

Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \leq x < \pi \end{cases}$, in a Fourier series. [102 中原生醫 4、104 中原土木 3]

$$[\text{解}] f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{1}{\pi} \left(\pi x - \frac{x^2}{2} \right) \Big|_0^{\pi} = \frac{\pi}{2}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx = \frac{1}{\pi} \left[\frac{\pi \sin nx}{n} \Big|_0^{\pi} - \frac{1}{n} (x \sin nx) \Big|_0^{\pi} - \int_0^{\pi} \sin nx dx \right] \\ &= \frac{1}{\pi} \left[0 - \frac{1}{n} (0 + \frac{\cos nx}{n} \Big|_0^{\pi}) \right] = -\frac{\cos n\pi - 1}{n^2 \pi} = -\frac{(-1)^n - 1}{n^2 \pi} = \frac{1 - (-1)^n}{n^2 \pi} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \sin nx dx = \frac{1}{\pi} \left[-\frac{\pi \cos nx}{n} \Big|_0^{\pi} + \frac{1}{n} (x \cos nx) \Big|_0^{\pi} - \int_0^{\pi} \cos nx dx \right] \\ &= \frac{1}{\pi} \left[-\frac{\pi(\cos n\pi - 1)}{n} + \frac{1}{n} (\pi \cos n\pi - 0) \right] = \frac{1}{n\pi} \end{aligned}$$

$$f(x) = \frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right], \quad -\pi < x < \pi$$