

Solve the first order ordinary differential equation $\frac{dy}{dx} = \frac{2(x^3 + xy)}{x^2 - y}$. [106 台科大機械 1(b)]

[解]原式 $\Rightarrow y'(x^2 - y) - 2(x^3 + xy) = 0 \dots\dots\dots(i)$

各項權值分別為 $(m-1)+2$, $(m-1)+m$, 3, $1+m$, 得 $m=2$

令 $y = ux^2 \Rightarrow y' = u'x^2 + 2ux$

$$(i) \Rightarrow (u'x^2 + 2ux)(x^2 - ux^2) - 2(x^3 + x \cdot ux^2) = 0$$

$$(u'x^4 - u'ux^4 + 2ux^3 - 2u^2x^3) - 2(x^3 + ux^3) = 0 \Rightarrow u'x - u'ux - 2u^2 - 2 = 0$$

$$u'x(1-u) - 2(u^2 + 1) = 0 \Rightarrow u'x(u-1) + 2(u^2 + 1) = 0$$

$$\frac{du}{dx} x(u-1) + 2(u^2 + 1) = 0 \Rightarrow \frac{u-1}{2(u^2 + 1)} du + \frac{dx}{x} = 0 \Rightarrow \frac{1}{2} \int \frac{u-1}{u^2 + 1} du + \int \frac{dx}{x} = k$$

$$\frac{1}{2} \int \left(\frac{u}{u^2 + 1} - \frac{1}{u^2 + 1} \right) du + \int \frac{dx}{x} = k \Rightarrow \frac{1}{2} \left[\frac{1}{2} \ln(u^2 + 1) - \tan^{-1} u \right] + \ln x = k$$

$$\ln(u^2 + 1) - 2 \tan^{-1} u + 4 \ln x = 4k \Rightarrow \ln[x^4(u^2 + 1)] - 2 \tan^{-1} u = C$$

$$\ln(y^2 + x^4) - 2 \tan^{-1} \frac{y}{x^2} = C$$