Solve the ordinary differential equation $y^2dx+(x^2-xy-y^2)dy=0$. [89 北科大自動化甲 1(a)]

[解]令 $y=ux \Rightarrow dy=xdu+udx$，原式為

$$(ux)^2dx+[x^2-x\cdot ux-(ux)^2](xdu+udx)=0 \Rightarrow u^2dx+(1-u-u^2)(xdu+udx)=0$$

$$[u^2+(1-u-u^2)u]dx+(1-u-u^2)xdu=0 \Rightarrow (-u^3+u)dx+(1-u-u^2)xdu=0$$

$$(u^3-u)dx+(u^2+u-1)xdu=0 \Rightarrow u(u+1)(u-1)dx+(u^2+u-1)xdu=0$$

$$\frac{dx}{x}+\frac{u^2+u-1}{u(u+1)(u-1)}du=0\cdots\cdots\cdots\text{(i)}$$

令 $\dfrac{u^2+u-1}{u(u+1)(u-1)}=\dfrac{A}{u}+\dfrac{B}{u+1}+\dfrac{C}{u-1}$

$$u^2+u-1=A(u+1)(u-1)+Bu(u-1)+Cu(u+1)$$

$$u=0 \Rightarrow -1=-A \Rightarrow A=1$$

$$u=-1 \Rightarrow -1=2B \Rightarrow B=-\frac{1}{2}$$

$$u=1 \Rightarrow 1=2C \Rightarrow C=\frac{1}{2}$$

$$\text{(i)} \Rightarrow \frac{dx}{x}+[\frac{1}{u}-\frac{1}{2(u+1)}+\frac{1}{2(u-1)}]du=0 \Rightarrow \int\frac{dx}{x}+\int[\frac{1}{u}-\frac{1}{2(u+1)}+\frac{1}{2(u-1)}]du=k$$

$$\ln x+\ln u-\frac{1}{2}\ln(u+1)+\frac{1}{2}\ln(u-1)=k \Rightarrow \ln\frac{xu\sqrt{u-1}}{\sqrt{u+1}}=k \Rightarrow \frac{xu\sqrt{u-1}}{\sqrt{u+1}}=C$$

$$\frac{x\cdot\frac{y}{x}\sqrt{\frac{y}{x}-1}}{\sqrt{\frac{y}{x}+1}}=C \Rightarrow \frac{y\sqrt{\frac{y}{x}-1}}{\sqrt{\frac{y}{x}+1}}=C \Rightarrow \frac{y\sqrt{y-x}}{\sqrt{y+x}}=C$$