Solve the initial-value problem $y''' - 2y'' + y' = xe^{-x} + 5$, $y(0) = 2$, $y'(0) = 2$, $y''(0) = -1$. [105 彰師大車輛機電電子電機甲資訊戊 2]

[解]特徵方程式$\lambda^3 - 2\lambda^2 + \lambda = 0 \Rightarrow \lambda(\lambda - 1)^2 = 0 \Rightarrow \lambda = 0, 1, 1$

$y_h = C_1 + (C_2 + C_3 x)e^x$

Let $y_p = (Ax + B)e^{-x} + (Cx^2 + Dx) \Rightarrow y_p' = [-Ax + (A - B)]e^{-x} + (2Cx + D)$

$y_p'' = [Ax + (-2A + B)]e^{-x} + 2C \Rightarrow y_p''' = [-Ax + (3A - B)]e^{-x}$

代入原式

$[-Ax + (3A - B)]e^{-x} - 2\{[Ax + (-2A + B)]e^{-x} + 2C\} + \{[-Ax + (A - B)]e^{-x} + (2Cx + D)\} = xe^{-x} + 5$

$[-4Ax + (8A - 4B)]e^{-x} + 2Cx + (D - 4C) = xe^{-x} + 5$

$$\begin{cases} -4A = 1 \\ 8A - 4B = 0 \\ 2C = 0 \\ D - 4C = 5 \end{cases} \Rightarrow A = -\frac{1}{4}, B = -\frac{1}{2}, C = 0, D = 5$$

得 $y(x) = y_h + y_p = C_1 + (C_2 + C_3 x)e^x - (\frac{1}{4}x + \frac{1}{2})e^{-x} + 5x$

$y' = [C_3 x + (C_2 + C_3)]e^x + (\frac{1}{4}x + \frac{1}{4})e^{-x} + 5$

$y'' = [C_3 x + (C_2 + 2C_3)]e^x - \frac{1}{4}xe^{-x}$

$$\begin{cases} y(0) = 2 \Rightarrow C_1 + C_2 - \frac{1}{2} = 2 \\ y'(0) = 2 \Rightarrow (C_2 + C_3) + \frac{1}{4} + 5 = 2 \Rightarrow C_1 = 8, C_2 = -\frac{11}{2}, C_3 = \frac{9}{4} \\ y''(0) = -1 \Rightarrow C_2 + 2C_3 = -1 \end{cases}$$

得 $y = 8 + (-\frac{11}{2} + \frac{9}{4}x)e^x - (\frac{1}{4}x + \frac{1}{2})e^{-x} + 5x$