Find the Laurent expansion of the complex function $f(z)=\dfrac{z+2}{z^2+4}$. [98 成大機械 6(1)]

[解] $f(z)=\dfrac{z+2}{(z+2i)(z-2i)}=\dfrac{A}{z+2i}+\dfrac{B}{z-2i}\Rightarrow z+2=A(z-2i)+B(z+2i)$

$$z=-2i\Rightarrow -2i+2=A(-4i)\Rightarrow A=\frac{-2i+2}{-4i}=\frac{2+2i}{4}=\frac{1+i}{2}$$

$$z=2i\Rightarrow 2i+2=B(4i)\Rightarrow B=\frac{2i+2}{4i}=\frac{-2+2i}{-4}=\frac{1-i}{2}$$

$$f(z)=\frac{1+i}{2}\cdot\frac{1}{z+2i}+\frac{1-i}{2}\cdot\frac{1}{z-2i}=\frac{1+i}{2}\cdot\frac{1}{(z-2i)+4i}+\frac{1-i}{2}\cdot\frac{1}{z-2i}$$

當 $|z-2i|<4$，

$$f(z)=\frac{1+i}{2}\cdot\frac{1}{\frac{z-2i}{4i}+1}+\frac{1-i}{2}\cdot\frac{1}{z-2i}$$

$$=\frac{1+i}{2}[1-(\frac{z-2i}{4i})+(\frac{z-2i}{4i})^2-(\frac{z-2i}{4i})^3+\cdots]+\frac{1-i}{2}\cdot\frac{1}{z-2i}$$

當 $|z-2i|>4$，

$$f(z)=\frac{1+i}{2}\cdot\frac{1}{1+\frac{4i}{z-2i}}+\frac{1-i}{2}\cdot\frac{1}{z-2i}$$

$$=\frac{1+i}{2}[1-(\frac{4i}{z-2i})+(\frac{4i}{z-2i})^2-(\frac{4i}{z-2i})^3+\cdots]+\frac{1-i}{2}\cdot\frac{1}{z-2i}$$