Apply residue theorem to evaluate $I=\int_0^{2\pi}\frac{d\theta}{2-\sin\theta}$. [88 台科大機械 4]

[解]令$z=e^{i\theta}\Rightarrow dz=ie^{i\theta}d\theta=izd\theta\Rightarrow d\theta=\frac{dz}{iz}, \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{z-1/z}{2i}=\frac{z^2-1}{2iz}$

$$\frac{d\theta}{2-\sin\theta}=\frac{\frac{dz}{iz}}{2-\frac{z^2-1}{2iz}}=\frac{2dz}{4iz-(z^2-1)}=-\frac{2dz}{z^2-4iz-1}$$

有單極點$z=\frac{4i\pm\sqrt{(-4i)^2+4}}{2}=\frac{4i\pm 2\sqrt{3}i}{2}=(2\pm\sqrt{3})i$，其中只有$(2-\sqrt{3})i$在單位圓內

$$R_{(2-\sqrt{3})i}=-\frac{2}{2z-4i}\bigg|_{z=(2-\sqrt{3})i}=-\frac{2}{2(2-\sqrt{3})i-4i}=-\frac{2}{-\sqrt{3}i}=-\frac{2i}{\sqrt{3}}$$

$$\int_0^{2\pi}\frac{d\theta}{2-\sin\theta}d\theta=2\pi i\,(R_{(2-\sqrt{3})i})=2\pi i\,(-\frac{2i}{\sqrt{3}})=\frac{4\pi}{\sqrt{3}}$$