

試求 $x^2y'' - 3xy' + 4y = x^2$ 的通解。[106 南大綠能 8]

[解] 令 $x = e^t \Rightarrow t = \ln x$ 原式為

$$\frac{d}{dt} \left(\frac{d}{dt} - 1 \right) y - 3 \frac{dy}{dt} + 4y = e^{2t} \Rightarrow \frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 4y = e^{2t} \quad (i)$$

特徵方程式 $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2, 2 \Rightarrow y_h = (C_1 + C_2 t)e^{2t}$

令 $y_p = At^2 e^{2t} \Rightarrow y_p' = (2At^2 + 2At)e^{2t} \Rightarrow y_p'' = (4At^2 + 8At + 2A)e^{2t}$

(i) $\Rightarrow (4At^2 + 8At + 2A)e^{2t} - 4(2At^2 + 2At)e^{2t} + 4At^2 e^{2t} = e^{2t}$

$$2Ae^{2t} = e^{2t} \Rightarrow A = \frac{1}{2}$$

$$y = y_h + y_p = (C_1 + C_2 t)e^{2t} + \frac{1}{2} t^2 e^{2t} = \left[C_1 + C_2 \ln x + \frac{1}{2} (\ln x)^2 \right] x^2$$