Find the general solution for the third-order ODE $x^2y''' + 4xy'' + 2y' = \ln x$. [101 交大機械丙 5]

[解]原式$\Rightarrow x^3y''' + 4x^2y'' + 2xy' = x\ln x \cdots\cdots\cdots$(i)，令 $x = e^t \Rightarrow t = \ln x$

$$\text{(i)} \Rightarrow \frac{d}{dt}\left(\frac{d}{dt}-1\right)\left(\frac{d}{dt}-2\right)y + 4\frac{d}{dt}\left(\frac{d}{dt}-1\right)y + 2\frac{dy}{dt} = te^t \Rightarrow \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} = te^t \cdots\cdots\cdots\text{(ii)}$$

特徵方程式$\lambda^3 + \lambda^2 = 0 \Rightarrow \lambda = 0, 0, -1 \Rightarrow y_h = C_1 + C_2t + C_3e^{-t}$

令 $y_p = (At+B)e^t \Rightarrow y_p' = [At + (A+B)]e^t \Rightarrow y_p'' = [At + (2A+B)]e^t \Rightarrow y_p''' = [At + (3A+B)]e^t$

$\text{(ii)} \Rightarrow [At + (3A+B)]e^t + [At + (2A+B)]e^t = te^t \Rightarrow [2At + (5A+2B)]e^t = te^t$

$A = \frac{1}{2}$, $B = -\frac{5}{4}$

$$y = y_h + y_p = C_1 + C_2t + C_3e^{-t} + (\frac{1}{2}t - \frac{5}{4})e^t = C_1 + C_2\ln x + C_3x^{-1} + (\frac{1}{2}\ln x - \frac{5}{4})x$$