Solve the ordinary differential equation $y'' - y' - 12y = 2\sinh^2 x$. [101 台科大自控 1(2)]

[解]原式$\Rightarrow y'' - y' - 12y = 2\left(\dfrac{e^x - e^{-x}}{2}\right)^2 \Rightarrow y'' - y' - 12y = \dfrac{e^{2x} + e^{-2x}}{2} - 1 \cdots\cdots\cdots(\text{i})$

特徵方程式$\lambda^2 - \lambda - 12 = 0 \Rightarrow (\lambda + 3)(\lambda - 4) = 0 \Rightarrow \lambda = -3,\ 4$

$y_h(x) = C_1 e^{-3x} + C_2 e^{4x}$

令 $y_p(x) = A + Be^{2x} + Ce^{-2x} \Rightarrow y_p' = 2Be^{2x} - 2Ce^{-2x} \Rightarrow y_p'' = 4Be^{2x} + 4Ce^{-2x}$

代入(i)式得

$$(4Be^{2x} + 4Ce^{-2x}) - (2Be^{2x} - 2Ce^{-2x}) - 12(A + Be^{2x} + Ce^{-2x}) = \frac{e^{2x} + e^{-2x}}{2} - 1$$

$$-10Be^{2x} - 6Ce^{-2x} - 12A = \frac{e^{2x} + e^{-2x}}{2} - 1 \Rightarrow A = \frac{1}{12},\ B = -\frac{1}{20},\ C = -\frac{1}{12}$$

解為 $y(x) = y_h(x) + y_p(x) = C_1 e^{-3x} + C_2 e^{4x} + \dfrac{1}{12} - \dfrac{1}{20} e^{2x} - \dfrac{1}{12} e^{-2x}$