, \end{cases}\tag{3.1}$$

where we choose the corrector $\epsilon\mu_\epsilon$ such that $\tilde{u}_0+\epsilon\mu_\epsilon$ satisfies the compatibility condition up to order $m+2$ for the regularized system (3.1).

The equation of vorticity $\tilde{w}_\epsilon=\partial_y\tilde{u}_\epsilon$ reads

$$\begin{cases} \partial_t\tilde{w}_\epsilon+(u^s+\tilde{u}_\epsilon)\partial_x\tilde{w}_\epsilon+v_\epsilon(u_{yy}^s+\partial_y\tilde{w}_\epsilon)=\partial_y^2\tilde{w}_\epsilon+\epsilon\partial_x^2\tilde{w}_\epsilon,\\ \partial_y\tilde{w}_\epsilon|_{y=0}=0,\\ \tilde{w}_\epsilon|_{t=0}=\tilde{w}_{0,\epsilon}=\tilde{w}_0+\epsilon\partial_y\mu_\epsilon. \end{cases}\tag{3.2}$$

Formally the solution sequence $(u^s+\tilde{u}_\epsilon,\tilde{v}_\epsilon)$ of above system is the approximate solution of the original Prandtl equation (1.1).

We give now the boundary data of the solution for the regularized nonlinear system (3.1) which deduce also the compatibility condition for the system (3.1).

Proposition 3.1. *Let $m\ge 6$ be an even integer, and assume that $\tilde{u}_0$ satisfies the compatibility conditions* (2.8) *and* (2.9) *for the system* (2.2)*, and $\mu_\epsilon\in H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)$ such that $\tilde{u}_0+\epsilon\mu_\epsilon$ satisfies the compatibility conditions up to order $m+2$ for the regularized system* (3.1)*. If $\tilde{u}_\epsilon\in L^\infty([0,T];H^{m+3}_{k+\ell}(\mathbb{R}^2_+))\cap Lip([0,T];H^{m+1}_{k+\ell}(\mathbb{R}^2_+))$ is a solution of the system* (3.1)*, then we have*

$$\begin{cases} \tilde{u}_\epsilon(t,x,0)=0,\ (\partial_y^2\tilde{u}_\epsilon)(t,x,0)=0,\ \forall(t,x)\in[0,T]\times\mathbb{R},\\ (\partial_y^4\tilde{u}_\epsilon)(t,x,0)=\big(u_y^s(t,0)+(\partial_y\tilde{u}_\epsilon)(t,x,0)\big)(\partial_y\partial_x\tilde{u}_\epsilon)(t,x,0),\forall(t,x)\in[0,T]\times\mathbb{R}, \end{cases}$$

and for $4 \le 2p \le m$,

$$
\begin{aligned}
(\partial_y^{2(p+1)}\tilde{u}_\epsilon)(t,x,0) = \sum_{q=2}^{p}\sum_{l=0}^{q-1}\epsilon^l \sum_{(\alpha^l,\beta^l)\in\Lambda_q^l} C_{\alpha^l,\beta^l} \\
\times \prod_{j=1}^{q}\partial_x^{\alpha_j^l}\partial_y^{\beta_j^l+1}\Big(u^s(t,0)+\tilde{u}_\epsilon(t,x,0)\Big),
\end{aligned}
\tag{3.3}
$$

for all $(t,x)\in[0,T]\times\mathbb{R}$, *where*

$$
\begin{aligned}
\Lambda_q^l = \Big\{ &(\alpha,\beta)=(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_p)\in\mathbb{N}^q\times\mathbb{N}^q; \\
&\alpha_j+\beta_j\le 2p-1, ,\ 1\le j\le q;\ \sum_{j=1}^{q}3\alpha_j+\beta_j = 2p+4l+1; \\
&\sum_{j=1}^{q}\beta_j \le 2p-2l-2,\ \ 0<\sum_{j=1}^{q}\alpha_j\le p+2l-1\Big\}.
\end{aligned}
$$

Remark that the condition $0<\sum_{j=1}^{q}\alpha_j^l$ implies that, for each terms of (3.3), there are at last one factor like $\partial_x^{\alpha_j^l}\partial_y^{\beta_j^l+1}\tilde{u}_\epsilon(t,x,0)$.

Proof. Firstly, for $p\le\frac{m}{2}$, we have $\partial_y^{2p+2}\tilde{u}_\epsilon\in L^\infty([0,T];H^1_{k+\ell+2p+1}(\mathbb{R}^2_+))$. So the trace of $\partial_y^{2p+2}\tilde{u}_\epsilon$ exists on $y=0$.

Using the boundary condition of (3.1), we have, for $0\le n\le m+2$,

$$
\partial_x^n\tilde{u}_\epsilon(t,x,0)=0,\quad \partial_x^n v_\epsilon(t,x,0)=0,\quad (t,x)\in[0,T]\times\mathbb{R},
$$

and for $0\le n\le m$

$$
(\partial_t\partial_x^n\tilde{u}_\epsilon)(t,x,0)=0,\quad (\partial_t\partial_x^n v_\epsilon)(t,x,0)=0,\quad (t,x)\in[0,T]\times\mathbb{R}.
$$

From the equation of (3.1), we get also

$$
(\partial_y^2\partial_x^n\tilde{u}_\epsilon)(t,x,0)=0,\quad (\partial_t\partial_y^2\partial_x^n\tilde{u}_\epsilon)(t,x,0)=0,\quad (t,x)\in[0,T]\times\mathbb{R}. \tag{3.4}
$$

On the other hand,

$$
\partial_t\partial_y\tilde{u}_\epsilon+\partial_y\big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{u}_\epsilon\big)+\partial_y\big(v_\epsilon(u^s_y+\partial_y\tilde{u}_\epsilon)\big)=\partial_y^3\tilde{u}_\epsilon+\epsilon\partial_x^2\partial_y\tilde{u}_\epsilon,
$$

observing

$$
\left(\partial_y\big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{u}_\epsilon\big)+\partial_y\Big(v_\epsilon(u^s_y+\partial_y\tilde{u}_\epsilon)\Big)\right)\Big|_{y=0}=0,
$$

we get

$$
(\partial_t\partial_y\tilde{u}_\epsilon)|_{y=0}=(\partial_y^3\tilde{u}_\epsilon)|_{y=0}+\epsilon(\partial_x^2\partial_y\tilde{u}_\epsilon)|_{y=0}.
$$

We have also

$$
\partial_t\partial_y^2\tilde{u}_\epsilon+\partial_y^2\Big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{u}_\epsilon\Big)+\partial_y^2\Big(v_\epsilon(u^s_y+\partial_y\tilde{u}_\epsilon)\Big)=\partial_y^4\tilde{u}_\epsilon+\epsilon\partial_x^2\partial_y^2\tilde{u}_\epsilon,
$$

using Leibniz formula

$$
\begin{aligned}
&\partial_y^2\Big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{u}_\epsilon\Big)+\partial_y^2\Big(v_\epsilon(u^s_y+\partial_y\tilde{u}_\epsilon)\Big)\\
&=(\partial_y^2(u^s+\tilde{u}_\epsilon))\partial_x\tilde{u}_\epsilon+(\partial_y^2 v_\epsilon)(u^s_y+\partial_y\tilde{u}_\epsilon)
\end{aligned}
$$

$$+(u^s+\tilde u_\epsilon)\partial_y^2\partial_x\tilde u_\epsilon+v_\epsilon\partial_y^2(u_y^s+\partial_y\tilde u_\epsilon)$$
$$+2(\partial_y(u^s+\tilde u_\epsilon))\partial_y\partial_x\tilde u_\epsilon+2(\partial_y v_\epsilon)\partial_y(u_y^s+\partial_y\tilde u_\epsilon),$$

thus,

$$(\partial_y^4\tilde u_\epsilon)(t,x,0)=\big(u_y^s(t,0)+(\partial_y\tilde u_\epsilon)(t,x,0)\big)\,(\partial_y\partial_x\tilde u_\epsilon)(t,x,0). \tag{3.5}$$

Applying ∂_t to (3.5), we have

$$(\partial_t\partial_y^4\tilde u_\epsilon)(t,x,0)=\big(\partial_y^3u^s(t,0)+(\partial_y^3\tilde u_\epsilon)(t,x,0)+\epsilon(\partial_x^2\partial_y\tilde u_\epsilon)(t,x,0)\big)\,(\partial_y\partial_x\tilde u_\epsilon)(t,x,0)$$
$$+\big(u_y^s(t,0)+(\partial_y\tilde u_\epsilon)(t,x,0)\big)\,\big((\partial_y^3\partial_x\tilde u_\epsilon)(t,x,0)+\epsilon(\partial_x^3\partial_y\tilde u_\epsilon)(t,x,0)\big).$$

On the other hand, we have

$$\partial_t\partial_y^4\tilde u_\epsilon+\partial_y^4\Big((u^s+\tilde u_\epsilon)\partial_x\tilde u_\epsilon\Big)+\partial_y^4\Big(v_\epsilon(u_y^s+\partial_y\tilde u_\epsilon)\Big)=\partial_y^6\tilde u_\epsilon+\epsilon\partial_x^2\partial_y^4\tilde u_\epsilon,$$

using Leibniz formula

$$\begin{aligned}&\partial_y^4\Big((u^s+\tilde u_\epsilon)\partial_x\tilde u_\epsilon\Big)+\partial_y^4\Big(v_\epsilon(u_y^s+\partial_y\tilde u_\epsilon)\Big)\\&=(\partial_y^4(u^s+\tilde u_\epsilon))\partial_x\tilde u_\epsilon+(\partial_y^4v_\epsilon)(u_y^s+\partial_y\tilde u_\epsilon)\\&\quad+(u^s+\tilde u_\epsilon)\partial_y^4\partial_x\tilde u_\epsilon+v_\epsilon\partial_y^4(u_y^s+\partial_y\tilde u_\epsilon)\\&\quad\quad+\sum_{1\le j\le 3}C_j^4\Big((\partial_y^j(u^s+\tilde u_\epsilon))\partial_y^{4-j}\partial_x\tilde u_\epsilon+(\partial_y^jv_\epsilon)\partial_y^{4-j}(u_y^s+\partial_y\tilde u_\epsilon)\Big),\end{aligned}$$

thus,

$$\begin{aligned}(\partial_y^6\tilde u_\epsilon)(t,x,0)&=(\partial_t\partial_y^4\tilde u_\epsilon)(t,x,0)-(\partial_y^3\partial_xu_\epsilon)(u_y^s+\partial_y\tilde u_\epsilon)(t,x,0)\\&\quad+\sum_{1\le j\le 3}C_j^4\Big((\partial_y^j(u^s+\tilde u_\epsilon))\partial_y^{4-j}\partial_x\tilde u_\epsilon+(\partial_y^jv_\epsilon)\partial_y^{4-j}(u_y^s+\partial_y\tilde u_\epsilon)\Big)(t,x,0)\\&\quad\quad-\underline{\epsilon\partial_x^2\partial_y^4\tilde u_\epsilon(t,x,0)}.\end{aligned}$$

Using (3.5), we get then

$$\begin{aligned}(\partial_y^6\tilde u_\epsilon)(t,x,0)&=\big(\partial_y^3u^s(t,0)+\partial_y^3\tilde u_\epsilon(t,x,0)\big)\partial_y\partial_x\tilde u_\epsilon(t,x,0)\\&\qquad-\underline{2\epsilon\partial_x\partial_y\tilde u_\epsilon(t,x,0)(\partial_y\partial_x^2\tilde u_\epsilon)(t,x,0)}\\&+\sum_{1\le j\le 3}C_j^4\Big((\partial_y^j(u^s+\tilde u_\epsilon))\partial_y^{4-j}\partial_x\tilde u_\epsilon-\partial_y^{j-1}\partial_x\tilde u_\epsilon\partial_y^{4-j}(u_y^s+\partial_y\tilde u_\epsilon)\Big)(t,x,0),\end{aligned} \tag{3.6}$$

Compared to (2.12), the underlined term is the new term.

This is the Proposition 3.1 for $p=2$. We can complete the proof of Proposition 3.1 by induction. □

The proof of the above Proposition implies also the following result.

Corollary 3.2. *Let $m\ge 6$ be an even integer, assume that $\tilde u_0$ satisfies the compatibility conditions (2.8) - (2.9) for the system (2.2) and $\|\tilde w_0\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\le 1$, then there exists $\epsilon_0>0$, and for any $0<\epsilon\le\epsilon_0$ there exists $\mu_\epsilon\in H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)$ such that $\tilde u_0+\epsilon\mu_\epsilon$ satisfies the compatibility condition up to order $m+2$ for the regularized system (3.1). Moreover,*

$$\|\tilde w_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\le\frac{3}{2}\|\tilde w_0\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)},$$

and

$$\lim_{\epsilon\to 0}\|\tilde w_{0,\epsilon}-\tilde w_0\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}=0.$$

Proof. We use the proof of the Proposition 3.1.

Taking the values at $t=0$ for (3.4), then (2.8) implies that the function μ_ϵ satisfies

$$(\partial_x^n\mu_\epsilon)(x,0)=0,\quad (\partial_y^2\partial_x^n\mu_\epsilon)(x,0)=0,\quad x\in\mathbb{R}\,.$$

Taking $t=0$ for (3.5), we have

$$\begin{aligned}(\partial_y^4\tilde u_0)(x,0)+\epsilon(\partial_y^4\mu_\epsilon)(x,0))=&\Big(\partial_y u_0^s(0)+(\partial_y\tilde u_0)(x,0)+\epsilon(\partial_y\mu_\epsilon)(x,0)\Big)\\ &\times\Big((\partial_y\partial_x\tilde u_0)(x,0)+\epsilon(\partial_y\partial_x\mu_\epsilon)(x,0)\Big),\end{aligned}$$

using (2.8), we have that μ_ϵ satisfies

$$\begin{aligned}(\partial_y^4\mu_\epsilon)(x,0))=&\big(\partial_y u_0^s(0)+(\partial_y\tilde u_0)(x,0)\big)(\partial_y\partial_x\mu_\epsilon)(x,0)\\ &+(\partial_y\mu_\epsilon)(x,0)(\partial_y\partial_x\tilde u_0)(x,0)\\ &+\epsilon(\partial_y\partial_x\mu_\epsilon)(x,0)(\partial_y\partial_x\mu_\epsilon)(x,0).\end{aligned}$$

We have also

$$\begin{aligned}(\partial_t\partial_y^4\tilde u_\epsilon)(0,x,0)=&\Big(\partial_y^3 u_0^s(0)+(\partial_y^3\tilde u_\epsilon)(0,x,0)+\epsilon(\partial_x^2\partial_y\tilde u_\epsilon)(0,x,0)\Big)\\ &\times\Big((\partial_y^3\partial_x\tilde u_\epsilon)(0,x,0)+\epsilon(\partial_x^3\partial_y\tilde u_\epsilon)(0,x,0)\Big).\end{aligned}$$

Taking the values at $t=0$ for (3.6), we obtain a restraint condition for $(\partial_y^6\mu_\epsilon)(x,0)$,

$$\begin{aligned}\partial_y^6\mu_\epsilon(x,0)=&((\partial_y^3 u_0^s+\partial_y^3\tilde u_0)\partial_y\partial_x\mu_\epsilon)|_{y=0}+\partial_y^3\mu_\epsilon\partial_y\partial_x\tilde u_0|_{y=0}+\epsilon\partial_y^3\mu_\epsilon\partial_y\partial_x\mu_\epsilon|_{y=0}\\ &-\underline{2\partial_x\partial_y\tilde u_0(x,0)(\partial_y\partial_x^2\tilde u_0)(x,0)}-2\epsilon\partial_x\partial_y\tilde u_0(x,0)(\partial_y\partial_x^2\mu_\epsilon)(t,x,0)\\ &-2\epsilon\partial_x\partial_y\mu_\epsilon(t,x,0)(\partial_y\partial_x^2\tilde u_0)(t,x,0)-2\epsilon^2\partial_x\partial_y\mu_\epsilon(x,0)(\partial_y\partial_x^2\mu_\epsilon)(x,0)\\ &+\sum_{1\le j\le 3}C_j^4\Big(\partial_y^j\big(u_0^s+\tilde u_0\big)\partial_y^{4-j}\partial_x\mu_\epsilon+\partial_y^j\mu\partial_y^{4-j}\partial_x\tilde u_0+\epsilon\partial_y^j\mu\partial_y^{4-j}\partial_x\mu_\epsilon\Big)\Big|_{y=0}\\ &-\sum_{1\le j\le 3}C_j^4\Big(\partial_y^{j-1}\partial_x\tilde u_0\partial_y^{4-j}\mu_\epsilon+\epsilon\partial_y^{j-1}\partial_x\mu_\epsilon\partial_y^{4-j}\partial_y\mu_\epsilon\Big)\Big|_{y=0}\\ &-\sum_{1\le j\le 3}C_j^4\partial_y^{j-1}\partial_x\mu_\epsilon\partial_y^{4-j}(\partial_y u_0^s+\partial_y\tilde u_0)\big|_{y=0},\end{aligned}$$

thus

$$\begin{aligned}\partial_y^6\mu_\epsilon(x,0)=&-\underline{2\partial_x\partial_y\tilde u_0(x,0)(\partial_y\partial_x^2\tilde u_0)(x,0)}\\ &+\sum_{\alpha_1,\beta_1;\alpha_2,\beta_2}C_{\alpha_1,\beta_1;\alpha_2,\beta_2}\partial_x^{\alpha_1}\partial_y^{\beta_1+1}(u_0^s+\tilde u_0)\partial_x^{\alpha_1}\partial_y^{\beta_1+1}\mu_\epsilon(x,0)\\ &+\sum_{\alpha_1,\beta_1;\alpha_2,\beta_2}C_{\alpha_1,\beta_1;\alpha_2,\beta_2}\partial_x^{\alpha_1}\partial_y^{\beta_1+1}\mu_\epsilon\partial_x^{\alpha_1}\partial_y^{\beta_1+1}\mu_\epsilon(x,0),\end{aligned}\tag{3.7}$$

where the summation is for the index $\alpha_2+\beta_2\le 3$; $\alpha_1+\beta_1+\alpha_2+\beta_2\le 3$. The underlined term in the above equality is deduced from the underlined term in (3.6). All these underlined terms are from the added regularizing term $\epsilon\partial_x^2\tilde u$ in the equation (3.1). This means that the regularizing term $\epsilon\partial_x^2\tilde{u}$ has an affect on the boundary. This is why we add a corrector term.

More generally, for $6 \le 2p \le m$, we have that $(\partial_y^{2(p+1)}\mu_\epsilon)(x,0)$ is a linear combination of the terms of the form

$$\prod_{j=1}^{q_1}\left(\partial_x^{\alpha_j^1}\partial_y^{\beta_j^1+1}(u_0^s+\tilde{u}_0)\right)\Big|_{y=0}, \quad \prod_{i=1}^{q_2}\left(\partial_x^{\alpha_i^2}\partial_y^{\beta_i^2+1}\mu_\epsilon\right)\Big|_{y=0},$$

and

$$\prod_{j=1}^{q_1}\left(\partial_x^{\alpha_j^1}\partial_y^{\beta_j^1+1}(u_0^s+\tilde{u}_0)\right)\Big|_{y=0} \times \prod_{i=1}^{q_2}\left(\partial_x^{\alpha_i^2}\partial_y^{\beta_i^2+1}\mu_\epsilon\right)\Big|_{y=0},$$

where the coefficients of the combination can be depends on ϵ but with a non-negative power. We have also $\alpha_j^l + \beta_j^l + 1 \le 2p, l = 1, 2$, thus $(\partial_y^{2(p+1)}\mu_\epsilon)(x,0)$ is determined by the low order derivatives of μ_ϵ and these of $\tilde{u}_0$.

We now construct a polynomial function $\tilde{\mu}_\epsilon$ on y by the following Taylor expansion,

$$\tilde{\mu}_\epsilon(x,y) = \sum_{p=3}^{\frac{m}{2}+1}\tilde{\mu}_\epsilon^{2p}(x)\frac{y^{2p}}{(2p)!},$$

where

$$\tilde{\mu}_\epsilon^6(x) = -2(\partial_x\partial_y\tilde{u}_0)(x,0)(\partial_y\partial_x^2\tilde{u}_0)(x,0),$$

and $\tilde{\mu}_\epsilon^{2p}(x)$ will give successively by $(\partial_y^{2q}\mu_\epsilon)(x,0)$ with $(\partial_y^{2q+1}\mu_\epsilon)(x,0) = 0, q = 0, \cdots, m$, and it is then determined by $(\partial_x^\alpha\partial_y^\beta\tilde{u}_0)|_{y=0}$. Finally we take $\mu_\epsilon = \chi(y)\tilde{\mu}_\epsilon$ with $\chi \in C^\infty([0,+\infty[); \chi(y) = 1,\ 0 \le y \le 1; \chi(y) = 0,\ y \ge 2$. Thus we complete the proof of the Corollary. □

Remark 3.3. *Suppose that $\tilde{u}_0$ satisfies the compatibility conditions up to order $m+2$ for the system (2.2) with $m \ge 4$, then for the regularized system (3.1), if we want to obtain the smooth solution $\tilde{w}_\epsilon$, we have to add a non-trivial corrector μ_ϵ to the initial data such that $\tilde{u}_0 + \epsilon\mu_\epsilon$ satisfies the compatibility conditions up to order $m+2$ for the system (3.1). In fact, if we take μ_ϵ with*

$$(\partial_y^j\mu_\epsilon)(x,0) = 0, \quad 0 \le j \le 5,$$

then (3.7) implies

$$(\partial_y^6\mu_\epsilon)(x,0) = -2(\partial_x\partial_y\tilde{u}_0)(x,0)(\partial_y\partial_x^2\tilde{u}_0)(x,0),$$

which is not equal to 0. So added a corrector is necessary for the initial data of the regularized system.

We will prove the following theorem for the existence of approximate solutions.

Theorem 3.4. *Let $\partial_y\tilde{u}_0 \in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$, and $m \ge 6$ be an even integer, $k > 1, 0 \le \ell < \frac{1}{2}, k+\ell > \frac{3}{2}$, assume that $\tilde{u}_0$ satisfies the compatibility conditions of order $m+2$ for the system (2.2). Suppose that the shear flow satisfies*

$$|\partial_y^{p+1}u^s(t,y)| \le C\langle y\rangle^{-k-p}, \quad (t,y) \in [0,T_1]\times\mathbb{R}_+,\ 0 \le p \le m+2.$$

Then, for any $0 < \epsilon \le \epsilon_0$ and $0 < \bar{\zeta} \le 1$, there exits $T_\epsilon > 0$ which depends on ϵ and $\bar{\zeta}$, such that if

$$\|\tilde{w}_0\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \bar{\zeta},$$

then the system (3.2) *admits a unique solution*

$$\tilde{w}_\epsilon \in L^\infty([0,T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+)),$$

which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \le \frac{4}{3}\|\tilde{w}_{0,\epsilon}\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \le 2\|\tilde{w}_0\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}. \tag{3.8}$$

Remark 3.5. *.*

(1) *We remark that T_ϵ depends on ϵ and $\bar{\zeta}$, and $T_\epsilon \to 0$ as $\epsilon \to 0$. So this is not a bounded estimate for the approximate solution sequences $\{u^s + \tilde{u}_\epsilon; 0 < \epsilon \le \epsilon_0\}$ where $\epsilon_0 > 0$ is given in Corollary 3.2. When the initial data $\tilde{u}_0$ is small enough, we observe that $u^s + \tilde{u}_\epsilon$ preserves the monotonicity and convexity of the shear flow on $[0, T_\epsilon]$.*

(2) *In this theorem, for the regularized Prandtl equation, there are not constrain conditions on the initial date, meaning that we don't need the monotonicity or convexcity of shear flow u^s, and $\bar{\zeta}$ is also arbitrary.*

We prove Theorem 3.4 by the following three Propositions, where the first one is devoted to the local existence of approximate solution $\tilde{w}_\epsilon$ of (3.2).

Proposition 3.6. *Let $\tilde{w}_{0,\epsilon} \in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$, $m \ge 6$ be an even integer, $k > 1, 0 \le \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$, and satisfy the compatibility conditions up to order $m+2$ for* (3.2). *Suppose that the shear flow satisfies*

$$|\partial_y^{p+1} u^s(t,y)| \le C\langle y\rangle^{-k-p}, \quad (t,y) \in [0,T_1] \times \mathbb{R}_+,\ 0 \le p \le m+2.$$

Then, for any $0 < \epsilon \le 1$ and $\bar{\zeta} > 0$, there exits $T_\epsilon > 0$ such that if

$$\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \bar{\zeta},$$

then the system (3.2) *admits a unique solution*

$$\tilde{w}_\epsilon \in L^\infty([0,T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+)).$$

Remark 3.7. *If $\tilde{w}_0 \in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$ is the initial data in Theorem 3.4, using Corollary 3.2, there exists $\epsilon_0 > 0$, and for any $0 < \epsilon \le \epsilon_0$, there exists $\mu_\epsilon \in H^{m+3}_{k+\ell}(\mathbb{R}^2_+)$ such that $\tilde{w}_{0,\epsilon} = \tilde{w}_0 + \epsilon\partial_y \mu_\epsilon$ satisfies the compatibility conditions up to order $m+2$ for the system* (3.2), *and*

$$\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \frac{3}{2}\|\tilde{w}_0\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.$$

Then, using Proposition 3.6, we obtain also the existence of the approximate solution under the assumption of Theorem 3.4.

The proof of this Proposition is standard since the equation in (3.2) is a parabolic type equation. Firstly, we establish *à priori* estimate and then prove the existence of solution by weak convergence methods. Because we work in the weighted Sobolev space and the computation is not so trivial, we give a detailed proof in the Appendix, to make the paper self-contained. So the rest of this section is devoted to proving the estimate (3.8).

3.2. **Uniform estimate with loss of x-derivative.** In the proof of the Proposition 3.6 (see Lemma C.2), we already get *à priori* estimate for $\tilde{w}_\epsilon$ by weak convergence methods. Now we try to prove the estimate (3.8) in a new way, and our object is to establish an uniform estimate with respect to $\epsilon > 0$. We first treat the easy part in this subsection.

We define the an-isotropic Sobolev norm,

$$\|f\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} = \sum_{|\alpha_1+\alpha_2|\leq m, \alpha_1\leq m-1} \|\langle y\rangle^{k+\ell+\alpha_2}\,\partial_x^{\alpha_1}\partial_y^{\alpha_2} f\|^2_{L^2(\mathbb{R}^2_+)}, \tag{3.9}$$

where we don't have the m-order derivative with respect to x-variable. Then

$$\|f\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} = \|f\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} + \|\partial_x^m f\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}.$$

Proposition 3.8. *Let $m \geq 6$ be an even integer, $k > 1, 0 \leq \ell < \frac{1}{2}, k+\ell > \frac{3}{2}$, and assume that $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$ is a solution to (3.2), then we have*

$$\begin{aligned} &\frac{d}{dt}\|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} + \|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} \\ &\quad + \epsilon\|\partial_x\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} \leq C_1\Big(\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon\|^m_{H^m_{k+\ell}(\mathbb{R}^2_+)}\Big), \end{aligned} \tag{3.10}$$

where $C_1 > 0$ is independent of ϵ.

Remark. The above estimate is uniform with respect to $\epsilon > 0$, but on the left hand of (3.10), there is no the estimate on the term $\|\partial_x^m\tilde{w}_\epsilon\|^2_{L^2_{k+\ell}}$. This is because that we can't control the term

$$\partial_x^m\tilde{v}_\epsilon(t,x,y) = -\int_0^y \partial_x^{m+1}\tilde{u}_\epsilon(t,x,\tilde{y})d\tilde{y}$$

which is the major difficulty in the study of the Prandtl equation. We will study this term in the next Proposition with a non-uniform estimate firstly, and then focus on proving the uniform estimate in the rest part of this paper.

Proof. For $|\alpha| = \alpha_1+\alpha_2 \leq m, \alpha_1 \leq m-1$, we have

$$\begin{aligned} &\partial_t\partial^\alpha\tilde{w}_\epsilon - \epsilon\partial_x^2\partial^\alpha\tilde{w}_\epsilon - \partial_y^2\partial^\alpha\partial\tilde{w}_\epsilon \\ &= -\partial^\alpha\Big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{w}_\epsilon\Big) - \partial^\alpha\Big(\tilde{v}_\epsilon(u^s_{yy}+\partial_y\tilde{w}_\epsilon)\Big). \end{aligned} \tag{3.11}$$

Multiplying the (3.11) with $\langle y\rangle^{2(k+\ell+\alpha_2)}\partial^\alpha\tilde{w}_\epsilon$, and integrating over $\mathbb{R}^2_+$,

$$\begin{aligned} &\int_{\mathbb{R}^2_+}(\partial_t\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy - \epsilon\int_{\mathbb{R}^2_+}(\partial_x^2\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy \\ &\qquad - \int_{\mathbb{R}^2_+}(\partial_y^2\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy \\ &= -\int_{\mathbb{R}^2_+}\partial^\alpha\Big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{w}_\epsilon - \tilde{v}_\epsilon(u^s_{yy}+\partial_y\tilde{w}_\epsilon)\Big)\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy. \end{aligned}$$

Remark that for $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$, all above integrations are in the classical sense. We deal with each term on the left hand respectively. After integration by part, we have

$$\int_{\mathbb{R}^2_+}(\partial_t\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2(\mathbb{R}^2_+)}\partial^\alpha\tilde{w}_\epsilon dxdy=\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)},$$

$$-\epsilon\int_{\mathbb{R}^2_+}(\partial^2_x\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2(\mathbb{R}^2_+)}\partial^\alpha\tilde{w}_\epsilon dxdy=\epsilon\|\partial_x\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)},$$

and

$$\begin{aligned}&-\int_{\mathbb{R}^2_+}\partial^2_y\partial^\alpha\tilde{w}_\epsilon\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy\\&=\|\partial_y\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)}+\int_{\mathbb{R}^2_+}\partial^\alpha\partial_y\tilde{w}_\epsilon(\langle y\rangle^{2(k+\ell)+2\alpha_2})'\partial^\alpha\tilde{w}_\epsilon dxdy\\&\quad+\int_{\mathbb{R}}(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon)\big|_{y=0}dx.\end{aligned}$$

Cauchy-Schwarz inequality implies

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+}\partial^\alpha\partial_y\tilde{w}_\epsilon(\langle y\rangle^{2(k+\ell)+2\alpha_2})'\partial^\alpha\tilde{w}_\epsilon dxdy\right|\\&\quad\le\frac{1}{16}\|\partial_y\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)}+C\|\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2-1}(\mathbb{R}^2_+)}.\end{aligned}$$

We study now the term

$$\int_{\mathbb{R}}(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon)\big|_{y=0}dx.$$

Case : $|\alpha|\le m-1$, using the trace Lemma A.2, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon)\big|_{y=0}dx\right|\\&\quad\le\|(\partial^\alpha\partial_y\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|(\partial^\alpha\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\\&\quad\le C\|\partial^\alpha\partial^2_y\tilde{w}_\epsilon\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\|\partial^\alpha\partial_y\tilde{w}_\epsilon\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\&\quad\le C\|\partial_y\tilde{w}_\epsilon\|_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}\|\tilde{w}_\epsilon\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\\&\quad\le\frac{1}{16}\|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+C\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

Case : $\alpha_1=m-1,\alpha_2=1$, using (3.4), we have

$$(\partial^\alpha\tilde{w}_\epsilon)|_{y=0}=(\partial^{\alpha_1}_x\partial^2_y\tilde{u}_\epsilon)|_{y=0}=0,$$

thus

$$\int_{\mathbb{R}}(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon)\,|_{y=0}dx=0.$$

Case : $\alpha_1=0,\alpha_2=m$. Only in this case, we need to suppose that m is even. Using again the trace Lemma A.2, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}\left(\partial^{m+1}_y\tilde{w}_\epsilon\partial^m_y\tilde{w}_\epsilon\right)|_{y=0}dx\right|\\&\quad\le\|(\partial^{m+2}_y\tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|(\partial^m_y\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\\&\quad\le C\|(\partial^{m+2}_y\tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|\partial^{m+1}_y\tilde{w}_\epsilon\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\&\quad\le\frac{1}{16}\|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+C\|(\partial^{m+2}_y\tilde{u}_\epsilon)|_{y=0}\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Using Proposition 3.1 and the trace Lemma A.2, we can estimate the above last term $\|(\partial_y^{m+2}\tilde{u}_\epsilon)|_{y=0}\|^2_{L^2(\mathbb{R})}$ by a finite summation of the following forms

$$\|\prod_{j=1}^{p}(\partial_x^{\alpha_j}\partial_y^{\beta_j+1}(u^s+\tilde{u}_\epsilon))|_{y=0}\|^2_{L^2(\mathbb{R})}\le C\|\partial_y\prod_{j=1}^{p}(\partial_x^{\alpha_j}\partial_y^{\beta_j+1}(u^s+\tilde{u}_\epsilon))\|^2_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}$$

with $2\le p\le \frac{m}{2}$, $\alpha_j+\beta_j\le m-1$ and $\{j;\alpha_j>0\}\neq\emptyset$. Then using Sobolev inequality and $m\ge 6$, we get

$$\|(\partial_y^{m+2}\tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\le C\|\tilde{w}_\epsilon\|^{m/2}_{H^m_{k+\ell}(\mathbb{R}^2_+)}.$$

Case : $1\le\alpha_1\le m-2, \alpha_1+\alpha_2=m, \alpha_2$ **even**, using the same argument to the precedent case, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}}(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon)|_{y=0}dx\Big| &= \Big|\int_{\mathbb{R}}(\partial_x^{\alpha_1}\partial_y^{\alpha_2+1}\tilde{w}_\epsilon\partial_x^{\alpha_1}\partial_y^{\alpha_2}\tilde{w}_\epsilon)|_{y=0}dx\Big|\\
&\le \|(\partial_x^{\alpha_1}\partial_y^{\alpha_2+1}\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|(\partial_x^{\alpha_1}\partial_y^{\alpha_2}\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\\
&\le \frac{1}{16}\|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+C\|(\partial_x^{\alpha_1}\partial_y^{\alpha_2+2}\tilde{u}_\epsilon)|_{y=0}\|^2_{L^2(\mathbb{R})}\\
&\le \frac{1}{16}\|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+C\|\tilde{w}_\epsilon\|^{\alpha_2}_{H^m_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

Case : $1\le\alpha_1\le m-2, \alpha_1+\alpha_2=m, \alpha_2$ **odd**, integration by part with respect to x variable implies

$$\begin{aligned}\Big|\int_{\mathbb{R}}(\partial_x^{\alpha_1}\partial_y^{\alpha_2+1}\tilde{w}_\epsilon\partial_x^{\alpha_1}\partial_y^{\alpha_2}\tilde{w}_\epsilon)|_{y=0}dx\Big| &= \Big|\int_{\mathbb{R}}(\partial_x^{\alpha_1-1}\partial_y^{\alpha_2+1}\tilde{w}_\epsilon\partial_x^{\alpha_1+1}\partial_y^{\alpha_2}\tilde{w}_\epsilon)|_{y=0}dx\Big|\\
&\le \|(\partial_x^{\alpha_1-1}\partial_y^{\alpha_2+1}\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|(\partial_x^{\alpha_1+1}\partial_y^{\alpha_2}\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\\
&\le \frac{1}{16}\|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+C\|(\partial_x^{\alpha_1+1}\partial_y^{\alpha_2+1}\tilde{u}_\epsilon)|_{y=0}\|^2_{L^2(\mathbb{R})}\\
&\le \frac{1}{16}\|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+C\|\tilde{w}_\epsilon\|^{\alpha_2-1}_{H^m_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

Finally, we have proven

$$\begin{aligned}&\int_{\mathbb{R}^2_+}\Big(\partial_t\partial^\alpha\tilde{w}_\epsilon-\partial_y^2\partial^\alpha\tilde{w}_\epsilon-\epsilon\partial_x^2\partial^\alpha\tilde{w}_\epsilon\Big)\langle y\rangle^{2(k+\ell+\alpha_2)}\partial^\alpha\tilde{w}_\epsilon dxdy\\
&\ge \frac12\frac{d}{dt}\|\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}}+\epsilon\|\partial_x\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}}+\|\partial_y\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}}\\
&\quad-\frac14\|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}-C\|\tilde{w}_\epsilon\|^m_{H^m_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

We estimate now the right hand of (3.11). For the first item, we need to split it into two parts

$$-\partial^\alpha\Big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{w}_\epsilon\Big)=-(u^s+\tilde{u}_\epsilon)\partial_x\partial^\alpha\tilde{w}_\epsilon+[(u^s+\tilde{u}_\epsilon),\partial^\alpha]\partial_x\tilde{w}_\epsilon.$$

Firstly, we have

$$\int_{\mathbb{R}^2_+}\big((u^s+\tilde{u}_\epsilon)\partial_x\partial^\alpha\tilde{w}_\epsilon\big)\langle y\rangle^{2(k+\ell+\alpha_2)}\partial^\alpha\tilde{w}_\epsilon dxdy\le\|\partial_x\tilde{u}_\epsilon\|_{L^\infty}\|\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}},$$

then using (A.2), we get

$$\left|\int_{\mathbb{R}^2_+}\big((u^s+\tilde u_\epsilon)\partial_x\partial^\alpha\tilde w_\epsilon\big)\langle y\rangle^{2(\ell+\alpha_2)}\partial^\alpha\tilde w_\epsilon dxdy\right|\le\|\tilde w_\epsilon\|_{H^3_1}\|\partial^\alpha\tilde w_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}}.$$

For the commutator operator, in fact, it can be written as

$$[(u^s+\tilde u_\epsilon),\partial^\alpha]\partial_x\tilde w_\epsilon=\sum_{\beta\le\alpha,\,1\le|\beta|}C^\beta_\alpha\;\partial^\beta(u^s+\tilde u_\epsilon)\partial^{\alpha-\beta}\partial_x\tilde w_\epsilon.$$

Then for $|\alpha|\le m, m\ge 4$, using the Sobolev inequality again and Lemma A.1,

$$\|[(u^s+\tilde u),\partial^\alpha]\partial_x\tilde w_\epsilon\|_{L^2_{k+\ell+\alpha_2}}\le C(\|\tilde w_\epsilon\|_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}).$$

Thus

$$\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2(k+\ell+\alpha_2)}\big([(u^s+\tilde u_\epsilon),\partial^\alpha]\partial_x\tilde w_\epsilon\big)\cdot\partial^\alpha\tilde w_\epsilon dxdy\right|\le C\bigg(\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}\bigg),$$

and

$$\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2(k+\ell+\alpha_2)}\bigg(\partial^\alpha\big((u^s+\tilde u_\epsilon)\partial_x\tilde w_\epsilon\big)\bigg)\partial^\alpha\tilde w_\epsilon dxdy\right|\le C\big(\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}\big),$$

where C is independent of ϵ.

For the next one, similar to the first term in (3.11), we have

$$\partial^\alpha\bigg(\tilde v_\epsilon(u^s_{yy}+\partial_y\tilde w_\epsilon)\bigg)=\tilde v_\epsilon\partial_y\partial^\alpha\tilde w_\epsilon-[\tilde v_\epsilon,\partial^\alpha]\partial_y\tilde w_\epsilon+\partial^\alpha(\tilde v_\epsilon u^s_{yy}).$$

Then

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+}\tilde v_\epsilon\langle y\rangle^{2(k+\ell+\alpha_2)}(\partial_y\partial^\alpha\tilde w_\epsilon)\cdot\partial^\alpha\tilde w_\epsilon dxdy\right|\\&\le\|\tilde v_\epsilon\|_{L^\infty(\mathbb{R}^2_+)}\|\partial_y\tilde w_\epsilon\|_{H^m_{k+\ell}}\|\tilde w_\epsilon\|_{H^m_{k+\ell}}\\&\quad\le\frac14\|\partial_y\tilde w_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}+C\|\tilde w_\epsilon\|^4_{H^m_{k+\ell}(\mathbb{R}^2_+)}\end{aligned}$$

where we have used

$$\begin{aligned}&\|\tilde v_\epsilon\|_{L^\infty(\mathbb{R}^2_+)}\le C\|\partial_x\tilde u_\epsilon\|_{L^\infty(\mathbb{R}_x;L^2_{\frac12+\delta}(\mathbb{R}_{y,+}))}\\&\le C\int_{\mathbb{R}^2_+}\langle y\rangle^{1+2\delta}(|\partial_x\tilde u_\epsilon|^2+|\partial_x^2\tilde u_\epsilon|^2)dxdy\\&\le C\int_{\mathbb{R}^2_+}\langle y\rangle^{3+2\delta}(|\partial_x\tilde w_\epsilon|^2+|\partial_x^2\tilde w_\epsilon|^2)dxdy\le C\|\tilde w_\epsilon\|_{H^2_{\frac32+\delta}},\end{aligned}$$

where $\delta>0$ is small.

Noticing that

$$[\tilde v_\epsilon,\partial^\alpha]\partial_y\tilde w_\epsilon=\sum_{\beta\le\alpha,1\le|\beta|}C^\beta_\alpha\;\partial^\beta\tilde v_\epsilon\;\partial^{\alpha-\beta}\partial_y\tilde w_\epsilon.$$

Since H^m_ℓ is an algebra for $m\ge 6$, we only need to pay attention to the order of derivative in the above formula. Firstly for $|\beta|\ge 1$, we have for $|\alpha-\beta|+1\le m$,

$$-\partial^\beta\tilde v_\epsilon=\partial_x^{\beta_1}\partial_y^{\beta_2}\int_0^y\tilde u_{\epsilon,x}d\tilde y=\begin{cases}\partial_x^{\beta_1+1}\partial_y^{\beta_2-1}\tilde u_\epsilon, & \beta_2\ge 1,\\ \int_0^y\partial_x^{\beta_1+1}\tilde u_\epsilon d\tilde y, & \beta_2=0.\end{cases}$$

Now using the hypothesis $\beta \le \alpha, 1 \le |\beta|$ and $\beta_1 \le \alpha_1 \le m-1$, using Lemma A.1, we get

$$\|[\tilde{v}_\epsilon, \partial^\alpha]\partial_y \tilde{w}_\epsilon\|_{L^2_{k+\ell+\alpha_2}} \le C\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}}.$$

On the other hand, if $\alpha_2 = 0$, using $-1+\ell < -\frac{1}{2}$, we can get

$$\|\partial_x^{m-1}(\tilde{v}_\epsilon u^s_{yy})\|_{L^2_{k+\ell}} \le C\|\partial_x^m \tilde{u}_\epsilon\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}\|u^s_{yy}\|_{L^2_{k+\ell}(\mathbb{R}_+)} \le C\|\tilde{w}_\epsilon\|_{H^m_{\frac{3}{2}+\delta}}.$$

Similar computation for other cases, we can get, for $\alpha_2 > 0, \alpha_1+\alpha_2 \le m$,

$$\|\partial^\alpha(\tilde{v}_\epsilon u^s_{yy})\|_{L^2_{k+\ell+\alpha_2}} \le C\|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}.$$

Combining the above estimates, we have finished the proof of the Proposition 3.8. □

3.3. **Smallness of approximate solutions.** To close the energy estimate, we still need to estimate the term $\partial_x^m \tilde{w}_\epsilon$.

Proposition 3.9. *Under the hypothesis of Theorem 3.4, and with the same notations as in Proposition 3.8, we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^m \tilde{w}_\epsilon\|^2_{L^2_{k+\ell}} + \frac{3\epsilon}{4}\|\partial_x^{m+1}\tilde{w}_\epsilon\|^2_{L^2_{k+\ell}} + \frac{3}{4}\|\partial_y\partial_x^m \tilde{w}_\epsilon\|^2_{L^2_{k+\ell}} \\
&\le C\big(\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}} + \|\tilde{w}_\epsilon\|^3_{H^m_{k+\ell}}\big) + \frac{32}{\epsilon}\big(\|\tilde{w}_\epsilon\|^4_{H^m_{k+\ell}} + \|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}}\big).
\end{aligned} \tag{3.12}$$

Proof. We have

$$\partial_t\partial_x^m \tilde{w}_\epsilon - \partial_y^2\partial_x^m \tilde{w}_\epsilon - \epsilon\partial_x^m\partial_x^2\tilde{w}_\epsilon = -\partial_x^m\bigg((u^s+\tilde{u}_\epsilon)\partial_x\tilde{w}_\epsilon\bigg) - \partial_x^m\big(\tilde{v}_\epsilon(\partial_y\tilde{w}_\epsilon + u^s_{yy})\big),$$

then the same computations as in Proposition 3.8 give

$$\begin{aligned}
&\frac{d}{2dt}\|\partial_x^m \tilde{w}_\epsilon\|^2_{L^2_{k+\ell}} + \epsilon\|\partial_x^{m+1}\tilde{w}_\epsilon\|^2_{L^2_{k+\ell}} + \frac{3}{4}\|\partial_y\partial_x^m \tilde{w}_\epsilon\|^2_{L^2_{k+\ell}} \\
&\le C(\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}} + \|\tilde{w}_\epsilon\|^3_{H^m_{k+\ell}}) \\
&+ \left|\int_{\mathbb{R}^2_+}\partial_x^m\big(\tilde{v}_\epsilon(\partial_y\tilde{w}_\epsilon + u^s_{yy})\big)\langle y\rangle^{2(k+\ell)}\partial_x^m\tilde{w}_\epsilon dxdy\right|,
\end{aligned} \tag{3.13}$$

where the boundary terms is more easy to control, since

$$(\partial_y\partial_x^m\tilde{w}_\epsilon)(t,x,0) = (\partial_y^2\partial_x^m\tilde{u}_\epsilon)(t,x,0) = 0, \ \ (t,x) \in [0,T]\times\mathbb{R}.$$

The estimate of the last term on right hand is the main obstacle for the study of the Prandtl equations.

$$\begin{aligned}
\partial_x^m\big(\tilde{v}_\epsilon(\partial_y\tilde{w}_\epsilon + u^s_{yy})\big) &= \tilde{v}_\epsilon\partial_x^m\partial_y\tilde{w}_\epsilon + (\partial_x^m\tilde{v}_\epsilon)(\partial_y\tilde{w}_\epsilon + u^s_{yy}) \\
&\quad + \sum_{1\le j\le m-1} C^j_m\,\partial_x^j\tilde{v}_\epsilon\partial_x^{m-j}\partial_y\tilde{w}_\epsilon.
\end{aligned}$$

For the first term

$$\begin{aligned}
&\int_{\mathbb{R}^2_+}\tilde{v}_\epsilon(\partial_x^m\partial_y\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)}(\partial_x^m\tilde{w}_\epsilon)dxdy \\
&\quad = \frac{1}{2}\int\tilde{v}_\epsilon\langle y\rangle^{2(k+\ell)}\partial_y(\partial_x^m\tilde{w}_\epsilon)^2dxdy \\
&\quad = \frac{1}{2}\int\tilde{u}_{\epsilon,x}\langle y\rangle^{2(k+\ell)}(\partial_x^m\tilde{w}_\epsilon)^2dxdy
\end{aligned}$$

$$
\begin{aligned}
&-\ell\int \tilde v_\epsilon \langle y\rangle^{2(k+\ell)-1}(\partial_x^m\tilde w_\epsilon)^2dxdy\\
&\le C\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}},
\end{aligned}
$$

where we have used $\tilde v_\epsilon|_{y=0}=0$, and

$$
\left|\int_{\mathbb{R}^2_+}\Big(\sum_{1\le j\le m-1}C_m^j\,\partial_x^j\tilde v_\epsilon\partial_x^{m-j}\partial_y\tilde w_\epsilon\Big)\langle y\rangle^{2(k+\ell)}(\partial_x^m\tilde w_\epsilon\big)dxdy\right|\le C\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}.
$$

Finally for the worst term, we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2_+}(\partial_x^m\tilde v_\epsilon)(\partial_y\tilde w_\epsilon+u^s_{yy})\langle y\rangle^{2(k+\ell)}(\partial_x^m\tilde w_\epsilon)dxdy\right|\\
&\le C\|\partial_x^m\tilde v_\epsilon\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))}\|\partial_y\tilde w_\epsilon\|_{L^\infty(\mathbb{R}_x;L^2_{k+\ell}(\mathbb{R}_+))}\|\tilde w_\epsilon\|_{H^m_{k+\ell}}\\
&\qquad+\|\partial_x^m\tilde v_\epsilon u^s_{yy}\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\|\tilde w_\epsilon\|_{H^m_{k+\ell}}.
\end{aligned}
$$

On the other hand, observing

$$
\partial_x^m\tilde v_\epsilon(t,x,y)=-\int_0^y\partial_x^{m+1}\tilde u_\epsilon(t,x,\tilde y)d\tilde y,
$$

then using Sobolev inequality and Lemma A.1

$$
\|\partial_x^m\tilde v_\epsilon\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))}\le C\|\partial_x^{m+1}\tilde u_\epsilon\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}\le C\|\partial_x^{m+1}\tilde w_\epsilon\|_{L^2_{\frac32+\delta}(\mathbb{R}^2_+)},
$$

we get

$$
\|\partial_x^m\tilde v_\epsilon\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))}\le C\|\partial_x^{m+1}\tilde w_\epsilon\|_{L^2_{\frac32+\delta}(\mathbb{R}^2_+)}.
$$

Using the hypothesis for the shear flow u^s and $-1+\ell<-\frac12$,

$$
\begin{aligned}
\|\partial_x^m(\tilde v_\epsilon u^s_{yy})\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}&\le\|\partial_x^m\tilde v_\epsilon\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))}\|u^s_{yy}\|_{L^2_{k+\ell}(\mathbb{R}_+)}\\
&\le C\|\partial_x^{m+1}\tilde w_\epsilon\|_{L^2_{\frac32+\delta}(\mathbb{R}^2_+)},
\end{aligned}
$$

and

$$
\|\partial_y\tilde w_\epsilon\|_{L^\infty(\mathbb{R}_x;L^2_{k+\ell}(\mathbb{R}_+))}\le C\|\partial_y\tilde w_\epsilon\|_{H^1(\mathbb{R}_x;L^2_{k+\ell}(\mathbb{R}_+))}\le C\|\tilde w_\epsilon\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}.
$$

Thus, we have

$$
\begin{aligned}
&\int\bigg(\partial_x^m\big(\tilde v_\epsilon(\partial_y\tilde w_\epsilon+u^s_{yy})\big)\bigg)\cdot\langle y\rangle^{2(k+\ell)}\partial_x^m\tilde w_\epsilon dxdy\\
&\le C\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}+\frac{32}{\epsilon}\big(\|\tilde w_\epsilon\|^4_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}\big)+\frac{\epsilon}{4}\|\partial_x^{m+1}\tilde w_\epsilon\|^2_{L^2_{\frac32+\delta}}.
\end{aligned}\tag{3.14}
$$

From (3.13) and (3.14), we have, if $k+\ell>\frac32$,

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\partial_x^m\tilde w_\epsilon\|^2_{L^2_{k+\ell}}+\frac{3\epsilon}{4}\|\partial_x^{m+1}\tilde w_\epsilon\|^2_{L^2_{k+\ell}}+\frac34\|\partial_y\partial_x^m\tilde w_\epsilon\|^2_{L^2_{k+\ell}}\\
&\le C\big(\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}\big)+\frac{32}{\epsilon}\big(\|\tilde w_\epsilon\|^4_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}\big).
\end{aligned}
$$

□

End of proof of Theorem 3.4. Combining (3.10) and (3.12), for $m \geq 6, k > 1, \frac{3}{2} - k < \ell < \frac{1}{2}$ and $0 < \epsilon \leq 1$, we get

$$\frac{d}{dt}\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq \frac{C}{\epsilon}\big(\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon\|^m_{H^m_{k+\ell}(\mathbb{R}^2_+)}\big), \tag{3.15}$$

with $C > 0$ independent of ϵ.

From (3.15), by the nonlinear Gronwall's inequality, we have

$$\|\tilde{w}_\epsilon(t)\|^{m-2}_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq \frac{\|\tilde{w}_\epsilon(0)\|^{m-2}_{H^m_{k+\ell}}}{e^{-\frac{C}{\epsilon}t(\frac{m}{2}-1)} - (\frac{m}{2}-1)\frac{C}{\epsilon}t\|\tilde{w}_\epsilon(0)\|^{m-2}_{H^m_{k+\ell}}},\ 0 < t \leq T_\epsilon,$$

where we choose $T_\epsilon > 0$ such that

$$\frac{1}{e^{-\frac{C}{\epsilon}T_\epsilon(\frac{m}{2}-1)} - (\frac{m}{2}-1)\frac{C}{\epsilon}T_\epsilon\bar{\zeta}^{m-2}} = \left(\frac{4}{3}\right)^{m-2}. \tag{3.16}$$

Finally, we get for any $\|\tilde{w}_\epsilon(0)\|_{H^m_{k+\ell}} \leq \bar{\zeta}$, and $0 < \epsilon \leq \epsilon_0$,

$$\|\tilde{w}_\epsilon(t)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq \frac{4}{3}\|\tilde{w}_\epsilon(0)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq 2\|\tilde{w}_0\|_{H^m_{k+\ell}(\mathbb{R}^2_+)},\ 0 < t \leq T_\epsilon.$$

□

The rest of this paper is dedicated to improve the results of Proposition 3.9, and try to get an uniform estimate with respect to ϵ. Of course, we have to recall the assumption on the shear flow in the main theorem 1.1.

4. Formal transformations

Since the estimate (3.10) is independent of ϵ, we only need to treat (3.12) in a new way to get an estimate which is also independent of ϵ. To simplify the notations, from now on, we drop the notation tilde and sub-index ϵ, that is, with no confusion, we take

$$u = \tilde{u}_\epsilon, \quad v = \tilde{v}_\epsilon, \quad w = \tilde{w}_\epsilon.$$

4.1. **The cut-off functions and a priori assumptions.** We will decompose the approximate solution into the monotone part and convex part, so we need to intoduce the cut-off functions. Choose $\phi_1, \phi_2, \psi \in C^\infty(\mathbb{R}_+), 0 \leq \phi_1, \phi_2, \psi \leq 1$ with

$$\begin{aligned}
&\phi_1(y) = 1\ \text{ for } 0 \leq y \leq a - 3a_0;\ \ \phi_1(y) = 0\ \text{ for } y \geq a - 2a_0;\\
&\phi_2(y) = 0\ \text{ for } 0 \leq y \leq a + 2a_0;\ \ \phi_2(y) = 1\ \text{ for } y \geq a + 3a_0;\\
&\psi(y) = 1\ \text{ for } |y-a| \leq 4a_0;\ \ \psi(y) = 0\ \text{ for } |y-a| \geq 5a_0.
\end{aligned}$$

Also the support of those cut-off functions are as follows

$$\begin{aligned}
&I_{\phi_1} = \{y;\, 0 \leq y \leq a - 2a_0\},\ I_{\phi_1'} = \{y;\, a - 3a_0 \leq y \leq a - 2a_0\},\\
&I_{\phi_2} = \{y;\, y \geq a + 2a_0\},\ I_{\phi_2'} = \{y;\, a + 2a_0 \leq y \leq a + 3a_0\},\\
&I_\psi = \{y;\ |y-a| \leq 5a_0\},\ \ I_{\psi'} = \{y;\ 4a_0 \leq |y-a| \leq 5a_0\}.
\end{aligned}$$

And we have

$$I_{\phi_1'} \cup I_{\phi_2'} \subset \{y;\, \psi(y) = 1\}, \quad I_{\psi'} \subset \{y;\, \phi_1(y) = 1\} \cup \{y;\ \phi_2(y) = 1\}.$$

Let $w \in L^\infty([0,T]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+)), m \geq 6, k > 1, 0 \leq \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$ be a classical solution of (3.2) which satisfies the following *à priori* condition

$$\|w\|_{L^\infty([0,T]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))} \leq \zeta. \tag{4.1}$$

Then (A.2) gives

$$\begin{aligned}\|\langle y\rangle^{k+\ell}w\|_{L^\infty([0,T]\times\mathbb{R}^2_+)} \le& C(\|\langle y\rangle^{\frac12+\delta}(\langle y\rangle^{k+\ell}w)_y\|_{L^\infty([0,T];L^2(\mathbb{R}^2_+))}\\ &+\|\langle y\rangle^{\frac12+\delta}(\langle y\rangle^{k+\ell}w)_{xy}\|_{L^\infty([0,T];L^2(\mathbb{R}^2_+))})\\ &\le C_m\|w\|_{L^\infty([0,T];H^{m+2}_{k+\ell}(\mathbb{R}^2_+))},\end{aligned}$$

which implies

$$|\partial_y u(t,x,y)|=|w(t,x,y)|\le C_m\,\zeta\,\langle y\rangle^{-k-\ell},\quad (t,x,y)\in[0,T]\times\mathbb{R}^2_+,$$

and similarly

$$|\partial_y^2 u(t,x,y)|=|\partial_y w(t,x,y)|\le \tilde C_m\,\zeta,\quad (t,x,y)\in[0,T]\times\mathbb{R}\times I_\psi.$$

We assume that ζ is small enough such that

$$C_m\,\zeta\le\frac{c_0}{2},\quad \tilde C_m\,\zeta\le\frac{c_0^2}{2},\tag{4.2}$$

where C_m is the above Sobolev embedding constant. Then we have for $\ell\ge0$,

$$|u^s_{yy}+\partial_y w|\ge\frac{c_0^2}{2},\quad (t,x,y)\in[0,T]\times\mathbb{R}\times I_\psi;\tag{4.3}$$

$$\frac{c_0}{4}\langle y\rangle^{-k}\le|u^s_y+u_y|\le 4c_0^{-1}\langle y\rangle^{-k},\quad (t,x,y)\in[0,T]\times\mathbb{R}\times(I_{\phi_1}\cup I_{\phi_2}).\tag{4.4}$$

4.2. **The formal transformation of equations.** Under the conditions (4.3) and (4.4), in this subsection, we will introduce two transformations of system (3.1), one to be used for the monotone domain, and the other for the convex domain.
Monotone parts. Set, for $0\le n\le m$

$$g_n=\left(\frac{\partial_x^n u}{u^s_y+u_y}\right)_y,\ \eta_1=\frac{u_{xy}}{u^s_y+u_y},\ \eta_2=\frac{u^s_{yy}+u_{yy}}{u^s_y+u_y},\ \forall(t,x,y)\in[0,T]\times\mathbb{R}\times(I_{\phi_1}\cup I_{\phi_2}).$$

Formally, we will use the following notations

$$\partial_y^{-1}g_n(t,x,y)=\frac{\partial_x^n u}{u^s_y+u_y}(t,x,y),\ \partial_y\partial_y^{-1}g_n=g_n,\ \forall(t,x,y)\in[0,T]\times\mathbb{R}\times(I_{\phi_1}\cup I_{\phi_2}).$$

Applying ∂_x^n to (3.1), we have

$$\begin{aligned}\partial_t\partial_x^n u+(u^s+u)\partial_x\partial_x^n u+(\partial_x^n v)(u^s_y+\partial_y u)&\\ =\partial_y^2\partial_x^n u+\epsilon\partial_x^2\partial_x^n u+A^1_n+A^2_n,&\end{aligned}\tag{4.5}$$

where

$$A^1_n=-[\partial_x^n,\,(u^s+u)]\partial_x u=-\sum_{i=1}^n C_n^i\partial_x^i u\,\partial_x^{n+1-i}u,$$

$$A^2_n=-[\partial_x^n,\,(u^s_y+\partial_y u)]v=-\sum_{i=1}^n C_n^i\partial_x^i w\,\partial_x^{n-i}v.$$

Dividing (4.5) with $(u^s_y+u_y)$ and performing ∂_y on the resulting equation, observing

$$\partial_x\partial_x^n u+\partial_y\partial_x^n v=\partial_x^n(\partial_x u+\partial_y v)=0,$$

we have for $j = 1, 2$,

$$\phi_j \partial_y \left(\frac{\partial_t \partial_x^n u}{u_y^s + u_y} \right) + \phi_j (u^s + u) \partial_y \left(\frac{\partial_x \partial_x^n u}{u_y^s + u_y} \right) = \phi_j \partial_y \left(\frac{\partial_y^2 \partial_x^n u + \epsilon \partial_x^2 \partial_x^n u}{u_y^s + u_y} \right) + \phi_j \partial_y \left(\frac{A_n^1 + A_n^2}{u_y^s + u_y} \right).$$

We compute each term on the support of ϕ_j,

$$\partial_y \left(\frac{\partial_t \partial_x^n u}{u_y^s + u_y} \right) = \partial_y \left(\partial_t \frac{\partial_x^n u}{u_y^s + u_y} + \partial_y^{-1} g_n \frac{\partial_t u_y + \partial_t u_y^s}{u_y^s + u_y} \right) = \partial_t g_n + \partial_y \left(\partial_y^{-1} g_n \frac{\partial_t u_y^s + \partial_t u_y}{u_y^s + \tilde{u}_y} \right),$$

$$(u^s + u) \partial_y \left(\frac{\partial_x \partial_x^n u}{u_y^s + u_y} \right) = (u^s + u) \Big\{ \partial_x \partial_y \left(\frac{\partial_x^n u}{u_y^s + u_y} \right) + \partial_y \left(\frac{\partial_x^n u}{u_y^s + u_y} \right) \frac{u_{xy}}{u_y^s + u_y} + \left(\frac{\partial_x^n u}{u_y^s + u_y} \right) \partial_y \left(\frac{u_{xy}}{u_y^s + u_y} \right) \Big\} = (u^s + u)(\partial_x g_n + g_n \, \eta_1 + \partial_y^{-1} g_n \, \partial_y \eta_1),$$

$$\frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} = \partial_y^2 \left(\frac{\partial_x^n u}{u_y^s + u_y} \right) + 2 \left(\frac{\partial_y u}{u_y^s + u_y} \right) \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} - \partial_x^n u \, \partial_y^2 \left(\frac{1}{u_y^s + u_y} \right),$$

$$\partial_y^2 \left(\frac{1}{u_y^s + u_y} \right) = -\partial_y \left(\frac{u_{yy}^s + u_{yy}}{(u_y^s + u_y)^2} \right) = -\frac{u_{yyy}^s + u_{yyy}}{(u_y^s + u_y)^2} + 2 \left(\frac{u_{yy}^s + u_{yy}}{(u_y^s + u_y)} \right)^2 \frac{1}{u_y^s + u_y},$$

$$\frac{\partial_y \partial_x^n u}{u_y^s + u_y} \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} = \left(\frac{\partial_x^n u}{u_y^s + u_y} \right)_y \frac{u_{yy}^s + u_{yy}}{u_y^s + u_y} - \frac{\partial_x^n u}{u_y^s + u_y} \left(\frac{u_{yy}^s + u_{yy}}{(u_y^s + u_y)} \right)^2.$$

So

$$\frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} = \partial_y g_n + 2(g_n \eta_2 - 2\partial_y^{-1} g_n \eta_2^2) + \partial_y^{-1} g_n \left(\frac{u_{yyy}^s + u_{yyy}}{(u_y^s + \tilde{u}_y)} \right),$$

$$\partial_y \left(\frac{\partial_y^2 \partial_x^n u}{u_y^s + u_y} \right) = \partial_y^2 g_n + 2(\partial_y g_n) \eta_2 + 2 g_n \partial_y \eta_2 - 4 g_n \eta_2^2 - 8 \partial_y^{-1} g_n \eta_2 \partial_y \eta_2 + \partial_y \left(\partial_y^{-1} g_n \frac{u_{yyy}^s + u_{yyy}}{u_y^s + u_y} \right).$$

Similarly, we have

$$\frac{\partial_x^2 \partial_x^n u}{u_y^s + u_y} = \partial_x^2 \left(\frac{\partial_x^n u}{u_y^s + u_y} \right) + 2 \left(\frac{\partial_x^n u}{u_y^s + u_y} \right)_x \frac{u_{xy}}{u_y^s + u_y} - 2 \frac{\partial_x^n u}{u_y^s + u_y} \left(\frac{u_{xy}}{(u_y^s + u_y)} \right)^2 + \frac{\partial_x^n u}{u_y^s + u_y} \frac{u_{xxy}}{(u_y^s + u_y)},$$

$$\partial_y \left(\frac{\partial_x^2 \partial_x^n u}{u_y^s + u_y} \right) = \partial_x^2 g_n + 2 \partial_x g_n \eta_1 + 2 \partial_x \partial_y^{-1} g_n \partial_y \eta_1$$

$$- 2g_n\eta_1^2 - 4\partial_y^{-1}g_n\eta_1\partial_y\eta_1 + \partial_y\Big(\partial_y^{-1}g_n\,\frac{u_{xxy}}{u^s_y+u_y}\Big).$$

For the boundary condition, we only need to pay attention to $j=1$. From (4.5) and the boundary condition for (u,v) in (3.1), we observe

$$\partial_x^n u|_{y=0}=0,\ \partial_y^2\partial_x^n u|_{y=0}=0,\ (u^s_y+u_y)|_{y=0}\neq 0.$$

At the same time,

$$\begin{aligned}0=\frac{\partial_y^2\partial_x^n u}{u^s_y+u_y}\Big|_{y=0}&=\partial_y g_n|_{y=0}+2(g_n\eta_2-2(\partial_y^{-1}g_n)\eta_2^2)|_{y=0}\\&\quad+\partial_y^{-1}g_n\left(\frac{u^s_{yyy}+u_{yyy}}{(u^s_y+\tilde u_y)}\right)\Big|_{y=0},\end{aligned}$$

and

$$\eta_2|_{y=0}=\frac{u^s_{yy}+u_{yy}}{u^s_y+u_y}\Big|_{y=0}=0,\quad \partial_y^{-1}g_n(t,x,y)|_{y=0}=\frac{\partial_x^n u}{u^s_y+u_y}(t,x,y)\Big|_{y=0}=0,$$

we get then

$$(\partial_y g_n)|_{y=0}=0,\quad 0\le n\le m.$$

Finally, we have, for $j=1,2$,

$$\begin{cases}\partial_t\phi_j g_n+(u^s+u)\phi_j\partial_x g_n-\phi_j\partial_y^2 g_n-\epsilon\phi_j\partial_x^2 g_n\\ \qquad\qquad -\epsilon\,2\,\phi_j(\partial_x\partial_y^{-1}g_n)\partial_y\eta_1=\phi_j M_n,\\ \phi_j(\partial_y g_n)|_{y=0}=0,\\ \phi_j g_n|_{t=0}=\phi_j g_{n,0},\end{cases}\tag{4.6}$$

with $M_n=\sum_{j=1}^6 M_j^n$,

$$\begin{aligned}M_1^n&=-(u^s+u)(g_n\eta_1+(\partial_y^{-1}g_n)\partial_y\eta_1),\\ M_2^n&=2(\partial_y g_n)\eta_2+2g_n(\partial_y\eta_2-2\eta_2^2)-8(\partial_y^{-1}g_n)\,\eta_2\partial_y\eta_2,\\ M_3^n&=\epsilon\big(2(\partial_x g_n)\eta_1-2g_n\,\eta_1^2-4(\partial_y^{-1}g_n)\eta_1\partial_y\eta_1\big),\\ M_4^n&=\partial_y\left(\partial_y^{-1}g_n\,\frac{(u^s+u)w_x+v(w_y+u^s_{yy})}{u^s_y+u_y}\right),\\ M_5^n&=-\partial_y\left(\frac{\sum_{i=1}^n C_n^i\partial_x^i u\cdot\partial_x^{n+1-i}u}{u^s_y+u_y}\right),\\ M_6^n&=-\partial_y\left(\frac{\sum_{i=1}^n C_n^i\partial_x^i w\cdot\partial_x^{n-i}v}{u^s_y+u_y}\right),\end{aligned}$$

where we have used the relation,

$$\partial_t u^s_y+\partial_t u_y-(u^s_{yyy}+u_{yyy})-\epsilon u_{xxy}=-(u^s+u)w_x+v(u^s_{yy}+w_y).$$

The convex part. Taking the equation in (3.2) with derivative ∂_x^m,

$$\begin{aligned}\partial_t\partial_x^m w+(u^s+u)\partial_x\partial_x^m w+(\partial_x^m v)(u^s_{yy}+\partial_y w)&=\partial_y^2\partial_x^m w+\epsilon\partial_x^2\partial_x^m w\\ -[\partial_x^m,\,(u^s+u)]\partial_x w-[\partial_x^m,\,(u^s_{yy}+\partial_y w)]v.&\end{aligned}\tag{4.7}$$

On the support of ψ, we have $u^s_{yy}+w_y>0$, then set

$$h_m=\frac{\partial_x^m w}{\sqrt{u^s_{yy}+w_y}},\ \eta_3=\frac{u^s_{yyy}+w_{yy}}{u^s_{yy}+w_y},\ \eta_4=\frac{w_{xy}}{u^s_{yy}+w_y}\,.$$

Dividing (4.7) with $\sqrt{u^s_{yy}+w_y}$, we have, on the support of ψ,

$$\begin{aligned}&\frac{\partial_t\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}+(u^s+u)\frac{\partial_x\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}+\sqrt{u^s_{yy}+w_y}\partial_x^m v\\&=\frac{\partial_y^2\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}+\epsilon\frac{\partial_x^2\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}+\frac{-[\partial_x^m,\,(u^s+u)]\partial_x w-[\partial_x^m,\,(u^s_{yy}+\partial_y w)]v}{\sqrt{u^s_{yy}+w_y}}\,.\end{aligned}$$

We calculate each term

$$\begin{aligned}\frac{\partial_t\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}&=\partial_t h_m+\frac{1}{2}h_m\frac{\partial_t(u^s_{yy}+w_y)}{(u^s_{yy}+w_y)},\\\frac{\partial_x\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}&=\partial_x h_m+\frac{1}{2}h_m\frac{w_{xy}}{(u^s_{yy}+w_y)},\end{aligned}$$

$$\begin{aligned}-\frac{\partial_y^2\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}&=-\partial_y^2\frac{\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}+2\partial_y\partial_x^m w\,\partial_y\frac{1}{\sqrt{u^s_{yy}+w_y}}+\partial_x^m w\,\partial_y^2\frac{1}{\sqrt{u^s_{yy}+w_y}}\\&=-\partial_y^2 h_m-\partial_y\partial_x^m w\,\frac{w_{yy}+u^s_{yyy}}{(\sqrt{u^s_{yy}+w_y})^3}-\partial_x^m w\,\partial_y\frac{w_{yy}+u^s_{yyy}}{2(\sqrt{u^s_{yy}+w_y})^3}\\&=-\partial_y^2 h_m-(\partial_y h_m)\,\eta_3+\frac{1}{2}h_m\,\eta_3^2-\frac{1}{2}h_m\partial_y\eta_3+\frac{1}{4}h_m\eta_3^2\\&=-\partial_y^2 h_m-(\partial_y h_m)\,\eta_3+\frac{5}{4}h_m\,\eta_3^2-\frac{1}{2}h_m\,\frac{w_{yyy}+u^s_{yyyy}}{u^s_{yy}+w_y}\,.\end{aligned}$$

Similarly, we have

$$-\frac{\partial_x^2\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}=-\partial_x^2 h_m+2\partial_x\partial_x^m w\,\partial_x\left(\frac{1}{\sqrt{u^s_{yy}+w_y}}\right)+\partial_x^m w\partial_x^2\left(\frac{1}{\sqrt{u^s_{yy}+w_y}}\right),$$

$$2\partial_x\partial_x^m w\,\partial_x\left(\frac{1}{\sqrt{u^s_{yy}+w_y}}\right)=-\frac{\partial_x\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}\,\eta_4=-\partial_x h_m\,\eta_4-\frac{1}{2}h_m\,\eta_4^2,$$

and

$$\begin{aligned}\partial_x^m w\,\partial_x^2\left(\frac{1}{\sqrt{u^s_{yy}+w_y}}\right)&=-\partial_x^m w\,\partial_x\left(\frac{w_{xy}}{2(\sqrt{u^s_{yy}+w_y})^3}\right)\\&=-\frac{1}{2}h_m\frac{w_{xxy}}{u^s_{yy}+w_y}+\frac{3}{4}h_m\eta_4^2.\end{aligned}$$

Then

$$-\frac{\partial_x^2\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}=-\partial_x^2 h_m-\partial_x h_m\,\eta_4+\frac{1}{4}h_m\,\eta_4^2-\frac{1}{2}h_m\frac{w_{xxy}}{u^s_{yy}+w_y}.$$

Taking the equation (3.2) with derivative y, we get the relation

$$\begin{aligned}&\partial_t(w_y)+(u^s+u)w_{xy}-w_{yyy}-\epsilon w_{xxy}\\&=-v(w_{yy}+u^s_{yyy})+u_x(u^s_{yy}+w_y)-(u^s_y+w)w_x,\end{aligned}$$

and also

$$\partial_t(u^s_{yy})-u^s_{yyyy}=0,$$

Finally we get

$$\begin{aligned}\partial_t(\psi h_m) + (u^s + u)\partial_x(\psi\, h_m) + &\sqrt{u^s_{yy} + w_y}\psi\, \partial_x^m v \\ &= \psi\partial_y^2 h_m + \epsilon\psi\, \partial_x^2 h_m + \psi N_m,\end{aligned} \tag{4.8}$$

where $N_m = N_1^m + N_2^m + N_3^m + N_4^m$

$$\begin{aligned}
N_1^m &= -\frac{1}{4}h_m\eta_3^2 + \partial_y h_m\, \eta_3 - \frac{\epsilon}{4}h_m\eta_4^2 + \epsilon\partial_x h_m\, \eta_4,\\
N_2^m &= h_m\, \frac{v(u^s_{yyy} + w_{yy}) - u_x(u^s_{yy} + w_y) - (u^s_y + w)w_x}{2\sqrt{u^s_{yy} + w_y}},\\
N_3^m &= -\frac{\sum_{p=0}^{m-1} C_m^p\, \partial_x^{m-p} u\, \partial_x^{p+1} w}{\sqrt{u^s_{yy} + w_y}},\\
N_4^m &= -\frac{\sum_{p=0}^{m-1} C_m^p\, \partial_x^{m-p}\partial_y w\, \partial_x^p v}{\sqrt{u^s_{yy} + w_y}}.
\end{aligned}$$

5. Uniform estimate for the monotone part

We first have

Lemma 5.1. *If $u \in H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+), m \geq 6, k > 1, 0 \leq \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$ which satisfies (4.1)-(4.2) with $0 < \zeta \leq 1$, then $\phi_2 g_m \in H^2_{k+\ell}(\mathbb{R}^2_+)$.*

In fact, observing

$$\phi_2 g_m = \phi_2\left(\frac{\partial_x^m u}{u^s_y + u_y}\right)_y = \phi_2\frac{\partial_y\partial_x^m u}{u^s_y + u_y} - \phi_2\frac{\partial_x^m u}{u^s_y + u_y}\eta_2,$$

then (4.4) implies

$$\langle y\rangle^{k+\ell}|\phi_2 g_m| \leq C\langle y\rangle^{2k+\ell}|\phi_2\partial_y\partial_x^m u| + C\langle y\rangle^{2k+\ell-1}|\phi_2\partial_x^m u|,$$

which finishes the proof of this Lemma.

Proposition 5.2. *Let $w \in L^\infty([0,T]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+)), m \geq 6, k > 1, 0 \leq \ell < \frac{1}{2}$ and $k + \ell > \frac{3}{2}$, be a solution to (3.2) which satisfies (4.1)-(4.2) with $0 < \zeta \leq 1$. Assume that the shear flow u^s satisfies Lemma 2.1, then we have the following estimates,*

$$\begin{aligned}\frac{d}{dt}\|\phi_1 g_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\phi_1\partial_y g_m\|^2_{L^2(\mathbb{R}^2_+)} + \epsilon\|\phi_1\partial_x g_m\|^2_{L^2(\mathbb{R}^2_+)}\\ \leq C_2(\|\phi_1 g_m\|^2_{L^2(\mathbb{R}^2_+)} + \|w\|^2_{H^m_{k+\ell}}),\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\frac{d}{dt}\sum_{n=1}^m\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \sum_{n=1}^m\|\phi_2\partial_y g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \epsilon\sum_{n=1}^m\|\phi_2\partial_x g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ \leq C_2(\sum_{n=1}^m\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|w\|^2_{H^m_{k+\ell}}),\end{aligned} \tag{5.2}$$

where C_2 is independent of ϵ.

Proof. With the boundary condition $(\partial_y g_n)|_{y=0} = 0$, the estimate of $\phi_1 g_n$ is simpler than $\phi_2 g_n$, since ϕ_1 is compactly supported, and then we can neglect the weight $\langle y\rangle^{2(k+\ell)}$. So we only give the proof of (5.2).

Multiplying the equation (4.6) by $\langle y\rangle^{2(k+\ell)}\phi_2 g_n$ and integrating over $\mathbb{R}\times\mathbb{R}^+$. We start to deal with the left hand of (4.6) first, we have

$$\int_{\mathbb{R}^2_+} \partial_t g_n \cdot \langle y\rangle^{2(k+\ell)}\phi_2^2 g_n dxdy = \frac{1}{2}\frac{d}{dt}\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)},$$

and

$$\begin{aligned}\int_{\mathbb{R}^2_+} (u^s+u)\partial_x g_n \cdot \langle y\rangle^{2(k+\ell)}\phi_2^2 g_n dxdy =& \frac{1}{2}\int_{\mathbb{R}^2_+} (u^s+u)\cdot\partial_x(\langle y\rangle^{2(k+\ell)}\phi_2^2 g_m^2)dxdy\\ &\le \frac{1}{2}\|u_x\|_{L^\infty(\mathbb{R}^2_+)}\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ &\le C\|w\|_{H^2_1(\mathbb{R}^2_+)}\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

Integrating by part, where the boundary value is vanish,

$$\begin{aligned}&-\int_{\mathbb{R}^2_+}\partial_y^2 g_n\cdot\langle y\rangle^{2(k+\ell)}\phi_2^2 g_n dxdy\\ &=\|\phi_2\partial_y g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}+\int_{\mathbb{R}^2_+}\partial_y g_n(\langle y\rangle^{2(k+\ell)}\phi_2^2)' g_n dxdy\\ &\ge\frac{3}{4}\|\phi_1\partial_y g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}-4\|(\langle y\rangle^{k+\ell}\phi_2)' g_n\|^2_{L^2(\mathbb{R}^2_+)},\end{aligned}$$

and

$$-\epsilon\int_{\mathbb{R}^2_+}\partial_x^2 g_n\cdot\langle y\rangle^{2(k+\ell)}\phi_2^2 g_n dxdy=\epsilon\|\phi_2\partial_x g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}.$$

We have also

$$\begin{aligned}&-\epsilon\int_{\mathbb{R}^2_+}\big(\partial_x\partial_y^{-1}g_n\big)\partial_y\eta_1\langle y\rangle^{2(k+\ell)}\phi_2^2 g_n dxdy\\ &=\epsilon\int_{\mathbb{R}^2_+}\partial_y^{-1}g_n\partial_y\eta_1\langle y\rangle^{2(k+\ell)}\phi_2^2\partial_x g_n dxdy\\ &\quad+\epsilon\int_{\mathbb{R}^2_+}\partial_y^{-1}g_n(\partial_y\partial_x\eta_1)\langle y\rangle^{2(k+\ell)}\phi_2^2 g_n dxdy\\ &\le\epsilon\|\phi_2\partial_y^{-1}g_n\partial_y\eta_1\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}+\frac{\epsilon}{4}\|\phi_2\partial_x g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ &\quad+\epsilon\|\phi_2\partial_y^{-1}g_n\partial_y\partial_x\eta_1\|^2_{L^2(\mathbb{R}^2_+)}+\epsilon\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

So by (4.6) and $0<\epsilon\le 1$, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}+\|\phi_2\partial_y g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}+\epsilon\|\phi_2\partial_x g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ &\le C\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}+\|\phi_2\partial_y^{-1}g_n\partial_y\eta_1\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ &\quad+\|\phi_2\partial_y^{-1}g_n\partial_y\partial_x\eta_1\|^2_{L^2(\mathbb{R}^2_+)}+C\|\langle y\rangle^{k+\ell}\phi_2' g_n\|^2_{L^2(\mathbb{R}^2_+)}\\ &\qquad+2\Big|\int_{\mathbb{R}^2_+}\phi_2 M^n\cdot\langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\Big|.\end{aligned}$$

Then we can finish the proof of the Proposition 5.2 by the following four Lemmas. □

Lemma 5.3. *Under the assumption of Proposition 5.2, we have*

$$\|\phi_2\partial_y^{-1}g_n\partial_y\eta_1\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|\phi_2\partial_y^{-1}g_n\partial_y\partial_x\eta_1\|^2_{L^2(\mathbb{R}^2_+)} + \|\langle y\rangle^{k+\ell}\phi_2'g_n\|^2_{L^2(\mathbb{R}^2_+)} \le \tilde{C}(\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}),$$

where $\tilde{C}$ is independent of ϵ.

Proof. Notice that (4.1) and (4.2) imply

$$|\mathbf{1}_{I_{\phi_2}}\eta_1| \le C\langle y\rangle^{-\ell}, \quad |\mathbf{1}_{I_{\phi_2}}\partial_x\eta_1| \le C\langle y\rangle^{-\ell},$$
$$|\mathbf{1}_{I_{\phi_2}}\partial_y\eta_1| \le C\langle y\rangle^{-\ell-1}, \quad |\mathbf{1}_{I_{\phi_2}}\partial_y\partial_x\eta_1| \le C\langle y\rangle^{-\ell-1}.$$

By Lemma A.1, we have

$$\begin{aligned}
\|\phi_2\partial_y^{-1}g_n(\partial_y\partial_x\eta_1)\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} &\le C\|\langle y\rangle^{k-1}(\phi_2\partial_y^{-1}g_n)\|^2_{L^2(\mathbb{R}^2_+)}\\
&\le C\|\langle y\rangle^{k}\partial_y(\phi_2\partial_y^{-1}g_n)\|^2_{L^2(\mathbb{R}^2_+)}\\
&\le C\|\phi_2 g_n\|^2_{L^2_k(\mathbb{R}^2_+)} + C\|\langle y\rangle^k\partial_y^{-1}g_n\phi_2'\|^2_{L^2}\\
&\le C\|\phi_2 g_n\|^2_{L^2_k(\mathbb{R}^2_+)} + C\|\partial_x^n u\phi_2'\|^2_{L^2}\\
&\le C\|\phi_2 g_n\|^2_{L^2_k(\mathbb{R}^2_+)} + C\|w\|^2_{H^n_1}.
\end{aligned}$$

Similarly, we also obtain

$$\|\phi_2\partial_y^{-1}g_n\partial_y\eta_1\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} \le C\|\phi_2 g_n\|^2_{L^2_k(\mathbb{R}^2_+)} + C\|w\|^2_{H^n_1}.$$

Finally, since $\operatorname{Supp}\phi_2' = [a+2c_0, a+3c_0]$, we have

$$\begin{aligned}
|\langle y\rangle^{k+\ell}\phi_2'g_n| &= \left|\langle y\rangle^{k+\ell}\phi_2'\left(\frac{\partial_x^n u}{u_y^s+w}\right)_y\right|\\
&\le \left|\langle y\rangle^{k+\ell}\phi_2'\left(\frac{\partial_x^n w}{u_y^s+w}\right)\right|\\
&\quad + \left|\langle y\rangle^{k+\ell}\phi_2'\left(\frac{\partial_x^n u(u_{yy}^s+w_y)}{(u_y^s+w)^2}\right)\right|\\
&\le C(|\phi_2'\partial_x^n w| + |\phi_2'\partial_x^n u|).
\end{aligned}$$

With the help of the above inequality, we conclude

$$\|\langle y\rangle^{k+\ell}\phi_2'g_n\|^2_{L^2(\mathbb{R}^2_+)} \le \tilde{C}\|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)},$$

which finishes the proof of Lemma 5.3. □

Lemma 5.4. *Under the assumption of Proposition 5.2, we have*

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2_+}\phi_2\sum_{j=1}^4 M_j^n\cdot\langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right|\\
&\quad\le \frac{1}{8}\|\phi_2\partial_y g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \frac{\epsilon}{4}\|\phi_2\partial_x g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\
&\qquad + \tilde{C}(\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|w\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}),
\end{aligned}$$

where $\tilde{C}$ is independent of ϵ.

Proof. Recall $\phi_2 M_1^n = \phi_2(u^s+u)(g_n\eta_1 + \partial_y^{-1}g_n\cdot\partial_y\eta_1)$

$$\left|\int_{\mathbb{R}^2_+}(u^s+u)g_n\eta_1\cdot\langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right| \le C\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}.$$

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+}(u^s+u)\partial_y^{-1}g_n\partial_y\eta_1\cdot\langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right|\\ &\le \|\phi_2\partial_y^{-1}g_n\partial_y\eta_1\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ &\le C(\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}).\end{aligned}$$

Then, we have

$$\left|\int_{\mathbb{R}^2_+}\phi_2 M_1^n\cdot\langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right| \le C(\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}).$$

The estimates of M_2^n and M_3^n need the following decay rate of η_2:

$$\begin{aligned}&|\mathbf{1}_{I_{\phi_2}}\eta_2| \le C\langle y\rangle^{-1}, \quad |\mathbf{1}_{I_{\phi_2}}\partial_x\eta_2| \le C\langle y\rangle^{-\ell-1},\\ &|\mathbf{1}_{I_{\phi_2}}\partial_y\eta_2| \le C\langle y\rangle^{-2}, \quad |\mathbf{1}_{I_{\phi_2}}\partial_y\partial_x\eta_2| \le C\langle y\rangle^{-\ell-2}.\end{aligned}$$

Recall $M_2^n = 2(\partial_y g_n)\eta_2 + 2g_n(\partial_y\eta_2 - 2\eta_2^2) - 8\partial_y^{-1}g_n\eta_2\partial_y\eta_2$. We have

$$\left|\int_{\mathbb{R}^2_+}g_n(\partial_y\eta_2-\eta_2^2)\cdot\langle y\rangle^{2(k+\ell)}\phi_2^2 g_n dxdy\right| \le C\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)},$$

$$\left|\int_{\mathbb{R}^2_+}(\partial_y g_n)\eta_2\cdot\langle y\rangle^{2(k+\ell)}\phi_2^2 g_n dxdy\right| \le C\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \frac{1}{8}\|\phi_2\partial_y g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)},$$

$$\begin{aligned}\left|2\int_{\mathbb{R}^2_+}\partial_y^{-1}g_n\eta_2\,\partial_y\eta_2\langle y\rangle^{2(k+\ell)}\phi_2^2 g_n dxdy\right| &\le C\|\langle y\rangle^{k+\ell-3}\phi_2\partial_y^{-1}g_n\|^2_{L^2} + \|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ &\le C(\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|w\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}).\end{aligned}$$

All together, we conclude

$$\left|\int_{\mathbb{R}^2_+}\eta_2 M_2^n\langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right| \le C(\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}+\|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)})+\frac{1}{8}\|\phi_2\partial_y g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)},$$

and exactly same computation gives also

$$\left|\int_{\mathbb{R}^2_+}\eta_2 M_3^n\langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right| \le C(\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}+\|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)})+\frac{\epsilon}{4}\|\phi_2\partial_x g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}.$$

Now for M_4^n, we have

$$\begin{aligned}M_4^n &= \left(\partial_y^{-1}g_n\,\frac{(u^s+u)w_x + v(w_y+u^s_{yy})}{u^s_y+u_y}\right)_y\\ &= g_n\frac{(u^s+u)w_x + v(w_y+u^s_{yy})}{u^s_y+u_y}\\ &\quad + (\partial_y^{-1}g_n)\left(\frac{(u^s+u)w_x + v(w_y+u^s_{yy})}{u^s_y+u_y}\right)_y.\end{aligned}$$

Now using (4.1)-(4.2) and $m \geq 6$, with the same computation as above, we can get

$$\left|\int_{\mathbb{R}^2_+} \eta_2 M_4^n \langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right| \leq C(\|\phi_2 g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}).$$

which finishes the proof of Lemma 5.4. □

Lemma 5.5. *Under the assumption of Proposition 5.2, we have*

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+} \phi_2 M_5^n \cdot \langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right|\\ &\quad\leq \tilde{C}(\sum_{p=1}^n \|\phi_2 g_p\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}),\end{aligned}$$

where $\tilde{C}$ is independent of ϵ.

Remark Since ϕ_1 is compactly supported, with similar proof, we can get

$$\left|\int_{\mathbb{R}^2_+} \phi_1 M_5^n\, \phi_1 g_n dxdy\right| \leq \tilde{C}(\|\phi_1 g_n\|^2_{L^2(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}).$$

Proof. We have, on the Supp ϕ_2,

$$\begin{aligned}M_5^n &= \left(\frac{\sum_{i=1}^n C_n^i \partial_x^i u\, \partial_x^{n+1-i} u}{u_y^s + u_y}\right)_y\\ &= \sum_{i\geq 4} C_n^i \big(\frac{\partial_x^i u}{u_y^s+u_y}\big)_y \partial_x^{n+1-i}u + \sum_{1\leq i\leq 3} C_n^i \partial_x^i u \big(\frac{\partial_x^{n+1-i}u}{u_y^s+u_y}\big)_y\\ &\quad+ \sum_{i\geq 4} C_n^i \big(\frac{\partial_x^i u}{u_y^s+u_y}\big)\partial_x^{n+1-i}\partial_y u + \sum_{1\leq i\leq 3} C_n^i (\partial_y \partial_x^i u)\big(\frac{\partial_x^{n+1-i}u}{u_y^s+u_y}\big)\\ &= \sum_{i\geq 4} C_n^i g_i\, \partial_x^{n+1-i} u + \sum_{1\leq i\leq 3} C_n^i \partial_x^i u\, g_{n+1-i}\\ &\quad+ \sum_{i\geq 4} C_n^i (\partial_y^{-1} g_i)\partial_x^{n+1-i} w + \sum_{1\leq i\leq 3} C_n^i \partial_x^i w\, (\partial_y^{-1} g_{n+1-i})\,.\end{aligned}$$

Here if $n \leq 3$, we have only the second term.

Then, for $\|w\|_{H^m_{k+\ell}} \leq \zeta \leq 1, m \geq 6$,

$$\begin{aligned}&\sum_{i\geq 4} C_n^i \|\phi_2 g_i\, \partial_x^{n+1-i} u\|_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \sum_{1\leq i\leq 3} \|\phi_2 \partial_x^i u\, g_{n+1-i}\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ &\leq \sum_{i\geq 4} C_n^i \|\phi_2 g_i\|_{L^2_{k+\ell}(\mathbb{R}^2_+)} \|\partial_x^{n+1-i} u\|_{L^\infty(\mathbb{R}^2_+)}\\ &\qquad+ \sum_{1\leq i\leq 3} C_n^i \|\partial_x^i u\|_{L^\infty(\mathbb{R}^2_+)}\, \|\phi_2 g_{n+1-i}\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ &\leq C\sum_{i\geq 4} C_n^i \|\phi_2 g_i\|_{L^2_{k+\ell}(\mathbb{R}^2_+)} \|w\|_{H_1^{n+3-i}}\\ &\qquad+ C\sum_{1\leq i\leq 3} C_n^i \|w\|_{H_1^{i+3}}\, \|\phi_2 g_{n+1-i}\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\end{aligned}$$

Similarly

$$\sum_{i\geq 4} C_n^i \|\phi_2(\partial_y^{-1}g_i)\partial_x^{n+1-i}w\|_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \sum_{1\leq i\leq 3} C_n^i \|\phi_2\partial_x^i w\,(\partial_y^{-1}g_{n+1-i})\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}$$
$$\leq \sum_{i\geq 4} C_n^i \|\phi_2(\partial_y^{-1}g_i)\|_{L^2_x(L^\infty_y)}\|\partial_x^{n+1-i}w\|_{L^\infty_x(L^2_{y,k+\ell})}$$
$$+ \sum_{1\leq i\leq 3} C_n^i \|\partial_x^i w\|_{L^\infty_x(L^2_{y,k+\ell})}\,\|\phi_2(\partial_y^{-1}g_{n+1-i})\|_{L^2_x(L^\infty_y)}\,.$$

On the other hand,

$$\|\phi_2(\partial_y^{-1}g_i)\|^2_{L^2_x(L^\infty_y)} = \int_{\mathbb{R}} \|\phi_2(\partial_y^{-1}g_i)\|^2_{L^\infty(\mathbb{R}_+)}dx$$
$$\leq \int_{\mathbb{R}} \left|\int_0^\infty |\partial_y(\phi_2\partial_y^{-1}g_i)|d\tilde{y}\right|^2 dx$$
$$\leq \int_{\mathbb{R}} \left|\int_0^\infty (|\phi_2 g_i)| + |\phi_2' \frac{\partial_x^i u}{u_y^s + w}|)d\tilde{y}\right|^2 dx$$
$$\leq C\|\phi_2 g_i\|^2_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)} + C\|\partial_x^i w\|^2_{L^2_1(\mathbb{R}^2_+)}.$$

We have proven Lemma 5.5. □

Lemma 5.6. *Under the assumption of Proposition 5.2, we have*

$$\left|\int_{\mathbb{R}^2_+} \phi_2 M_6^n \,\langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right|$$
$$\leq \frac{1}{8}\|\phi_2\partial_y g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \tilde{C}(\sum_{p=1}^n \|\phi_2 g_p\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}), \tag{5.3}$$

where $\tilde{C}$ is independents of ϵ.

Remark Since ϕ_1 is compactly supported, with similar proof, we can get

$$\left|\int_{\mathbb{R}^2_+} \phi_1 M_6^n\,\phi_1 g_n dxdy\right| \leq \frac{1}{8}\|\phi_1\partial_y g_n\|^2_{L^2(\mathbb{R}^2_+)} + \tilde{C}(\|\phi_1 g_n\|^2_{L^2(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}).$$

Proof.

$$M_6^n = \left(\frac{\sum_{i=1}^n C_n^i\partial_x^i w\,\partial_x^{n-i}v}{u_y^s + u_y}\right)_y$$
$$= \sum_{i=1}^n C_n^i\left(\frac{\partial_x^i w}{u_y^s+u_y}\right)_y \partial_x^{n-i}v + \sum_{i=1}^n C_n^i\left(\frac{\partial_x^i w}{u_y^s+u_y}\right)\partial_x^{n+1-i}u$$
$$= \sum_{i=1}^n C_n^i\left(g_i + \partial_y^{-1}g_i\,\eta_2\right)_y \partial_x^{n-i}v + \sum_{i=1}^n C_n^i\left(g_i + \partial_y^{-1}g_i\,\eta_2\right)\partial_x^{n+1-i}u.$$

Compared with the computation before, the only term to be studied is

$$\left|\int_{\mathbb{R}^2_+} \phi_2(\partial_y g_n)\,v\,\langle y\rangle^{2(k+\ell)}\phi_2 g_n dxdy\right| \leq \|v\|_{L^\infty(\mathbb{R}^2_+)}\|\partial_y g_n\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\|g_n\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}$$
$$\leq \frac{1}{8}\|\phi_2\partial_y g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + 4\|w\|^2_{H^2_1(\mathbb{R}^2_+)}\|g_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)},$$

which gives (5.3). □

6. Uniform estimate for the convex part

In this section we consider the convex part. The goal is to obtain the L^2 estimate of $\partial_x^m w$ on the support of ψ.

Proposition 6.1. *Let $w \in L^\infty([0,T]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+)), m \geq 6$ be a solution of (3.2) such that it satisfies (4.1)-(4.2) with $0 < \zeta \leq 1$. Assume that the shear flow u^s satisfies Lemma 2.1, then h_m satisfies the following estimate,*

$$\frac{1}{2}\frac{d}{dt}\|\psi h_m\|^2_{L^2} + \frac{3}{4}\|\psi\partial_y h_m\|^2_{L^2} + \frac{3\epsilon}{4}\|\psi\partial_x h_m\|^2_{L^2} \leq C\|w\|^3_{H^m_{k+\ell}} - \int_{\mathbb{R}^2_+}\psi^2\sqrt{u^s_{yy}+w_y}\,\partial_x^m\, v\, h_m dxdy,$$

where C is independent of $0 < \epsilon \leq 1$.

Proof. Multiplying (4.8) by ψh_m, integrating over $\mathbb{R}^2_+$, then integrating by part, where the boundary value is vanish, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\psi h_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\psi\partial_y h_m\|^2_{L^2(\mathbb{R}^2_+)} + \epsilon\|\psi\partial_x h_m\|^2_{L^2(\mathbb{R}^2_+)}\\ &= -\int_{\mathbb{R}^2_+}\psi^2\sqrt{u^s_{yy}+w_y}(\partial_x^m v)h_m dxdy + \int_{\mathbb{R}^2_+}\psi^2 N_m h_m dxdy\\ &\quad + \int_{\mathbb{R}^2_+}(\partial_x u)|\psi\, h_m|^2 dxdy - 2\int_{\mathbb{R}^2_+}(\partial_y\psi)\psi\, h_m\partial_y h_m dxdy.\end{aligned}$$

Now, for $m \geq 6$, and $\|w\|_{H^m_{k+\ell}} \leq \zeta$, we have

$$\left|\frac{1}{\sqrt{u^s_{yy}+w_y}}\mathbf{1}_{I_\psi}\right| \leq c_0^{-1}, \quad |\eta_3\,\mathbf{1}_{I_\psi}| + |\eta_4\,\mathbf{1}_{I_\psi}| \leq C, \tag{6.1}$$

and also

$$|h_m\mathbf{1}_{I_\psi}| = \left|\frac{\partial_x^m w}{\sqrt{u^s_{yy}+w_y}}\mathbf{1}_{I_\psi}\right| \leq c_0^{-1}|\mathbf{1}_{I_\psi}\partial_x^m w|. \tag{6.2}$$

Thus,

$$\left|\int_{\mathbb{R}^2_+}(\partial_x u)|\psi\, h_m|^2 dxdy\right| + \left|2\int_{\mathbb{R}^2_+}(\partial_y\psi)\psi\, h_m\partial_y h_m dxdy\right| \leq C\|w\|^2_{H^m_{k+\ell}} + \frac{1}{8}\|\psi\partial_y h_m\|^2_{L^2(\mathbb{R}^2_+)}.$$

For the terms $N_m = N_1^m + N_2^m + N_3^m + N_4^m$ with

$$\begin{aligned} N_1^m &= -\frac{1}{4}h_m\eta_3^2 + \partial_y h_m\,\eta_3 - \frac{\epsilon}{4}h_m\eta_4^2 + \epsilon\partial_x h_m\,\eta_4,\\ N_2^m &= h_m\,\frac{v(u^s_{yyy}+w_{yy}) - u_x(u^s_{yy}+w_y) - (u^s_y+w)w_x}{2\sqrt{u^s_{yy}+w_y}},\end{aligned}$$

using (6.1) and (6.2), we can get directly

$$\left|\int_{\mathbb{R}^2_+}\psi(N_1^m+N_2^m)\psi h_m dxdy\right|$$

$$\le \frac{1}{8}\|\psi\partial_y h_m\|_{L^2}^2 + \frac{\epsilon}{4}\|\psi\partial_x h_m\|_{L^2}^2 + C\|w\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}^2.$$

For the term

$$N_3^m = -\frac{\sum_{p=0}^{m-1} C_m^p\, \partial_x^{m-p} u\, \partial_x^{p+1} w}{\sqrt{u^s_{yy} + w_y}},$$

using (6.1) and (A.3), we can get

$$\left|\int_{\mathbb{R}^2_+} \psi N_3^m \psi h_m dxdy\right| \le C\|w\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}^3.$$

For the term

$$N_4^m = -\frac{\sum_{p=0}^{m-1} C_m^p\, (\partial_x^{m-p}\partial_y w)\, \partial_x^p v}{\sqrt{u^s_{yy} + w_y}},$$

observing that

$$\partial_x^p v = -\int_0^y \partial_x^{p+1} u d\tilde{y},$$

we can also get, by using (6.1) and (A.4),

$$\left|\int_{\mathbb{R}^2_+} \psi^2 \frac{\sum_{p=1}^{m-1} C_m^p\, (\partial_x^{m-p}\partial_y w)\partial_x^p v}{\sqrt{u^s_{yy} + w_y}} h_m dxdy\right| \le C\|w\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}^3.$$

We study finally the term

$$\begin{aligned}\left|\int_{\mathbb{R}^2_+} \psi^2 \frac{(\partial_x^m \partial_y w)\, v}{\sqrt{u^s_{yy} + w_y}} h_m dxdy\right| &= \left|\int_{\mathbb{R}^2_+} \psi^2 \frac{v(\partial_x^m \partial_y w)\partial_x^m w}{u^s_{yy} + w_y} dxdy\right| \\ &= \frac{1}{2}\left|\int_{\mathbb{R}^2_+} \psi^2 \frac{v\partial_y(\partial_x^m w)^2}{u^s_{yy} + w_y} dxdy\right| \\ &\le \frac{1}{2}\left|\int_{\mathbb{R}^2_+} \left(\frac{v\,\psi^2}{u^s_{yy} + w_y}\right)_y (\partial_x^m w)^2 dxdy\right| \\ &\le C\|w\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}^3,\end{aligned}$$

which conclude the Proposition 6.1. □

We study the worst term

$$-\int_{\mathbb{R}^2_+} \psi^2 \sqrt{u^s_{yy} + w_y}\, (\partial_x^m\, v)\, h_m dxdy$$

which is the main difficulty for the study of the Prandtl equation. We have

Proposition 6.2. *Under the assumption of Proposition 6.1, we have*

$$\begin{aligned}&-\int_{\mathbb{R}^2_+} \psi^2 \sqrt{u^s_{yy} + w_y}(\partial_x^m v) h_m dxdy \\ &\quad \le C\|w\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}^2 + \epsilon C\big(\|\phi_1 \partial_x g_m\|_{L^2(\mathbb{R}^2_+)}^2 + \|\phi_2 \partial_x g_m\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}^2\big) \\ &\qquad - \frac{d}{dt}\int_{\mathbb{R}^2_+} (\partial_x^m u)^2\, \frac{\psi\psi'}{u^s_y + u_y} dxdy\end{aligned}$$

where C is independent of ϵ.

Proof. By the definition of h_m, w and v, firstly, we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^2_+}\psi^2\sqrt{u^s_{yy}+w_y}(\partial^m_x v)\,h_m dxdy\\
&=-\int_{\mathbb{R}^2_+}\psi^2\sqrt{u^s_{yy}+w_y}\partial^m_x v\,\frac{\partial^m_x w}{\sqrt{u^s_{yy}+w_y}}dxdy\\
&=-\int_{\mathbb{R}^2_+}\psi^2\partial^m_x v\,\partial^m_x w dxdy=-\int_{\mathbb{R}^2_+}\psi^2\partial^m_x v\,\partial_y\partial^m_x u dxdy\\
&=-\int_{\mathbb{R}^2_+}\psi^2\partial^{m+1}_x u\,\partial^m_x u dxdy+\int_{\mathbb{R}^2_+}\partial_y(\psi^2)\partial^m_x v\,\partial^m_x u dxdy\\
&=2\int_{\mathbb{R}^2_+}\psi\,\psi'\partial^m_x v\,\partial^m_x u dxdy,
\end{aligned}
$$

where we use $\partial_y\partial^m_x v=-\partial^{m+1}_x u$, and the fact,

$$
\int_{\mathbb{R}^2_+}\psi^2\partial^{m+1}_x u\cdot\partial^m_x u dxdy=\frac{1}{2}\int_{\mathbb{R}^2_+}\psi^2\partial_x\big(\partial^m_x u\big)^2 dxdy=0.
$$

Taking the first equation of (3.1) with derivative ∂^m_x, we have

$$
\partial_t\partial^m_x u+(u^s+u)\partial^{m+1}_x u+(u^s_y+u_y)\partial^m_x v=\partial^m_x\partial^2_y u+\epsilon\partial^{m+2}_x u+\mathbf{A}_1+\mathbf{A}_2, \tag{6.3}
$$

where $\mathbf{A}_1=[u^s+u,\partial^m_x]\partial_x u,\ \mathbf{A}_2=[u^s_y+u_y,\partial^m_x]\,v$. Observing

$$
\mathrm{Supp}\,\psi'\subset\mathrm{Supp}\,\psi\cap\big(\{y;\,\phi_1(y)=1\}\cup\{y;\ \phi_2(y)=1\}\big), \tag{6.4}
$$

multiplying (6.3) with $\frac{2\psi\psi'\partial^m_x u}{u^s_y+u_y}$, we get

$$
\begin{aligned}
2\int_{\mathbb{R}^2_+}\psi\psi'\partial^m_x v\,\partial^m_x u dxdy=&-\int_{\mathbb{R}^2_+}\partial_t\partial^m_x u\,\frac{2\psi\psi'\partial^m_x u}{u^s_y+u_y}dxdy\\
&-\int_{\mathbb{R}^2_+}(u^s+u)\partial^{m+1}_x u\,\frac{2\psi\psi'\partial^m_x u}{u^s_y+u_y}dxdy\\
&+\int_{\mathbb{R}^2_+}\partial^2_y\partial^m_x u\,\frac{2\psi\psi'\partial^m_x u}{u^s_y+u_y}dxdy\\
&+\epsilon\int_{\mathbb{R}^2_+}\partial^{m+2}_x u\,\frac{2\psi\psi'\partial^m_x u}{u^s_y+u_y}dxdy\\
&+\int_{\mathbb{R}^2_+}\big(\mathbf{A}_1+\mathbf{A}_2\big)\,\frac{2\psi\psi'\partial^m_x u}{u^s_y+u_y}dxdy\\
=&\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3+\mathbf{I}_4+\mathbf{I}_5.
\end{aligned} \tag{6.5}
$$

Now, we study each term of (6.5).

$$
\begin{aligned}
\mathbf{I}_1&=-\int_{\mathbb{R}^2_+}\partial_t\partial^m_x u\cdot\frac{2\psi\psi'\partial^m_x u}{u^s_y+u_y}=-\int_{\mathbb{R}^2_+}\partial_t(\partial^m_x u)^2\cdot\frac{\psi\psi'}{u^s_y+u_y}\\
&=-\frac{d}{dt}\int_{\mathbb{R}^2_+}(\partial^m_x u)^2\cdot\frac{\psi\psi'}{u^s_y+u_y}dxdy+\int_{\mathbb{R}^2_+}(\partial^m_x u)^2\cdot\partial_t\left(\frac{\psi\psi'}{u^s_y+u_y}\right)dxdy.
\end{aligned}
$$

For the second term in the above equality, according to (3.2) and (6.4), we obtain, for $m \geq 6$,

$$\begin{aligned}\left\|\partial_t\left(\frac{\psi\,\psi'}{u_y^s+u_y}\right)\right\|_{L^\infty(\mathbb{R}^2_+)} &= \left\|\psi\,\psi'\left(\frac{\partial_t u_y^s+\partial_t u_y}{(u_y^s+u_y)^2}\right)\right\|_{L^\infty(\mathbb{R}^2_+)}\\ &\leq c_0^{-2}\|\psi\,\psi'(\partial_t u_y^s+\partial_t u_y)\|_{L^\infty(\mathbb{R}^2_+)}\\ &\leq C(1+\|w\|^2_{H^m_1\mathbb{R}^2_+)}).\end{aligned} \tag{6.6}$$

Then under the assumption $\|w\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq \zeta$, we get

$$\mathbf{I}_1 \leq -\frac{d}{dt}\int_{\mathbb{R}^2_+}(\partial_x^m u)^2\cdot\left(\frac{\psi\psi'}{u_y^s+u_y}\right)dxdy + C\|w\|^2_{H^m_{k+\ell}\mathbb{R}^2_+)}.$$

For the second term

$$\begin{aligned}\mathbf{I}_2 &= \int_{\mathbb{R}^2_+}(u^s+u)\partial_x\partial_x^m u\,\frac{2\psi\psi'\partial_x^m u}{u_y^s+u_y}dxdy\\ &= \int_{\mathbb{R}^2_+}\frac{\psi\psi'(u^s+u)}{u_y^s+u_y}\,\partial_x(\partial_x^m u)^2 dxdy\\ &= -\int_{\mathbb{R}^2_+}\left(\frac{\psi\psi'(u^s+u)}{u_y^s+u_y}\right)_x(\partial_x^m u)^2 dxdy.\end{aligned}$$

Similar to (6.6), we can get

$$\left\|\left(\frac{\psi\psi'(u^s+u)}{u_y^s+u_y}\right)_x\right\|_{L^\infty(\mathbb{R}^2_+)} \leq C(1+\|w\|^2_{H^m_1\mathbb{R}^2_+)}).$$

Again with the help of the assumption $\|w\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq \zeta$, we get

$$|\mathbf{I}_2| \leq C\|w\|^2_{H^m_{k+\ell}\mathbb{R}^2_+)}.$$

For the term $\mathbf{I}_3$, using $w=\partial_y u$, and integrating by part,

$$\begin{aligned}\mathbf{I}_3 &= \int_{\mathbb{R}^2_+}\partial_y^2\partial_x^m u\,\frac{2\psi\psi'\partial_x^m u}{u_y^s+u_y} = \int_{\mathbb{R}^2_+}\partial_y\partial_x^m w\,\frac{2\psi\psi'\partial_x^m u}{u_y^s+u_y}\\ &= -\int_{\mathbb{R}^2_+}(\partial_x^m w)^2\,\frac{2\psi\psi'}{u_y^s+u_y}dxdy - \int_{\mathbb{R}^2_+}\partial_x^m w\,\partial_x^m u\,\partial_y\left(\frac{2\psi\psi'}{u_y^s+u_y}\right)dxdy,\end{aligned}$$

we get

$$|\mathbf{I}_3| \leq C\|w\|^2_{H^m_{k+\ell}\mathbb{R}^2_+)}.$$

For the term $\mathbf{I}_5$, recalling

$$\mathbf{A}_1 = \sum_{i=1}^m C_m^i\partial_x^i u\,\partial_x^{m+1-i}u, \quad \mathbf{A}_2 = \sum_{i=1}^{m-1} C_m^i\partial_x^i w\,\partial_x^{m-i}v\,,$$

we don't need to worry about it, since the order of derivative is easy to be controlled. So we have

$$|\mathbf{I}_5| \leq C\|w\|^3_{H^m_{k+\ell}\mathbb{R}^2_+)}.$$

For the term $\mathbf{I}_4$, we have to be careful, since the order of derivative with respect to x is up to $m+2$. Integrating by parts with respect to x, we have

$$\begin{aligned}\mathbf{I}_4 &= \epsilon\int_{\mathbb{R}^2_+} \partial_x^{m+2}u\frac{2\psi\psi'\partial_x^m u}{u^s_y+u_y}dxdy\\ &= -\epsilon\int_{\mathbb{R}^2_+}\frac{2\psi\psi'}{u^s_y+u_y}(\partial_x^{m+1}u)^2dxdy+\epsilon\int_{\mathbb{R}^2_+}\big(\frac{\psi\psi'}{u^s_y+u_y}\big)_{xx}(\partial_x^m u)^2dxdy\\ &= \mathbf{I}_4^1+\mathbf{I}_4^2.\end{aligned}$$

For the second term, observing

$$\Big\|\big(\frac{\psi\psi'}{u^s_y+u_y}\big)_{xx}\Big\|_{L^\infty(\mathbb{R}^2_+)}\leq C\|w\|_{H^4_1(\mathbb{R}^2_+)},$$

it implies

$$|\mathbf{I}_4^2| = \left|\epsilon\int_{\mathbb{R}^2_+}\big(\frac{\psi\psi'}{u^s_y+u_y}\big)_{xx}\cdot(\partial_x^m u)^2dxdy\right|\leq C\|w\|^3_{H^m_1(\mathbb{R}^2_+)}.$$

For the first term, recalling

$$\begin{aligned}&\mathrm{Supp}\,\psi' = [a-5a_0, a-4a_0]\cup[a+4a_0,a+5a_0] = J_1\cup J_2;\\ &\phi_1(y)=1,\ \forall y\leq a-3a_0;\quad \phi_2(y)=1,\ \forall y\geq a+3a_0,\end{aligned}$$

and

$$\psi'\frac{\partial_x^{m+1}u}{u^s_y+u_y} = \psi'\partial_x\left(\frac{\partial_x^m u}{u^s_y+u_y}\right)+\psi'\frac{\partial_x^m u}{u^s_y+u_y}\eta_1 = \psi'\partial_x\partial_y^{-1}g_m+\psi'\frac{\partial_x^m u}{u^s_y+u_y}\eta_1,$$

we have

$$\begin{aligned}\epsilon&\left|\int_{\mathbb{R}^2_+}\frac{2\psi\psi'(\partial_x^{m+1}u)^2}{u^s_y+u_y}dxdy\right| = \epsilon\left|\int_{\mathbb{R}^2_+}2\psi\psi'\big(u^s_y+u_y\big)\big(\frac{\partial_x^{m+1}u}{u^s_y+u_y}\big)^2dxdy\right|\\ &\leq \epsilon\int_{\mathbb{R}^2_+}|\psi\psi'|\big(|u^s_y+u_y|\big)(\partial_x\partial_y^{-1}g_m)^2dxdy+\epsilon\int_{\mathbb{R}^2_+}\frac{|\psi\psi'|}{|u^s_y+u_y|}|\eta_1^2|(\partial_x^m u)^2dxdy\\ &\leq \epsilon C\int_{\mathbb{R}^2_+}\big((\mathbf{1}_{J_1}\partial_x\partial_y^{-1}g_m)^2+(\mathbf{1}_{J_2}\partial_x\partial_y^{-1}g_m)^2\big)dxdy+C\|w\|^2_{H^m_1(\mathbb{R}^2_+)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathbf{1}_{J_1}\partial_x\partial_y^{-1}g_m)^2(t,x,y) &= \mathbf{1}_{J_1}(y)\left(\int_0^{\min\{y,a-4a_0\}}\partial_x g_m(t,x,\tilde y)d\tilde y\right)^2\\ &= \mathbf{1}_{J_1}(y)\left(\int_0^{\min\{y,a-4a_0\}}\phi_1(\tilde y)\partial_x g_m(t,x,\tilde y)d\tilde y\right)^2\\ &\leq \mathbf{1}_{J_1}(y)C\|\phi_1\partial_x g_m(t,x)\|^2_{L^2(\mathbb{R}_+)},\end{aligned}$$

and

$$|(\mathbf{1}_{J_2}\partial_x\partial_y^{-1}g_m)^2(t,x,y)| = \mathbf{1}_{J_2}(y)\left(\int_{\max\{y,a+4a_0\}}^{+\infty}\partial_x g_m(t,x,\tilde y)d\tilde y\right)^2$$

$$
\begin{aligned}
&= \mathbf{1}_{J_2}(y)\left(\int_{\min\{y,a+4a_0\}}^{+\infty}\phi_2(\tilde{y})\partial_x g_m(t,x,\tilde{y})d\tilde{y}\right)^2\\
&\le C\|\phi_2\partial_x g_m(t,x)\|^2_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+)}.
\end{aligned}
$$

We get finally

$$
|\mathbf{I}_4^1| \le C\|w\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} + \epsilon C\big(\|\phi_1\partial_x g_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\phi_2\partial_x g_m\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\big),
$$

with C is independent of ϵ, which finishes the proof of Proposition 6.2. □

Combining Proposition 6.1 and Proposition 6.2, for the convex part, we get the following results.

Proposition 6.3. *Under the assumption of Proposition 6.1, we have*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\psi h_m\|^2_{L^2} + \frac{3}{4}\|\psi\partial_y h_m\|^2_{L^2} + \frac{3\epsilon}{4}\|\psi\partial_x h_m\|^2_{L^2}\\
&\qquad \le C_3\|w\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} + \epsilon C_4\big(\|\phi_1\partial_x g_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\phi_2\partial_x g_m\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\big)\\
&\qquad\qquad - \frac{d}{dt}\int_{\mathbb{R}^2_+}(\partial_x^m u)^2\,\frac{\psi\psi'}{u^s_y+u_y}dxdy,
\end{aligned}
\tag{6.7}
$$

where C_3, C_4 are independent of ϵ.

7. Existence of the solution

Now, we can conclude the following energy estimate for the sequence of approximate solutions.

Theorem 7.1. *Assume u^s satisfies Lemma 2.1. Let $m \ge 6$ be an even integer, $k+\ell > \frac{3}{2}, 0 \le \ell < \frac{1}{2}$, and $\tilde{u}_0 \in H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)$ which satisfies the compatibility conditions (2.8)-(2.9). Suppose that $\tilde{w}_\epsilon \in L^\infty([0,T]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$ is a solution to (3.2) such that*

$$
\|\tilde{w}_\epsilon\|_{L^\infty([0,T];H^m_{k+\ell}(\mathbb{R}^2_+))} \le \zeta
$$

with

$$
0 < \zeta \le 1, \quad C_m\zeta \le \frac{c_0^2}{2},
$$

where $0 < T \le T_1$ and T_1 is the lifespan of shear flow u^s in the Lemma 2.1, C_m is the Sobolev embedding constant in (4.2). Then there exists $C_T > 0, \tilde{C}_T > 0$ such that,

$$
\|\tilde{w}_\epsilon\|_{L^\infty([0,T];H^m_{k+\ell}(\mathbb{R}^2_+))} \le C_T\|\tilde{u}_0\|_{H^{m+1}_{2k+\ell-1}(\mathbb{R}^2_+)}, \tag{7.1}
$$

and

$$
\begin{aligned}
&\|\phi_1 g^\epsilon_m(t)\|^2_{L^2(\mathbb{R}^2_+)} + \sum_{n=1}^m\|\phi_2 g^\epsilon_n(t)\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon(t)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}\\
&\le \tilde{C}_T\big(\|\phi_1 g^\epsilon_m(0)\|^2_{L^2(\mathbb{R}^2_+)} + \sum_{n=1}^m\|\phi_2 g^\epsilon_n(0)\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon(0)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}\big).
\end{aligned}
\tag{7.2}
$$

Moreover, $C_T > 0, \tilde{C}_T > 0$ is increasing with respect to $0 < T \le T_1$ and independent of $0 < \epsilon \le 1$.

Firstly, we collect some results to be used from Section 3 - 6. We come back to the notations with tilde and the sub-index ϵ. Then $g^\epsilon_m, h^\epsilon_m$ are the the functions defined by $\tilde u_\epsilon$. Under the hypothesis of Theorem 7.1, we have proven the estimates (3.10), (5.1), (5.2) and (6.7),

$$\begin{aligned}\frac{d}{dt}\|\tilde w_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} &+ \|\partial_y \tilde w_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}\\ &+ \epsilon\|\partial_x \tilde w_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} \le C_1\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)},\end{aligned}\tag{7.3}$$

$$\begin{aligned}\frac{d}{dt}\|\phi_1 g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} &+ \|\phi_1\partial_y g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \epsilon\|\phi_1\partial_x g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)}\\ &\le C_2\big(\|\phi_1 g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\tilde w_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}\big),\end{aligned}\tag{7.4}$$

and

$$\begin{aligned}\frac{d}{dt}\sum_{n=1}^m\|\phi_2 g^\epsilon_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} &+ \sum_{n=1}^m\|\phi_2\partial_y g^\epsilon_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \epsilon\sum_{n=1}^m\|\phi_2\partial_x g^\epsilon_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\ &\le C_2\Big(\sum_{n=1}^m\|\phi_2 g^\epsilon_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|\tilde w_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}\Big),\end{aligned}\tag{7.5}$$

$$\begin{aligned}\frac12\frac{d}{dt}\|\psi h^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} &+ \frac34\|\psi\partial_y h^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \frac{3\epsilon}{4}\|\psi\partial_x h^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)}\\ &\le C_3\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} + \epsilon C_4\big(\|\phi_1\partial_x g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\phi_2\partial_x g^\epsilon_m\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\big)\\ &\quad - \frac{d}{dt}\int_{\mathbb{R}^2_+}\frac{\psi\psi'(\partial^m_x\tilde u_\epsilon)^2}{u^s_y+\tilde w_\epsilon}dxdy,\end{aligned}\tag{7.6}$$

where C_1, C_2, C_3, C_4 are independent of $0<\epsilon\le 1$.

Lemma 7.2. *Under the assumption of Theorem 7.1, we have*

$$\left|\int_{\mathbb{R}^2_+}\frac{\psi\psi'(\partial^m_x\tilde u_\epsilon)^2}{u^s_y+\tilde w_\epsilon}dxdy\right| \le C_5\big(\|\phi_1 g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\phi_2 g^\epsilon_m\|^2_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}\big),\tag{7.7}$$

where C_5 is independent of $0<\epsilon\le 1$.

Proof. In fact,

$$\begin{aligned}\left|\int_{\mathbb{R}^2_+}\frac{\psi\psi'(\partial^m_x\tilde u_\epsilon)^2}{u^s_y+\tilde w_\epsilon}dxdy\right| &\le c_0\int_{\mathbb{R}^2_+}|\psi\psi'|\left(\frac{\partial^m_x\tilde u_\epsilon}{u^s_y+\tilde w_\epsilon}\right)^2dxdy\\ &\le C\int_{\mathbb{R}^2_+}\big((\mathbf{1}_{J_1}\partial_y^{-1}g^\epsilon_m)^2 + (\mathbf{1}_{J_2}\partial_y^{-1}g^\epsilon_m)^2\big)dxdy\\ &\le C\int_{\mathbb{R}^2_+}\left(\mathbf{1}_{J_1}(y)\int_0^{\min\{y,a-4a_0\}}\phi_1(\tilde y)g^\epsilon_m(t,x,\tilde y)d\tilde y\right)^2dxdy\\ &\quad + C\int_{\mathbb{R}^2_+}\left(\mathbf{1}_{J_2}(y)\int_{\max\{y,a+4a_0\}}^{+\infty}\phi_2(\tilde y)g^\epsilon_m(t,x,\tilde y)d\tilde y\right)^2dxdy\\ &\le C_5\big(\|\phi_1 g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\phi_2 g^\epsilon_m\|^2_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}\big).\end{aligned}$$

The proof of the Lemma is complete. □

We have also directly

$$\left|\int_{\mathbb{R}^2_+}\frac{\psi\psi'(\partial_x^m\tilde{u}_\epsilon)^2}{u_y^s+\tilde{w}_\epsilon}dxdy\right|\le C_6\|\tilde{w}_\epsilon\|^2_{H^m_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)},\tag{7.8}$$

and

$$\|\psi h_m^\epsilon\|^2_{L^2(\mathbb{R}^2_+)}\le C_7\|\partial_x^m\tilde{w}_\epsilon\|^2_{L^2(\mathbb{R}^2_+)}.\tag{7.9}$$

Denoting $\tilde{C}_4=\max\{C_4,2C_5\}$, and taking

$$\tilde{C}_4\times\big\{(7.4)+(7.5)\big\}+(7.3)+(7.6),$$

we get

$$\begin{aligned}\frac{d}{dt}\bigg\{\tilde{C}_4\Big(\|\phi_1 g_m^\epsilon\|^2_{L^2(\mathbb{R}^2_+)}+\sum_{n=1}^m\|\phi_2 g_n^\epsilon\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\Big)+\frac{1}{2}\|\psi h_m^\epsilon\|^2_{L^2(\mathbb{R}^2_+)}\\+\|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+\int_{\mathbb{R}^2_+}\frac{\psi\psi'(\partial_x^m\tilde{u}_\epsilon)^2}{u_y^s+\tilde{w}_\epsilon}dxdy\bigg\}\\\le\tilde{C}_4C_2\left(\|\phi_1 g_m^\epsilon\|^2_{L^2(\mathbb{R}^2_+)}+\sum_{n=1}^m\|\phi_2 g_n^\epsilon\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+}\right)\\+(2\tilde{C}_4C_2+C_1+C_3)\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

Then we have

$$\begin{aligned}\frac{d}{dt}\bigg\{e^{-C_2t}\bigg(\tilde{C}_4\Big(\|\phi_1 g_m^\epsilon\|^2_{L^2(\mathbb{R}^2_+)}+\sum_{n=1}^m\|\phi_2 g_n^\epsilon\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\Big)+\frac{1}{2}\|\psi h_m^\epsilon\|^2_{L^2(\mathbb{R}^2_+)}\\+\|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+\int_{\mathbb{R}^2_+}\frac{\psi\psi'(\partial_x^m\tilde{u}_\epsilon)^2}{u_y^s+\tilde{w}_\epsilon}dxdy\bigg)\bigg\}\\\le(2\tilde{C}_4C_2+C_1+C_3)e^{-C_2t}\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}-\frac{C_2}{2}e^{-C_2t}\|\psi h_m^\epsilon\|^2_{L^2(\mathbb{R}^2_+)}\\-C_2e^{-C_2t}\|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}-C_2e^{-C_2t}\int_{\mathbb{R}^2_+}\frac{\psi\psi'(\partial_x^m\tilde{u}_\epsilon)^2}{u_y^s+\tilde{w}_\epsilon}dxdy.\end{aligned}$$

According to (7.8) and (7.9), we obtain

$$\begin{aligned}\frac{d}{dt}\bigg\{e^{-C_2t}\bigg(\tilde{C}_4\Big(\|\phi_1 g_m^\epsilon\|^2_{L^2(\mathbb{R}^2_+)}+\sum_{n=1}^m\|\phi_2 g_n^\epsilon\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\Big)+\frac{1}{2}\|\psi h_m^\epsilon\|^2_{L^2(\mathbb{R}^2_+)}\\+\|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+\int_{\mathbb{R}^2_+}\frac{\psi\psi'(\partial_x^m\tilde{u}_\epsilon)^2}{u_y^s+\tilde{w}_\epsilon}dxdy\bigg)\bigg\}\\\le C_8e^{-C_2t}\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

Using (7.8) and (7.9) again , we have for any $t \in]0, T]$

$$
\begin{aligned}
&\tilde{C}_4\Big(\|\phi_1 g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \sum_{n=1}^m \|\phi_2 g^\epsilon_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\Big) + \frac{1}{2}\|\psi h^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)}\\
&\quad + \|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} + \int_{\mathbb{R}^2_+} \frac{\psi\psi'(\partial^m_x \tilde{u}_\epsilon)^2}{u^s_y + \tilde{w}_\epsilon} dxdy\\
&\le C_8 e^{C_2 t}\int_0^t e^{-C_2\tau}\|\tilde{w}_\epsilon(\tau)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} d\tau\\
&\quad + e^{C_2 t}\tilde{C}_4\Big(\|\phi_1 g^\epsilon_m(0)\|^2_{L^2(\mathbb{R}^2_+)} + \sum_{n=1}^m \|\phi_2 g^\epsilon_n(0)\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+}\Big)\\
&\quad + (1 + C_6 + C_7)e^{C_2 t}\|\tilde{w}_\epsilon(0)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}.
\end{aligned}
\tag{7.10}
$$

We study now

$$
T^\epsilon_m(g, w)(t) = \|\phi_1 g^\epsilon_m(t)\|^2_{L^2(\mathbb{R}^2_+)} + \sum_{n=1}^m \|\phi_2 g^\epsilon_n(t)\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon(t)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)},
$$

firstly, we have the following upper bounded estimate.

Lemma 7.3. *For the inital date, we have*

$$
\begin{aligned}
T^\epsilon_m(g, w)(0) &= \|\phi_1 g^\epsilon_m(0)\|^2_{L^2(\mathbb{R}^2_+)} + \sum_{n=1}^m \|\phi_2 g^\epsilon_n(0)\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon(0)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}\\
&\le C\|\tilde{u}_0\|^2_{H^{m+1}_{2k+\ell-1}(\mathbb{R}^2_+)},
\end{aligned}
$$

where C is independent of ϵ.

Proof. Notice for any $1 \le n \le m$,

$$
\phi_2 g^\epsilon_n = \phi_2\big(\frac{\partial^n_x \tilde{u}_\epsilon}{u^s_y + \tilde{w}_\epsilon}\big)_y = \phi_2\frac{\partial^n_x\partial_y \tilde{u}_\epsilon}{u^s_y + \tilde{w}_\epsilon} - \phi_2\frac{\partial^n_x \tilde{u}_\epsilon}{u^s_y + \tilde{w}_\epsilon}\eta_2,
$$

and $\tilde{u}_\epsilon(0) = \tilde{u}_0$, then we deduce, for any $1 \le n \le m$,

$$
\begin{aligned}
&\|\phi_2 g^\epsilon_n(0)\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} \le 2\left\|\phi_2\frac{\partial^n_x\partial_y\tilde{u}_0}{u^s_y + \tilde{w}_0}\right\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + 2\left\|\phi_2\frac{\partial^n_x\tilde{u}_0}{u^s_y + \tilde{w}_0}\eta_2(0)\right\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\
&\le C\big(\|\partial^n_x\partial_y\tilde{u}_0\|^2_{L^2_{2k+\ell}(\mathbb{R}^2_+)} + \|\partial^n_x\tilde{u}_0\|^2_{L^2_{2k+\ell-1}(\mathbb{R}^2_+)}\big) \le C\|\tilde{u}_0\|^2_{H^{m+1}_{2k+\ell-1}(\mathbb{R}^2_+)}.
\end{aligned}
$$

The estimate of $\|\phi_1 g^\epsilon_m(0)\|^2_{L^2(\mathbb{R}^2_+}$ is easier, since there is no weight. And

$$
\|\tilde{w}_\epsilon(0)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} = \|\partial_y\tilde{u}_0\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \le \|\tilde{u}_0\|^2_{H^{m+1}_{k+\ell-1}(\mathbb{R}^2_+)}.
$$

This completes the proof of Lemma 7.3. □

Using (7.7), Lemma 7.3 and $2C_5 \le \tilde{C}_4$, we can deduce from (7.10) that for any $t \in]0, T]$

$$
\begin{aligned}
&C_5\Big(\|\phi_1 g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\phi_2 g^\epsilon_m\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\Big) + \frac{1}{2}\|\psi h^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}\\
&\le C_8 e^{C_2 t}\int_0^t e^{-C_2\tau}\|\tilde{w}_\epsilon(\tau)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} d\tau + C_9 e^{C_2 t}\|\tilde{u}_0\|^2_{H^{m+1}_{2k+\ell-1}(\mathbb{R}^2_+)}.
\end{aligned}
\tag{7.11}
$$

We recall the definition of the cut-off functions

$$\phi_1(y) = 1 \ \text{for}\, 0 \le y \le a - 3a_0;\ \ \phi_1(y) = 0 \ \text{for}\, y \ge a - 2a_0;$$
$$\phi_2(y) = 0 \ \text{for}\, 0 \le y \le a + 2a_0;\ \ \phi_2(y) = 1 \ \text{for}\, y \ge a + 3a_0;$$
$$\psi(y) = 1 \ \text{for}\, |y - a| \le 4a_0;\ \ \psi(y) = 0 \ \text{for}\, |y - a| \ge 5a_0.$$

We have also the following low bounded estimate.

Lemma 7.4. *We have also the following estimate :*

$$\|\partial_x^m \tilde{w}_\epsilon\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} \le \tilde{C}\Big(\|\phi_1 g^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} + \|\phi_2 g^\epsilon_m\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} + \|\psi h^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)}\Big),$$

where $\tilde{C}$ is independent of ϵ.

Proof. Firstly,

$$\psi(y) h^\epsilon_m(t,x,y) = \psi(y) \frac{\partial^n_x \tilde{w}_\epsilon}{\sqrt{u^s_{yy} + \partial_y \tilde{w}_\epsilon}}.$$

Since

$$\mathbf{1}_{I_\psi}(y) \langle y\rangle^{2(k+\ell)} |u^s_{yy} + \partial_y \tilde{w}_\epsilon| \le C(c_0^{-1} + \frac{c_0}{2}),$$

we have

$$\|\psi h^\epsilon_m\|^2_{L^2(\mathbb{R}^2_+)} \ge \frac{1}{\tilde{C}} \|\psi \partial^m_x \tilde{w}_\epsilon\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}.$$

On the other hand,

$$\partial^m_x \tilde{u}_\epsilon(t,x,y) = (u^s_y + \tilde{w}_\epsilon) \int_y^{+\infty} g^\epsilon_m(t,x,\tilde{y}) d\tilde{y}, \quad y \in I_{\phi_2},$$

Therefore,

$$\partial^m_x \tilde{w} = (u^s_{yy} + (\tilde{w}_\epsilon)_y) \int_y^{+\infty} g^\epsilon_m(t,x,\tilde{y}) d\tilde{y} - (u^s_y + \tilde{w}_\epsilon) g^\epsilon_m(t,x,y),\ y \in I_{\phi_2}.$$

Denoting

$$J_{\phi_2} = \{y \in \mathbb{R}_+; \phi_2(y) = 1\} = \{y \in \mathbb{R}_+; y \ge a + 3a_0\},$$

we have, for $\ell - 1 < -\frac{1}{2}$,

$$\begin{aligned}
\|\mathbf{1}_{J_{\phi_2}} \partial^m_x \tilde{w}_\epsilon\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} \le & \int_{\mathbb{R}^2_+} \langle y\rangle^{2(k+\ell)} (u^s_{yy} + (\tilde{w}_\epsilon)_y)^2 \\
& \qquad \times \mathbf{1}_{J_{\phi_2}}(y) \left| \int_y^{+\infty} \phi_2(\tilde{y}) g^\epsilon_m(t,x,\tilde{y}) d\tilde{y} \right|^2 dxdy \\
& + \int_{\mathbb{R}^2_+} \langle y\rangle^{2(k+\ell)} \mathbf{1}_{J_{\phi_2}} \big((u^s_y + \tilde{w}_\epsilon) \phi_2 g^\epsilon_m(t,x,y)\big)^2 dxdy \\
\le & \, C \int_{\mathbb{R}^2_+} \langle y\rangle^{2(k+\ell)} (u^s_{yy} + (\tilde{w}_\epsilon)_y)^2 \|\phi_2\, g^\epsilon_m(t,x)\|^2_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+)} dxdy \\
& \quad + C\|\phi_2 g^\epsilon_m\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} \\
\le & \, C\big(\|\langle y\rangle^{k+\ell} u^s_{yy}\|_{L^2(\mathbb{R}_+)} + \|\langle y\rangle^{k+\ell} (\tilde{w}_\epsilon)_y\|_{L^\infty_x(L^2_y)}\big) \|\phi_2\, g^\epsilon_m\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} \\
& \quad + C\|\phi_2 g^\epsilon_m\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} \\
\le & \, C\|\phi_2 g^\epsilon_m\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}.
\end{aligned}$$

Similarly, for the other side,

$$J_{\phi_1} = \{y \in \mathbb{R}_+; \phi_1(y) = 1\} = \{y \in \mathbb{R}_+; y \le a - 3a_0\},$$

we have,

$$\partial_x^m \tilde{w}_\epsilon = (u_{yy}^s + (\tilde{w}_\epsilon)_y) \int_0^y g(t,x,\tilde{y})d\tilde{y} + (u_y^s + \tilde{w}_\epsilon)g(t,x,y),\ y \in I_{\phi_1}.$$

Thus

$$\|\mathbf{1}_{J_{\phi_1}} \partial_x^m \tilde{w}_\epsilon\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)} \le C\|\phi_1 g\|^2_{L^2(\mathbb{R}^2_+)}.$$

Then we can finish the proof of Lemma 7.4 by using

$$1 \le \mathbf{1}_{J_{\phi_1}}(y) + \psi^2(y) + \mathbf{1}_{J_{\phi_2}}(y), \quad \forall y \in \mathbb{R}_+.$$

□

End of proof of Theorem 7.1. Combining (7.11), Lemma 7.3 and Lemma 7.4, we get, for any $t \in]0,T]$,

$$\begin{aligned}\|\tilde{w}_\epsilon(t)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} \le& \tilde{C}_8 e^{C_2 t}\int_0^t e^{-C_2\tau}\|\tilde{w}_\epsilon(\tau)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} d\tau \\ &+ \tilde{C}_9 e^{C_2 t}\|\tilde{u}_0\|^2_{H^{m+1}_{2k+\ell-1}(\mathbb{R}^2_+)},\end{aligned}$$

with $\tilde{C}_8, \tilde{C}_9$ independent of $0 < \epsilon \le 1$. We have by Gronwell's inequality that, for any $t \in]0,T]$,

$$\|\tilde{w}_\epsilon(t)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} \le \tilde{C}_9 e^{(C_2+\tilde{C}_8)t}\|\tilde{u}_0\|^2_{H^{m+1}_{2k+\ell-1}(\mathbb{R}^2_+)}. \tag{7.12}$$

So it is enough to take

$$C_T^2 = \tilde{C}_9 e^{(C_2+\tilde{C}_8)T}$$

which gives (7.1), and C_T is increasing with respect to T.

On the other hand, from (7.11), (7.12) and Lemma 7.4

$$T^\epsilon_m(g,w)(t) \le \tilde{C}_T T^\epsilon_m(g,w)(0) \le \tilde{C}\tilde{C}_T\|\tilde{u}_0\|^2_{H^{m+1}_{2k+\ell-1}(\mathbb{R}^2_+)}, \quad \forall t \in]0,T].$$

this gives (7.2), and $\tilde{C}_T$ is also increasing with respect to T. We finish the proof of Theorem 7.1. □

Theorem 7.5. *Assume u^s satisfies Lemma 2.1, and let $\tilde{u}_0 \in H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)$, $m \ge 6$ be an even integer, $k > 1, 0 \le \ell < \frac{1}{2}, k+\ell > \frac{3}{2}$, and*

$$0 < \zeta \le 1 \ \textit{with}\ C_m\zeta \le \frac{c_0^2}{2},$$

where C_m is the Sobolev embedding constant. If there exists $0 < \zeta_0$ small enough such that,

$$\|\tilde{u}_0\|_{H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)} \le \zeta_0,$$

then, there exists $\epsilon_0 > 0$ and for any $0 < \epsilon \le \epsilon_0$, the system (3.2) admits a unique solution $\tilde{w}_\epsilon$ such that

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_1];H^m_{k+\ell}(\mathbb{R}^2_+))} \le \zeta,$$

where T_1 is the lifespan of shear flow u^s in the Lemma 2.1.

Remark 7.6. *Under the uniform monotonic assumption* (1.6)*, some results of above theorem holds for any fixed $T > 0$. But ζ_0 decreases as T increases, according to the* (2.7)*.*

Proof. We fix $0 < \epsilon \le 1$, then for any $\tilde{w}_0 \in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$ with $m \ge 6, k > 1, 0 \le \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$, Theorem 3.4 ensures that, there exists $\epsilon_0 > 0$ and for any $0 < \epsilon \le \epsilon_0$, there exits $T_\epsilon > 0$ such that the system (3.2) admits a unique solution $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \le \frac{4}{3}\|\tilde{w}_\epsilon(0)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \le 2\|\tilde{u}_0\|_{H^{m+1}_{k+\ell-1}(\mathbb{R}^2_+)}.$$

Now choose ζ_0 such that

$$\max\{2, C_{T_1}\}\zeta_0 \le \frac{\zeta}{2}.$$

On the other hand, taking $\tilde{w}_\epsilon(T_\epsilon)$ as initial data for the system (3.2), Theorem 3.4 ensures that there exits $T'_\epsilon > 0$, which is defined by (3.16) with $\bar{\zeta} = \frac{\zeta}{2}$, such that the system (3.2) admits a unique solution $\tilde{w}'_\epsilon \in L^\infty([T_\epsilon, T_\epsilon + T'_\epsilon]; H^m_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}'_\epsilon\|_{L^\infty([T_\epsilon,T_\epsilon+T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \le \frac{4}{3}\|\tilde{w}_\epsilon(T_\epsilon)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \le \zeta.$$

Now, we extend $\tilde{w}_\epsilon$ to $[0, T_\epsilon + T'_\epsilon]$ by $\tilde{w}'_\epsilon$, then we get a solution $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon + T'_\epsilon]; H^m_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon+T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \le \zeta.$$

So if $T_\epsilon + T'_\epsilon < T_1$, we can apply Theorem 7.1 to $\tilde{w}_\epsilon$ with $T = T_\epsilon + T'_\epsilon$, and use (7.1), this gives

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon+T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \le C_{T_1}\|\tilde{u}_0\|_{H^{m+1}_{k+\ell-1}(\mathbb{R}^2_+)} \le \frac{\zeta}{2}.$$

Now taking $\tilde{w}_\epsilon(T_\epsilon + T'_\epsilon)$ as initial data for the system (3.2), applying again Theorem 3.4, for the same $T'_\epsilon > 0$, the system (3.2) admits a unique solution $\tilde{w}'_\epsilon \in L^\infty([T_\epsilon + T'_\epsilon, T_\epsilon + 2T'_\epsilon]; H^m_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}'_\epsilon\|_{L^\infty([T_\epsilon+T'_\epsilon,T_\epsilon+2T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \le \frac{4}{3}\|\tilde{w}_\epsilon(T_\epsilon + T'_\epsilon)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \le \zeta.$$

Now, we extend $\tilde{w}_\epsilon$ to $[0, T_\epsilon + 2T'_\epsilon]$ by $\tilde{w}'_\epsilon$, then we get a solution $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon + 2T'_\epsilon]; H^m_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon+2T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \le \zeta.$$

So if $T_\epsilon + 2T'_\epsilon < T_1$, we can apply Theorem 7.1 to $\tilde{w}_\epsilon$ with $T = T_\epsilon + 2T'_\epsilon$, and use (7.1), this gives again

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon+2T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \le C_{T_1}\|\tilde{u}_0\|_{H^{m+1}_{k+\ell-1}(\mathbb{R}^2_+)} \le \frac{\zeta}{2}.$$

Then by recurrence, we can extend the solution $\tilde{w}_\epsilon$ to $[0, T_1]$, and then the lifespan of approximate solution is equal to that of shear flow if the initial date $\tilde{u}_0$ is small enough. □

We have obtained the following estimate, for $m \ge 6$ and $0 < \epsilon \le \epsilon_0$,

$$\|\tilde{w}_\epsilon(t)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \le \zeta, \quad t \in [0, T_1].$$

By using the equation (3.2) and the Sobolev inequality, we get, for $0<\delta<1$
$$\|\tilde w_\epsilon\|_{Lip([0,T_1];C^{2,\delta}(\mathbb{R}^2_+))}\le M<+\infty.$$
Then taking a subsequence, we have, for $0<\delta'<\delta$,
$$\tilde w_\epsilon\to\tilde w\ (\epsilon\to 0),\ \text{locally strong in}\ \ C^0([0,T_1];C^{2,\delta'}(\mathbb{R}^2_+))\,,$$
and
$$\partial_t\tilde w\in L^\infty([0,T_1];H^{m-2}_{k+\ell}(\mathbb{R}^2_+)),\quad \tilde w\in L^\infty([0,T_1];H^m_{k+\ell}(\mathbb{R}^2_+)),$$
with
$$\|\tilde w\|_{L^\infty([0,T_1];H^m_{k+\ell}(\mathbb{R}^2_+))}\le\zeta.$$
Then we have
$$\tilde u=\partial_y^{-1}w\in L^\infty([0,T_1];H^m_{k+\ell-1}(\mathbb{R}^2_+)),$$
where we use the Hardy inequality (A.1), since
$$\lim_{y\to+\infty}\tilde u(t,x,y)=-\lim_{y\to+\infty}\int_y^{+\infty}\tilde w(t,x,\tilde y)d\tilde y=0.$$
In fact, we also have
$$\lim_{y\to 0}\tilde u(t,x,y)=\lim_{y\to 0}\int_0^y\tilde w(t,x,\tilde y)d\tilde y=0.$$
Using the condition $k+\ell-1>\frac12$, we have also
$$\tilde v=-\int_0^y\tilde u_x\,d\tilde y\in L^\infty([0,T_1];L^\infty(\mathbb{R}_{+,y});H^{m-1}(\mathbb{R}_x)).$$
We have proven that, $\tilde w$ is a classical solution to the following vorticity Prandtl equation
$$\begin{cases}\partial_t\tilde w+(u^s+\tilde u)\partial_x\tilde w+\tilde v\partial_y(u^s_y+\tilde w)=\partial_y^2\tilde w,\\ \partial_y\tilde w|_{y=0}=0,\\ \tilde w|_{t=0}=\tilde w_0,\end{cases}$$
and $(\tilde u,\tilde v)$ is a classical solution to (2.2). Finally, $(u,v)=(u^s+\tilde u,\tilde v)$ is a classical solution to (1.1), and satisfies (7.13). In conclusion, we have proved the following theorem

Theorem 7.7. *Let $m\ge 6$ be an even integer, $k>1, 0\le\ell<\frac12, k+\ell>\frac32$, assume that u^s_0 satisfies (1.2), the initial date $\tilde u_0\in H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)$ and $\tilde u_0$ satisfies the compatibility condition (2.8)-(2.9) up to order $m+2$. Then there exists $T>0$ such that if*
$$\|\tilde u_0\|_{H^{m+3}_{2k+\ell-1}(\mathbb{R}^2_+)}\le\delta_0,$$
for some $\delta_0>0$ small enough, then the initial-boundary value problem (2.2) admits a solution $(\tilde u,\tilde v)$ with
$$\tilde u\in L^\infty([0,T];H^m_{k+\ell-1}(\mathbb{R}^2_+)),\quad \partial_y\tilde u\in L^\infty([0,T];H^m_{k+\ell}(\mathbb{R}^2_+)).$$
Moreover, we have the following energy estimate,
$$\|\partial_y\tilde u\|_{L^\infty([0,T];H^m_{k+\ell}(\mathbb{R}^2_+))}\le C\|\tilde u_0\|^2_{H^{m+1}_{2k+\ell-1}(\mathbb{R}^2_+)}.\tag{7.13}$$

Remark 7.8. *Under the uniform monotonic assumption (1.6), the result of the above theorem hold for any fixed $T>0$, but $\delta_0\to 0$ when $T\to+\infty$.*

8. Uniqueness and stability

Now, we study the stability of solutions which implies immediately the uniqueness of solution.

Let $\tilde{u}^1, \tilde{u}^2$ be two solutions obtained in Theorem 7.7 with respect to the initial date $\tilde{u}^1_0, \tilde{u}^2_0$ respectively. Denote $\bar{u} = \tilde{u}^1 - \tilde{u}^2$ and $\bar{v} = \tilde{v}^1 - \tilde{v}^2$, then

$$\begin{cases} \partial_t \bar{u} + (u^s + \tilde{u}_1)\partial_x \bar{u} + (u^s_y + \tilde{u}_{1,y})\bar{v} = \partial^2_y \bar{u} - \tilde{v}_2 \partial_y \bar{u} - (\partial_x \tilde{u}_2)\bar{u}, \\ \partial_x \bar{u} + \partial_y \bar{v} = 0, \\ \bar{u}|_{y=0} = \bar{v}|_{y=0} = 0, \\ \bar{u}|_{t=0} = \tilde{u}^1_0 - \tilde{u}^2_0. \end{cases}$$

So it is a linear equation for $\bar{u}$. We also have for the vorticity $\bar{w} = \partial_y \bar{u}$,

$$\begin{cases} \partial_t \bar{w} + (u^s + \tilde{u}_1)\partial_x \bar{w} + (u^s_{yy} + \tilde{w}_{1,y})\bar{v} = \partial^2_y \bar{w} - \tilde{v}_2 \partial_y \bar{w} - (\partial_x \tilde{w}_2)\bar{u}, \\ \partial_y \bar{w}|_{y=0} = 0, \\ \bar{w}|_{t=0} = \tilde{w}^1_0 - \tilde{w}^2_0. \end{cases} \tag{8.1}$$

8.1. Estimate with a loss of x-derivative.

Firstly, for the vorticity $\bar{w} = \partial_y \bar{u}$, we deduce an energy estimate with a loss of x-derivative with the anisotropic norm defined by (3.9).

Proposition 8.1. *Let $\tilde{u}^1, \tilde{u}^2$ be two solutions obtained in Theorem 7.7 with respect to the initial date $\tilde{u}^1_0, \tilde{u}^2_0$, then we have*

$$\frac{d}{dt}\|\bar{w}\|^2_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)} + \|\partial_y \bar{w}\|^2_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)} \le \bar{C}_1 \|\bar{w}\|^2_{H^{m-2}_{k+\ell}}, \tag{8.2}$$

where the constant $\bar{C}_1$ depends on the norm of $\tilde{w}^1, \tilde{w}^2$ in $L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))$.

Proof. The proof of this Proposition is similar to the proof of the Proposition 3.8, and we need to use that $m-2$ is even. We only give the calculation for the terms which need a different argument. Moreover we also explain why we only get the estimate on $\|\bar{w}\|^2_{H^{m-2}_{k+\ell}}$ but require the norm of $\tilde{w}^1, \tilde{w}^2$ in $L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))$. With out loss of the generality, we suppose that $\|\bar{w}\|_{H^{m-2}_{k+\ell}} \le 1, \|\tilde{w}^1\|_{H^m_{k+\ell}} \le 1$ and $\|\tilde{w}^2\|_{H^m_{k+\ell}} \le 1$.

Derivating the equation of (8.1) with $\partial^\alpha = \partial^{\alpha}_x \partial^{\alpha_2}_y$, for $|\alpha| = \alpha_1 + \alpha_2 \le m-2, \alpha_1 \le m-3$,

$$\begin{aligned} \partial_t \partial^\alpha \bar{w} - \partial^2_y \partial^\alpha \partial \bar{w} = -\partial^\alpha \big(& (u^s + \tilde{u}_1)\partial_x \bar{w} + \tilde{v}_2 \partial_y \bar{w} \\ & + (u^s_{yy} + \tilde{w}_{1,y})\bar{v} + (\partial_x \tilde{w}_2)\bar{u} \big). \end{aligned} \tag{8.3}$$

Multiplying the above equation with $\langle y \rangle^{2k+\ell+\alpha_2} \partial^\alpha \bar{w}$, the same computation as in the proof of the Proposition 3.8, in particular, the reduction of the boundary-data are the same, gives

$$\begin{aligned} & \int_{\mathbb{R}^2_+} \Big(\partial_t \partial^\alpha \bar{w} - \partial^2_y \partial^\alpha \bar{w} \Big) \langle y \rangle^{2(k+\ell+\alpha_2)} \partial^\alpha \bar{w} dx dy \\ & \ge \frac{1}{2}\frac{d}{dt}\|\partial^\alpha \bar{w}\|^2_{L^2_{k+\ell+\alpha_2}} + \frac{3}{4}\|\partial_y \bar{w}\|^2_{H^{m-2,m-3}_{k+\ell}} - C\|\bar{w}\|^2_{H^{m-2}_{k+\ell}}. \end{aligned}$$

As for the right hand of (8.3), for the first item, we split it into two parts

$$-\partial^\alpha\Big((u^s+\tilde u_1)\partial_x\bar w\Big) = -(u^s+\tilde u_1)\partial_x\partial^\alpha\bar w + [(u^s+\tilde u_1),\partial^\alpha]\partial_x\bar w.$$

Firstly, we have

$$\left|\int_{\mathbb{R}^2_+}\big((u^s+\tilde u_1)\partial_x\partial^\alpha\bar w\big)\langle y\rangle^{2(\ell+\alpha_2)}\partial^\alpha\bar w dxdy\right| \le \|\tilde w_1\|_{H^3_1}\|\partial^\alpha\bar w\|^2_{L^2_{k+\ell+\alpha_2}}.$$

For the commutator operator, we have,

$$\|[(u^s+\tilde u_1),\partial^\alpha]\partial_x\tilde w_\epsilon\|_{L^2_{k+\ell+\alpha_2}} \le C\|\tilde w_1\|_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)}\|\bar w\|_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)}.$$

Notice that for this term, we don't have the loss of x-derivative.

With the similar method for the terms $\tilde v_2\partial_y\bar w$, we get

$$\left|\int_{\mathbb{R}^2_+}\tilde v_2\partial_y\bar w\langle y\rangle^{2(\ell+\alpha_2)}\partial^\alpha\bar w dxdy\right| \le \|\tilde w_2\|_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)}\|\bar w\|^2_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)}.$$

For the next one, we have

$$\partial^\alpha\Big((u^s_{yy}+\partial_y\tilde w_1)\bar v\Big) = \sum_{\beta\le\alpha} C^\alpha_\beta\,\partial^\beta(u^s_{yy}+\partial_y\tilde w_1)\partial^{\alpha-\beta}\bar v,$$

and thus

$$\begin{aligned}\left\|\sum_{\beta\le\alpha,1\le|\beta|<|\alpha|} C^\alpha_\beta\,\partial^\beta(u^s_{yy}+\partial_y\tilde w_1)\partial^{\alpha-\beta}\bar v\right\|_{L^2_{k+\ell+\alpha_2}}&\\ \le C\|\tilde w_1\|_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)}\|\bar w\|_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)}.&\end{aligned}$$

On the other hand, using Lemma A.1 and $\frac32-k<\ell<\frac12$,

$$\begin{aligned}\big\|\big(\partial^\alpha(u^s_{yy}+\partial_y\tilde w_1)\big)\bar v\big\|_{L^2_{k+\ell+\alpha_2}} &\le \big\|\big(\partial^\alpha u^s_{yy}\big)\bar v\big\|_{L^2_{k+\ell+\alpha_2}} + \big\|\big(\partial^\alpha\partial_y\tilde w_1\big)\bar v\big\|_{L^2_{k+\ell+\alpha_2}}\\ &\le C\,\|\bar v\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))} + C\|\tilde w_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\,\|\bar v\|_{L^\infty(\mathbb{R}^2_+)}\\ &\le C\,\|\bar u_x\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)} + C\|\tilde w_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\big(\|\bar u_x\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)} + \|\bar u_{xx}\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}\big)\\ &\le C(1+\|\tilde w_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)})\,\|\bar w\|_{H^2_{\frac12+\delta}(\mathbb{R}^2_+)}\\ &\le C(1+\|\tilde w_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)})\,\|\bar w\|_{H^2_{k+\ell}(\mathbb{R}^2_+)}\,.\end{aligned}$$

So this term requires the norms $\|\tilde w_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)})$.

Moreover, if $\alpha_2\neq 0$

$$\begin{aligned}\big\|(u^s_{yy}+\partial_y\tilde w_1)\partial^\alpha\bar v\big\|_{L^2_{k+\ell+\alpha_2}} &= \big\|(u^s_{yy}+\partial_y\tilde w_1)\partial^{\alpha_1}_x\partial^{\alpha_2-1}\bar u_x\big\|_{L^2_{k+\ell+\alpha_2}}\\ &\le C(1+\|\tilde w_1\|_{H^{m-1}_{k+\ell}(\mathbb{R}^2_+)})\,\|\bar w\|_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)}\,,\end{aligned}$$

and also if $\alpha_2=0$

$$\begin{aligned}\big\|(u^s_{yy}+\partial_y\tilde w_1)\partial^{\alpha_1}_x\bar v\big\|_{L^2_{k+\ell}} &= \big\|(u^s_{yy}+\partial_y\tilde w_1)\partial^{-1}_y\partial^{\alpha_1}_x\bar u_x\big\|_{L^2_{k+\ell}}\\ &\le C(1+\|\tilde w_1\|_{H^{m-1}_{k+\ell}(\mathbb{R}^2_+)})\,\big\|\partial^{\alpha_1+1}_x\bar w\big\|_{L^2_{\frac32+\delta}(\mathbb{R}^2_+)}\,.\end{aligned}$$

These two cases imply the loss of x-derivative.

Similar argument also gives

$$\left|\int_{\mathbb{R}^2_+}\big(\partial^\alpha(\partial_x\tilde{w}_2)\bar{u}\big)\langle y\rangle^{2(\ell+\alpha_2)}\partial^\alpha\bar{w}dxdy\right|\le C\|\tilde{w}_2\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\|\bar{w}\|^2_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)},$$

which finishes the proof of the Proposition 8.1. □

8.2. **Estimate on the loss term.** To close the estimate (7.13), we need to study the terms $\|\partial_x^{m-2}\bar{w}\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}$ which is missing in the left hand side of (8.2).

Similar to the argument in Section 7, we will recover this term by the estimate of functions

$$\bar{g}_n=\left(\frac{\partial_x^n\bar{u}}{u^s_y+\tilde{u}_{1,y}}\right)_y,\quad \forall(t,x,y)\in[0,T]\times\mathbb{R}\times(I_{\phi_1}\cup I_{\phi_2}),$$

$$\bar{h}_n=\frac{\partial_x^n\bar{w}}{\sqrt{u^s_{yy}+\tilde{w}_{1,y}}},\quad \forall(t,x,y)\in[0,T]\times\mathbb{R}\times(I_\psi).$$

Proposition 8.2. *Let $\tilde{u}^1,\tilde{u}^2$ be two solutions obtained in Theorem 7.7 with respect to the initial date $\tilde{u}^1_0,\tilde{u}^2_0$, then we have*

$$\frac{d}{dt}\|\phi_1\bar{g}_{m-2}\|^2_{L^2(\mathbb{R}^2_+)}+\|\phi_1\partial_y\bar{g}_{m-2}\|^2_{L^2(\mathbb{R}^2_+)}\le\bar{C}_2(\|\phi_1\bar{g}_{m-2}\|^2_{L^2(\mathbb{R}^2_+)}+\|\bar{w}\|^2_{H^{m-2}_{k+\ell}}),$$

and

$$\frac{d}{dt}\sum_{n=1}^{m-2}\|\phi_2\bar{g}_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}+\sum_{n=1}^{m-2}\|\phi_2\partial_y\bar{g}_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\le C_2\Big(\sum_{n=1}^{m-2}\|\phi_2\bar{g}_n\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}+\|\bar{w}\|^2_{H^{m-2}_{k+\ell}}\Big),$$

where the constant $\bar{C}_2$ depends on the norm of $\tilde{w}^1,\tilde{w}^2$ in $L^\infty([0,T];H^m_{k+\ell}(\mathbb{R}^2_+))$.

Proposition 8.3. *With same assumption as in Proposition 8.2, we have*

$$\frac{d}{2dt}\|\psi\bar{h}_{m-2}\|^2_{L^2}+\frac{3}{4}\|\psi\partial_y\bar{h}_{m-2}\|^2_{L^2}\le\bar{C}_3\|\bar{w}\|^2_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)}-\frac{d}{dt}\int_{\mathbb{R}^2_+}(\partial_x^{m-2}\bar{u})^2\,\frac{\psi\psi'}{u^s_y+u_y}dxdy,$$

where $\bar{C}_3$ depends on the norm of $\tilde{w}^1,\tilde{w}^2$ in $L^\infty([0,T];H^m_{k+\ell}(\mathbb{R}^2_+))$.

These two Propositions can be proven by using exactly the same calculation and also the same nonlinear cancellation as in Section 5 and Section 6. The only difference is that when we use the Leibniz formula, for the term where the order of derivatives is $|\alpha|=m-2$, it acts on the coefficient which depends on $\tilde{u}^1,\tilde{u}^2$. Therefore, we need their norm in the order of $(m-2)+1$. So we omit the proof of the two Proposition here.

With the similar argument to the proof of Theorem 7.1, we get

$$\|\bar{w}\|_{L^\infty([0,T];H^{m-2}_{k+\ell}(\mathbb{R}^2_+))}\le C\|\bar{u}_0\|_{H^m_{2k+\ell-1}(\mathbb{R}^2_+)},$$

which finishes the proof of Theorem 1.1.

Appendix A. Some inequalities

We will use the following Hardy type inequalities.

Lemma A.1. *Let $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$. Then*

(i) *if $\lambda > -\frac{1}{2}$ and $\lim\limits_{y\to\infty} f(x,y) = 0$, then*

$$\|\langle y\rangle^{\lambda} f\|_{L^2(\mathbb{R}^2_+)} \le C_\lambda \|\langle y\rangle^{\lambda+1}\partial_y f\|_{L^2(\mathbb{R}^2_+)}; \tag{A.1}$$

(ii) *if $-1 \le \lambda < -\frac{1}{2}$ and $f(x,0) = 0$, then*

$$\|\langle y\rangle^{\lambda} f\|_{L^2(\mathbb{R}^2_+)} \le C_\lambda \|\langle y\rangle^{\lambda+1}\partial_y f\|_{L^2(\mathbb{R}^2_+)}.$$

Here $C_\lambda \to +\infty$, as $\lambda \to -\frac{1}{2}$.

We need the following trace theorem in the weighted Sobolev space.

Lemma A.2. *Let $\lambda > \frac{1}{2}$, then there exists $C > 0$ such that for any function f defined on $\mathbb{R}^2_+$, if $\partial_y f \in L^2_\lambda(\mathbb{R}^2_+)$, it admits a trace on $\mathbb{R}_x \times \{0\}$, and satisfies*

$$\|\gamma_0(f)\|_{L^2(\mathbb{R}_x)} \le C\|\partial_y f\|_{L^2_\lambda(\mathbb{R}^2_+)},$$

where $\gamma_0(f)(x) = f(x,0)$ is the trace operator.

The proof of the above two Lemmas is elementary, so we leave it to the reader.
We use also the following Sobolev inequality and algebraic properties of $H^m_{k+\ell}(\mathbb{R}^2_+)$,

Lemma A.3. *For the suitable functions f, g, we have:*
1) If the function f satisfies $f(x,0) = 0$ or $\lim_{y\to+\infty} f(x,y) = 0$, then for any small $\delta > 0$,

$$\|f\|_{L^\infty(\mathbb{R}^2_+)} \le C(\|f_y\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)} + \|f_{xy}\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}). \tag{A.2}$$

2) For $m \ge 6, k+\ell > \frac{3}{2}$, and any $\alpha, \beta \in \mathbb{N}^2$ with $|\alpha| + |\beta| \le m$, we have

$$\|(\partial^\alpha f)(\partial^\beta g)\|_{L^2_{k+\ell+\alpha_2+\beta_2}(\mathbb{R}^2_+)} \le C\|f\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\|g\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}. \tag{A.3}$$

3) For $m \ge 6, k+\ell > \frac{3}{2}$, and any $\alpha \in \mathbb{N}^2, p \in \mathbb{N}$ with $|\alpha| + p \le m$, we have,

$$\|(\partial^\alpha f)(\partial^p_x(\partial_y^{-1} g))\|_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)} \le C\|f\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\|g\|_{H^m_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}, \tag{A.4}$$

where ∂_y^{-1} is the inverse of derivative ∂_y, meaning, $\partial_y^{-1} g = \int_0^y g(x,\tilde y)\, d\tilde y$.

Proof. For (1), using $f(x,0) = 0$, we have

$$\begin{aligned}\|f\|_{L^\infty(\mathbb{R}^2_+)} &= \left\|\int_0^y (\partial_y f)(x,\tilde y)\, d\tilde y\right\|_{L^\infty(\mathbb{R}^2_+)} \le C\|\partial_y f\|_{L^\infty(\mathbb{R}_x; L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+))} \\ &\le C(\|\partial_y f\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)} + \|\partial_x\partial_y f\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}).\end{aligned}$$

If $\lim_{y\to+\infty} f(x,y) = 0$, we use

$$f(x,y) = -\int_y^\infty (\partial_y f)(x,\tilde y)\, d\tilde y.$$

For (2), firstly, $m \geq 6$ and $|\alpha| + |\beta| \leq m$ imply $|\alpha| \leq m-2$ or $|\beta| \leq m-2$, without loss of generality, we suppose that $|\alpha| \leq m-2$. Then, using the conclusion of (1), we have

$$\begin{aligned}\|(\partial^\alpha f)(\partial^\beta g)\|_{L^2_{k+\ell+\alpha_2+\beta_2}(\mathbb{R}^2_+)} &\leq \|\langle y\rangle^{\alpha_2}(\partial^\alpha f)\|_{L^\infty(\mathbb{R}^2_+)}\|\partial^\beta g\|_{L^2_{k+\ell+\beta_2}(\mathbb{R}^2_+)}\\ &\leq C\|f\|_{H^{|\alpha|+2}_{\frac12+\delta}(\mathbb{R}^2_+)}\|\partial^\beta g\|_{L^2_{k+\ell+\beta_2}(\mathbb{R}^2_+)},\end{aligned}$$

which give (A.3).

For (3), if $|\alpha| \leq m-2$, we have

$$\begin{aligned}&\|(\partial^\alpha f)(\partial^p_x(\partial_y^{-1} g))\|_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)}\\ &\quad\leq \|\langle y\rangle^{k+\ell+\alpha_2}(\partial^\alpha f)\|_{L^2(\mathbb{R}_{y,+};L^\infty(\mathbb{R}_x))}\|\partial^p_x(\partial_y^{-1} g)\|_{L^\infty(\mathbb{R}_{y,+};L^2(\mathbb{R}_x))}\\ &\quad\leq C\|f\|_{H^{|\alpha|+2}_{k+\ell}(\mathbb{R}^2_+)}\|\partial^p_x g\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}.\end{aligned}$$

If $p \leq m-2$, we have

$$\begin{aligned}&\|(\partial^\alpha f)(\partial^p_x(\partial_y^{-1} g))\|_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)}\\ &\quad\leq \|\langle y\rangle^{k+\ell+\alpha_2}(\partial^\alpha f)\|_{L^2(\mathbb{R}^2_+)}\|\partial^p_x(\partial_y^{-1} g)\|_{L^\infty(\mathbb{R}^2_+)}\\ &\quad\leq C\|f\|_{H^{|\alpha|}_{k+\ell}(\mathbb{R}^2_+)}\|\partial^p_x g\|_{L^\infty(\mathbb{R}_x;L^2_{\frac12+\delta}(\mathbb{R}_{y,+}))}\\ &\quad\leq C\|f\|_{H^{|\alpha|}_{k+\ell}(\mathbb{R}^2_+)}\|g\|_{H^m_{\frac12+\delta}(\mathbb{R}^2_+)}.\end{aligned}$$

We have completed the proof of the Lemma. □

Appendix B. Some example of initial data

We construct now some example of initial data $u_0 = u_0^s + \tilde{u}_0$ with non degenerate critical points. Let u_0^s be the initial date of shear profile satisfying (2.3), we suppose that u_0^s is convex, then we have

$$\partial_y u_0^s(y) < 0,\ 0 \leq y < a; \quad \partial_y u_0^s(y) > 0,\ a < y.$$

For $c_0 << 1$, set

$$\tilde{u}_0 = \begin{cases} 0, & y \leq a - 6a_0\\ c_0^j f_1(y) b(x) & a-6a_0 \leq y \leq a-5a_0\\ c_0^j (y+6a_0-a)^2 b(x), & a-5a_0 \leq y \leq a+5a_0\\ c_0^j f_2(y) b(x) & y \geq a_0 + 5a_0.\end{cases}$$

where j is a big integer, f_1, f_2 are two smooth joint functions with compact supports, and $b \in H^{m+1}(\mathbb{R}_x), 0 < b(x) \leq 1$ for all $x \in \mathbb{R}$.

One can check that if j is big enough, we have that

$$\|\partial_y \tilde{u}_0\|_{H^m_{2k+\ell}(\mathbb{R}^2_+)} << 1,$$

which imply

$$\partial_y^2 u_0(x,y) = \partial_y^2 u_0^s(y) + \partial_y^2 \tilde{u}_0(x,y) \geq c_0^2, \quad (x,y) \in \mathbb{R}\times[a-6a_0, a+6a_0]. \tag{B.1}$$

If $j >> 1$ we have also

$$0 < \partial_y \tilde{u}_0 \leq 2c_0^j(y+6a_0-a)b(x) \leq 24a_0 c_0^j b(x) \leq c_0^6,\ a-5a_0 \leq y \leq a+5a_0.$$

For the shear profile, according to (2.3),

$$\partial_y u_0^s(y) \leq -2c_0,\ \ 0 \leq y \leq a - a_0.$$

Then we have

$$\partial_y u_0^s(a-a_0)+\partial_y\tilde u_0(x,a-a_0)\le -2c_0+c_0^6\ <0,$$
$$\partial_y u_0^s(a)+\partial_y\tilde u_0(x,a)=\partial_y\tilde u_0(x,a)>0.$$

By the intermediate value theorem, there exist a point $a(x)$ such that

$$\partial_y u_0(x,a(x))=\partial_y u_0^s(a(x))+\partial_y\tilde u_0(x,a(x))=0,\ \ a(x)\in]a-a_0,a[.$$

In detail, it is equal to

$$2c_0^j\bigg(a(x)+6a_0-a\bigg)b(x)=-\partial_y u_0^s(a(x)),\ \ a(x)\in]a-a_0,a[.$$

So the smoothness of the curve $a(x)$ can be deduced by implicit function theorem and (B.1).

Appendix C. The proof of Proposition 3.6

Now, we prove the existence of solution to the vorticity equation $\tilde w_\epsilon=\partial_y\tilde u_\epsilon$ and suppose that m,k,ℓ and $u^s(t,y)$ satisfy the assumption of Proposition 3.6,

$$\begin{cases}\partial_t\tilde w_\epsilon+(u^s+\tilde u_\epsilon)\partial_x\tilde w_\epsilon+v_\epsilon(u^s_{yy}+\partial_y\tilde w_\epsilon)=\partial_y^2\tilde w_\epsilon+\epsilon\partial_x^2\tilde w_\epsilon,\\ \partial_y\tilde w_\epsilon|_{y=0}=0\\ \tilde w_\epsilon|_{t=0}=\tilde w_{0,\epsilon},\end{cases}\tag{C.1}$$

where $\tilde u_\epsilon=\partial_y^{-1}\tilde w_\epsilon$ and $\tilde v_\epsilon=-\partial_y^{-1}\tilde u_{\epsilon,x}$. We will use the following iteration process to prove the existence of solution, where $w^0=\tilde w_{0,\epsilon}$,

$$\begin{cases}\partial_t w^n+(u^s+u^{n-1})\partial_x w^n+(u^s_{yy}+\partial_y w^{n-1})v^n=\partial_y^2 w^n+\epsilon\partial_x^2 w^n,\\ \partial_y w^n|_{y=0}=0\\ w^n|_{t=0}=\tilde w_{0,\epsilon}.\end{cases}\tag{C.2}$$

Here for the boundary data, we have

$$\partial_y^3 w^n|_{y=0}=((u^s_y+w^{n-1})\partial_x w^n)|_{y=0},$$

$$\begin{aligned}&(\partial_y^5 w^n)(t,x,0)\\ =&\big(\partial_y^3 u^s(t,0)+\partial_y^2 w^{n-1}(t,x,0)+\epsilon(\partial_x^2 w^{n-1})(t,x,0)\big)(\partial_x w^n)(t,x,0)\\ &+\big(u^s_y(t,0)+(w^{n-1})(t,x,0)\big)\big((\partial_y^2\partial_x w^n)(t,x,0)+\epsilon(\partial_x^3 w^n)(t,x,0)\big)\\ &\qquad-(\partial_y\partial_x w^n)(u^s_y+w^{n-1})(t,x,0)\\ &+\sum_{1\le j\le 3}C_j^4\bigg((\partial_y^j(u^s+u^{n-1}))\partial_y^{4-j}\partial_x u^n-(\partial_y^{j-1}\partial_x\tilde u^n)\partial_y^{4-j}(u^s_y+w^{n-1})\bigg)(t,x,0)\\ &\qquad-\epsilon\partial_x^2\bigg(\big(u^s_y(t,0)+(w^{n-1})(t,x,0)\big)(\partial_x w^n)(t,x,0)\bigg).\end{aligned}$$

and also for $3\le p\le\frac{m}{2}+1$, $\partial_y^{2p+1}w^n|_{y=0}$ is a linear combination of the terms of the form:

$$\prod_{j=1}^{q_1}\bigg(\partial_x^{\alpha_j}\partial_y^{\beta_j+1}\big(u^s+u^n\big)\bigg)\bigg|_{y=0}\times\prod_{l=1}^{q_2}\bigg(\partial_x^{\tilde\alpha_l}\partial_y^{\tilde\beta_l+1}\big(u^s+u^{n-i}\big)\bigg)\bigg|_{y=0},\tag{C.3}$$

where $2\le q_1+q_2\le p,\ 1\le i\le\min\{n,\,p\}$ and

$$\alpha_j+\beta_j\le 2p-1,\ 1\le j\le q_1;\ \tilde\alpha_l+\tilde\beta_l\le 2p-1,\ 1\le l\le q_2;$$

$$\sum_{j=1}^{q_1}(3\alpha_j+\beta_j)+\sum_{l=1}^{q_2}(3\tilde{\alpha}_l+\tilde{\beta}_l)=2p+1\,;$$
$$\sum_{j=1}^{q_1}\beta_j+\sum_{l=1}^{q_2}\tilde{\beta}_l\le 2p-2;\ \ \sum_{j=1}^{q_1}\alpha_j+\sum_{l=1}^{q_2}\tilde{\alpha}_l\le p-1,\ \ 0<\sum_{j=1}^{q_1}\alpha_j.$$

Remark that the condition $0<\sum_{j=1}^{q_1}\alpha_j$ implies that, in (C.3), there are at last one factor like $\partial_x^{\alpha_j}\partial_y^{\beta_j+1}u^n(t,x,0)$.

For given w^{n-1}, we have $u^{n-1}=\partial_y^{-1}w^{n-1}$ and $v^n=-\partial_y^{-1}u^n_x$. We will prove the existence and boundness of the sequence $\{w^n, n\in\mathbb{N}\}$ in $L^\infty([0,T_\epsilon];H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$ to the linear equation (C.2) firstly, then the existence of solution to (C.1) is guaranteed by using the standard weak convergence methods.

Lemma C.1. *Assume that* $w^{n-i}\in L^\infty([0,T];H^{m+2}_{k+\ell}(\mathbb{R}^2_+)), 1\le i\le \min\{n,\frac{m}{2}+1\}$ *and* $\tilde{w}_{0,\epsilon}$ *satisfies the compatibility condition up to order* $m+2$ *for the system* (C.1), *then the initial-boundary value problem* (C.2) *admit a unique solution* w^n *such that, for any* $t\in[0,T]$,

$$\frac{d}{dt}\|w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\le B^{n-1}_T\|w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}+D^{n-1}_T\|w^n\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)},\tag{C.4}$$

where

$$B^{n-1}_T=C\Big(1+\sum_{i=1}^{\min\{n,m/2+1\}}\|w^{n-i}\|_{L^\infty([0,T];H^{m+2}_{k+\ell}(\mathbb{R}^2_+))}+(1+\frac{1}{\epsilon})\sum_{i=1}^{\min\{n,m/2+1\}}\|w^{n-i}\|^2_{L^\infty([0,T];H^{m+2}_{k+\ell}(\mathbb{R}^2_+))}\Big),$$

and

$$D^{n-1}_T=C\sum_{i=1}^{\min\{n,m/2+1\}}\|w^{n-i}\|^{m+2}_{L^\infty([0,T];H^{m+2}_{k+\ell}(\mathbb{R}^2_+))}\,.$$

Proof. Once we get *à priori* estimate for this linear problem, the existence of solution is guaranteed by the Hahn-Banach theorem. So we only prove the *à priori* estimate of the smooth solutions.

For any $\alpha\in\mathbb{N}^2,|\alpha|\le m+2$, taking the equation (C.2) with derivative ∂^α, multiplying the resulting equation by $\langle y\rangle^{2k+2\ell+2\alpha_2}\partial^\alpha w^n$ and integrating by part over $\mathbb{R}^2_+$, one obtains that

$$\begin{aligned}&\frac12\frac{d}{dt}\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}+\|\partial_y w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}+\epsilon\|\partial_x w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\\&=\sum_{|\alpha|\le m+2}\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}\partial^\alpha\big((u^s+u^{n-1})\partial_x w^n\\&\qquad-(\partial_y^{-1}u^n_x)(u^s_{yy}+\partial_y w^{n-1})\big)\partial^\alpha w^n dxdy\\&\quad+\sum_{|\alpha|\le m+2}\int_{\mathbb{R}^2_+}(\langle y\rangle^{2k+2\ell+2\alpha_2})'\partial^\alpha\partial_y w^n\partial^\alpha\partial_y w^n dxdy\\&\qquad+\sum_{|\alpha|\le m+2}\int_{\mathbb{R}}(\partial^\alpha\partial_y w^n\partial^\alpha\partial_y w^n)\big|_{y=0}dx,\end{aligned}\tag{C.5}$$

With similar analysis to Section 5, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}(u^s+u^{n-1})\partial_x\partial^\alpha w^n\partial^\alpha w^n dxdy\right|\\&=\left|-\frac{1}{2}\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}\partial_x(u^s+u^{n-1})\partial^\alpha w^n\partial^\alpha w^n dxdy\right|\\&\le C\|u^{n-1}\|_{L^\infty(\mathbb{R}^2_+)}\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)},\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}[\partial^\alpha,(u^s+u^{n-1})]\partial_x w^n\partial^\alpha w^n dxdy\right|\\&\le C(1+\|w^{n-1}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)})\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

For the second term on the right hand of (C.5), by using the Leibniz formula, we need to pay more attention to the following two terms

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}\big(\partial^\alpha\partial_y^{-1}u^n_x\big)(u^s_{yy}+\partial_y w^{n-1})\partial^\alpha w^n dxdy\right|\\&\le C(1+\|w^{n-1}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)})\|\partial_x w^n\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\|w^n\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\\&\le\frac{\epsilon}{2}\|\partial_x w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}+\frac{C}{\epsilon}(1+\|w^{n-1}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)})^2\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)},\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}v^n\big)\big(\partial^\alpha\partial_y w^{n-1}\big)\partial^\alpha w^n dxdy\\&=-\int_{\mathbb{R}^2_+}\partial_y\big(\langle y\rangle^{2k+2\ell+2\alpha_2}(\partial_y^{-1}u^n_x)\big)\big(\partial^\alpha w^{n-1}\big)\partial^\alpha w^n dxdy\\&\quad-\int_{\mathbb{R}^2_+}\big(\langle y\rangle^{2k+2\ell+2\alpha_2}(\partial_y^{-1}u^n_x)\big)\big(\partial^\alpha w^{n-1}\big)\partial_y\partial^\alpha w^n dxdy,\end{aligned}$$

here we have used $v^n|_{y=0}=0$, thus

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}v^n\big)\big(\partial^\alpha\partial_y w^{n-1}\big)\partial^\alpha w^n dxdy\right|\\&\le C\|w^{n-1}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\big(\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}+\|\partial_y w^n\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\|w^n\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\big).\end{aligned}$$

For the boundary term, similar to the proof of Proposition 3.8, we can get

$$\begin{aligned}&\sum_{|\alpha|\le m+2}\left|\int_{\mathbb{R}}(\partial^\alpha\partial_y w^n\partial^\alpha\partial_y w^n)\big|_{y=0}dx\right|\\&\le\frac{1}{16}\|\partial_y w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}+C\|w^{n-1}\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\|w^n\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

We get finally

$$\begin{aligned}&\frac{d}{dt}\|w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}+\|\partial_y w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}+\epsilon\|\partial_x w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\\&\qquad\le B^{n-1}_T\|w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}+D^{n-1}_T\|w^n\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

□

Lemma C.2. *Suppose that* m, k, ℓ *and* $u^s(t,y)$ *satisfy the assumption of Proposition 3.6,* $\bar{\zeta} > 0$*, then for any* $0 < \epsilon \le 1$*, there exists* $T_\epsilon > 0$ *such that for any* $\tilde{w}_{0,\epsilon} \in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$ *with*

$$\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \bar{\zeta},$$

the iteration equations (C.2) admit a sequence of solution $\{w^n, n \in \mathbb{N}\}$ *such that, for any* $t \in [0, T_\epsilon]$*,*

$$\|w^n(t)\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \frac{4}{3}\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}, \quad \forall n \in \mathbb{N}.$$

Proof. Integrating (C.4) over $[0,t]$, for $0 < t \le T$ and $T > 0$ small,

$$\|w^n(t)\|^m_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \frac{\|\tilde{w}_{0,\epsilon}\|^m_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}}{e^{-\frac{m}{2}B^{n-1}_T t} - \frac{m}{2}D^{n-1}_T t\|\tilde{w}_{0,\epsilon}\|^m_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}}.$$

We prove the Lemma by induction. For $n = 1$, we have

$$\begin{aligned} B^0_T &= C\left(1 + \|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + (1 + \frac{1}{\epsilon})\|\tilde{w}_{0,\epsilon}\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\right) \\ &\le C\left(1 + \bar{\zeta} + (1 + \frac{1}{\epsilon})\bar{\zeta}^2\right), \end{aligned}$$

and

$$D^{n-1}_T = C\|\tilde{w}_{0,\epsilon}\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le C\bar{\zeta}^{m+2}.$$

Choose $T_\epsilon > 0$ small such that

$$\frac{1}{e^{-\frac{m}{2}C\left(1+2\bar{\zeta}+4(1+\frac{1}{\epsilon})\bar{\zeta}^2\right)T_\epsilon} - \frac{m}{2}C(2\bar{\zeta})^{m+2}T_\epsilon(2\bar{\zeta})^m} = \left(\frac{4}{3}\right)^m,$$

we get

$$\|w^1(t)\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \frac{4}{3}\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.$$

The induction hypothesis is for $0 \le t \le T_\epsilon$,

$$\|w^{n-1}(t)\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \frac{4}{3}\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)},$$

and the choose of T_ϵ implies

$$\frac{1}{e^{-\frac{m}{2}B^{n-1}_{T_\epsilon}T_\epsilon} - \frac{m}{2}D^{n-1}_{T_\epsilon}T_\epsilon\|\tilde{w}_{0,\epsilon}\|^m_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}} \le \left(\frac{4}{3}\right)^m$$

for any $t \in [0, T_\epsilon]$, then we finish the proof of the Lemma C.2. □

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