

Apply residue theorem to evaluate $I = \int_0^{2\pi} \frac{d\theta}{2 - \sin \theta}$. [88 台科大機械 4]

[解] 令 $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = izd\theta \Rightarrow d\theta = \frac{dz}{iz}$, $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i} = \frac{z^2 - 1}{2iz}$

$$\frac{d\theta}{2 - \sin \theta} = \frac{\frac{dz}{iz}}{2 - \frac{z^2 - 1}{2iz}} = \frac{2dz}{4iz - (z^2 - 1)} = -\frac{2dz}{z^2 - 4iz - 1}$$

有單極點 $z = \frac{4i \pm \sqrt{(-4i)^2 + 4}}{2} = \frac{4i \pm 2\sqrt{3}i}{2} = (2 \pm \sqrt{3})i$, 其中只有 $(2 - \sqrt{3})i$ 在單位圓內

$$R_{(2-\sqrt{3})i} = -\frac{2}{2z - 4i} \Big|_{z=(2-\sqrt{3})i} = -\frac{2}{2(2-\sqrt{3})i - 4i} = -\frac{2}{-\sqrt{3}i} = -\frac{2i}{\sqrt{3}}$$

$$\int_0^{2\pi} \frac{d\theta}{2 - \sin \theta} = 2\pi i (R_{(2-\sqrt{3})i}) = 2\pi i \left(-\frac{2i}{\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}}$$