Chapter 14 Modelling the Variance ARCH-GARCH Models

Introduction
Conventional econometric analysis views the variance of the disturbance terms as constant over time. However, mainly financial but also many economic time series exhibit periods of usually **high volatility** followed by more tranquil periods of low volatility.
The assumption of homoskedasticity is very limiting.

The ARCH model
The first ARCH model was presented by Engle (1982). The model suggests that the variance of the residuals at time $t$ depends on the squared error term from past periods.
Consider the simple model:

$$ Y_t = a + \beta'X_t + u_t $$

(14.1)

where $X_t$ is a $k \times 1$ vector of explanatory variables and $\beta$ is a $k \times 1$ vector of coefficients.

$$ u_t \sim iid \ N(0, \sigma^2) $$

(14.2)

$$ \sigma_t^2 = \gamma_0 + \gamma_1 u_{t-1}^2 $$

(14.3)

The ARCH(1) model
For example ARCH(2) process will be:

$$ h_t = \gamma_0 + \gamma_1 u_{t-1}^2 + \gamma_2 u_{t-2}^2 $$

(14.5)

Here equation (14.4) is called the mean equation and equation (14.5) is called the variance equation.

The ARCH(q) model
For example ARCH(2) process will be:

$$ h_t = \gamma_0 + \gamma_1 u_{t-1}^2 + \gamma_2 u_{t-2}^2 $$

(14.6)

In general the ARCH(q) process will be given by:

$$ h_t = \gamma_0 + \gamma_1 u_{t-1}^2 + \gamma_2 u_{t-2}^2 + \ldots + \gamma_q u_{t-q}^2 = \gamma_0 + \sum_{j=1}^{q} \gamma_j u_{t-j}^2 $$

(14.8)
The ARCH(q) model will simultaneously examine the mean and the variance of series according to the following specification:

\[
Y_t = a + \beta' X_t + u_t
\]

\[
u_t \mid \Omega_t \sim iid N(0, h_t)
\]

\[
h_t = \gamma_0 + \sum_{j=1}^{q} \gamma_j \mu_{t-j}^2
\]

**Testing for ARCH effects**

Before estimating ARCH(q) models it is important to check for the possible presence of ARCH effects in order to know which models require the ARCH estimation method instead of the OLS. The test can be done along the line of the Breusch-Pagan test, which entails estimation of the mean equation:

\[
Y_t = a + \beta' X_t + u_t
\]

\[
\hat{\mu}_t^2 = \gamma_0 + \gamma_1 \hat{\mu}_{t-1}^2 + ... + \gamma_q \hat{\mu}_{t-q}^2 + w_t
\]

Run auxiliary regression of the squared residuals the lagged squared terms and then compute the \(R^2\) time \(T\). Null hypothesis of heteroskedasticity the resulting test statistic follows a \(\chi^2\) distribution with \(q\) degrees of freedom.

**Estimation of ARCH models by iteration**

This model can no long be estimated using a simple technique such as OLS, instead we must solve a non-linear maximization problem, which requires an iterative computer algorithm to search for the solution to the problem.

**Estimating ARCH models in EViews**

A more mathematical approach

\[
Y_t = a + \beta' X_t + u_t
\]

\[
u_t = z_t \sqrt{h_t}
\]

Where \(z_t\) follows a standard normal distribution with zero mean and variance one, and \(h_t\) is a scaling factor.
The GARCH model

A new idea was born which was to include the lagged conditional variance terms as autoregressive terms. This idea was worked out by Tim Bollerslev.

The GARCH \((p, q)\) model

The general GARCH\((p, q)\) model

\[
Y_t = a + \beta' X_t + u_t, \\
u_t, \Omega_t \sim iid N(0, h_t)
\]

\[
h_t = \gamma_0 + \sum_{i=1}^{p} \delta_i h_{t-i} + \sum_{j=1}^{q} \gamma_j u_{t-j}^2
\]

The simplest form of the GARCH\((p, q)\) model is the GARCH(1,1) model for which the variance equation has the form:

\[
h_t = \gamma_0 + \delta h_{t-1} + \gamma_1 u_{t-1}^2
\]

The GARCH(1,1) as an infinite ARCH\((p)\) process

\[
h_t = \gamma_0 + \delta h_{t-1} + \gamma_1 u_{t-1}^2 = \gamma_0 + \delta (\gamma_0 + \delta h_{t-2} + \gamma_1 u_{t-2}^2) + \gamma_1 u_{t-1}^2 = \gamma_0 + \gamma_1 u_{t-1}^2 + \delta \gamma_0 + \delta^2 h_{t-2} + \delta \gamma_1 u_{t-2}^2 = \frac{\gamma_0}{1-\delta} + \gamma_1 (u_{t-1}^2 + \delta u_{t-2}^2 + \delta^2 u_{t-3}^2 + \ldots) = \frac{\gamma_0}{1-\delta} + \gamma_1 \sum_{j=1}^{\infty} \delta^{j-1} u_{t-j}^2
\]

The GARCH(1,1) we have less parameters to estimate and therefore lose fewer degrees of freedom.

Estimating GARCH models in EViews

Alternative specifications

The GARCH in mean or GARCH-M model

The GARCH-M\((p, q)\) model has the following form:

\[
Y_t = a + \beta' X_t + \theta h_t + u_t, \\
u_t, \Omega_t \sim iid N(0, h_t)
\]

\[
h_t = \gamma_0 + \sum_{i=1}^{p} \delta_i h_{t-i} + \sum_{j=1}^{q} \gamma_j u_{t-j}^2
\]
\[
Y_t = a + \beta X_t + \theta \sqrt{h_t} + u_t
\]

\[u_t \mid \Omega_t \sim iid \ N(0, h_t)\] (14.29)

\[h_t = \gamma_0 + \sum_{i=1}^{p} \delta_i h_{t-i} + \sum_{j=1}^{q} \gamma_j u_{t-j}^2\] (14.30)

Estimating GARCH-M models in EViews

The threshold GARCH(TGARCH) model

The threshold GARCH model was introduced by the works of Zakoian(1990) and Glosten, Jaganathan and Runkle(1993).

The main target of this model is to capture asymmetries in terms of negative and positive shock.

The specification of the conditional variance equation (for a TGARCH(1,1)) is given by:

\[h_t = \gamma_0 + \sum_{i=1}^{p} \delta_i h_{t-i} + \sum_{j=1}^{q} \gamma_j u_{t-j}^2\] (14.31)

Where \(d_t\) takes the value of 1 for \(u_t < 0\) and 0 otherwise. If \(\theta > 0\) we conclude that there is asymmetry, while if \(\theta = 0\) the new impact is symmetric.

Estimating TGARCH models in EViews

The exponential GARCH (EGARCH) model

The exponential GARCH or EGARCH model was first developed by Nelson (1991), and the variance equation for this model is given by:

\[\log(h_t) = \gamma + \sum_{j=1}^{q} \xi_j \frac{u_{t-j}}{\sqrt{h_{t-j}}} + \sum_{j=1}^{q} \xi_j \frac{u_{t-j}}{\sqrt{h_{t-j}}} + \sum_{i=1}^{p} \delta_i \log(h_{t-i})\] (14.33)

The left-hand side is the log of the variance series. This makes the leverage effect exponential instead of quadratic, and therefore the estimates of the conditional variance are guaranteed to be non-negative.

Estimating EGARCH models in EViews
Adding explanatory variables in the variance equation

GARCH models also allow us to add explanatory variables in the specification of the conditional variance. We can have an augment GARCH\((q, p)\) specification such as the following:

\[
h_t = \gamma_0 + \sum_{i=1}^{\mu} \delta_i h_{t-i} + \sum_{j=1}^{q} \gamma_j \mu_{t-j}^2 + \sum_{k=1}^{m} \mu_k X_k
\]

where \(X_k\) is a set of explanatory variables that might help to explain the variance.

Asteriou and Price(2001) estimated the following model:

\[
\Delta \ln(Y_t) = a_0 + a_1 \sum_{i=0}^{4} \Delta \ln(Y_{t-i}) + a_2 \sum_{i=0}^{4} \Delta \ln(I_{t-i}) + \sum_{j=1}^{6} d_j X_{j} u_t + u_t
\]

\[
u_t \sim N(0, h_t)
\]

\[
h_t = b_1 e_{t-1}^2 + b_2 h_{t-1}
\]