

(a) Find the Fourier series of the function $f(x)$, where $f(x) = x^2$, $-\pi \leq x \leq \pi$. (b) Use the results in (a) to prove $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$. (c) Use the results in (a) to calculate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = ?$

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[解](a) 令 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, 因 $f(x)$ 為偶函數 $b_n = 0$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx = \frac{2}{n\pi} (x^2 \sin nx \Big|_0^{\pi} - 2 \int_0^{\pi} x \sin nx dx) \\ &= \frac{4}{n^2 \pi} (x \cos nx \Big|_0^{\pi} - \int_0^{\pi} \cos nx dx) = (-1)^n \frac{4}{n^2} \end{aligned}$$

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

(b) $x = \pi$ 代入 (a) 得

$$\frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n \Rightarrow \pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(c) $x = 0$ 代入 (a) 得

$$\frac{f(0^-) + f(0^+)}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow 0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$