

Evaluate $\int_0^{\infty} \frac{x^{1/3}}{x(x^2+1)} dx$. [86 台科大機械 5]

[解] 令 $f(z) = \frac{z^{1/3}}{z(z^2+1)}$, $f(z)$ 有單極點在 $z=0, i, -i$

$$\text{而 } z^{1/3} = |z|^{1/3} e^{i\theta/3}$$

$$\text{在 } L_1 \text{ 上, } f(z) \rightarrow \frac{x^{1/3}}{x(x^2+1)} \quad \text{在 } L_2 \text{ 上, } f(z) \rightarrow \frac{x^{1/3} e^{i2\pi/3}}{x(x^2+1)}$$

$$R_i = \frac{i^{1/3}}{3i^2+1} = \frac{e^{i\pi/6}}{-2} \quad R_{-i} = \frac{(-i)^{1/3}}{3(-i)^2+1} = \frac{e^{i\pi/2}}{-2}$$

$$\oint f(z) dz = 2\pi i (R_i + R_{-i})$$

$$\int_{C_R} f(z) dz + \int_R^r \frac{x^{1/3} e^{i2\pi/3}}{x(x^2+1)} dx + \int_{C_r} f(z) dz + \int_r^R \frac{x^{1/3}}{x(x^2+1)} dx = \frac{e^{i\pi/6} + e^{i\pi/2}}{-2}$$

當 $r \rightarrow 0, R \rightarrow \infty$ 時

$$0 + e^{i2\pi/3} \int_0^{\infty} \frac{x^{1/3}}{x(x^2+1)} dx + 0 + \int_0^{\infty} \frac{x^{1/3}}{x(x^2+1)} dx = \frac{e^{i\pi/6} + e^{i\pi/2}}{-2}$$

$$(1 - e^{i2\pi/3}) \int_0^{\infty} \frac{x^{1/3}}{x(x^2+1)} dx = \frac{e^{i\pi/6} + e^{i\pi/2}}{-2}$$

$$\frac{3 - \sqrt{3}i}{2} \int_0^{\infty} \frac{x^{1/3}}{x(x^2+1)} dx = \frac{\pi(3 - \sqrt{3}i)}{2} \Rightarrow \int_0^{\infty} \frac{x^{1/3}}{x(x^2+1)} dx = \pi$$

