$\mathbf{F}=(2x-y)\mathbf{i}-yz^2\mathbf{j}-y^2z^2\mathbf{k}$, calculate $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}ds$ using Stoke's theorem, where S is the half upper surface of the sphere $x^2+y^2+z^2=1$, $(0\le z\le 1)$. [98 嘉大土木 5]

[解](1)令 $f=x^2+y^2+z^2 \Rightarrow \mathbf{n}=\dfrac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z^2\end{vmatrix}=2y(-z^2+z)\mathbf{i}+\mathbf{k}$$

$(\nabla\times\mathbf{F})\cdot\mathbf{n}=2xy(-z^2+z)+z,\ \ \mathbf{n}\cdot\mathbf{k}=z$

$$\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}ds=\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}=\iint_S[2xy(-z^2+z)+z]\frac{dxdy}{z}$$

其中$2xy(-z^2+z)$為x(或y)的奇函數$\Rightarrow \iint_S 2xy(-z^2+z)\dfrac{dxdy}{z}=0$

$$\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}ds=\iint_S dxdy=\pi\cdot 1^2=\pi$$

(2)由Stokes定理知$\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{s}=\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $C: x^2+y^2=1$

Let $x=\cos\theta,\ y=\sin\theta$

$\mathbf{F}\cdot d\mathbf{r}=(2\cos\theta-\sin\theta)\mathbf{i}+0\mathbf{j}+0\mathbf{k})\cdot(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})d\theta=(-\sin 2\theta+\sin^2\theta)d\theta$

$$\oint_C\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}(-\sin 2\theta+\sin^2\theta)\,d\theta=\int_0^{2\pi}(-\sin 2\theta+\frac{1-\cos 2\theta}{2})\,d\theta$$

$$=(\frac{\cos 2\theta}{2}+\frac{\theta}{2}-\frac{\sin 2\theta}{4})\Big|_0^{2\pi}=\pi$$